NUMBER: *no. 4518*
TITLE: *Δ-Modulated Feedback in Discretization of Sliding Mode Control*
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1) Does the Introduction state the purpose of the paper? Yes
2) Is the significance of the paper explained relative to previous work? Yes
3) Is the paper clearly written and well organized?

The paper is well organized, but the proofs of the Theorem are too long.

4) Briefly, the contribution of the paper appears to be the following:

In the paper, the global attracting properties of a scalar discrete-time system under Δ-modulated feedback is first investigated. Then, the obtained results are applied to a zero-order hold discretized sliding mode control system with relative degree one. The study shows that the considered system can have only 2-periodic orbits regulated by the amplitude of the sampling period. The presented results are interesting, but the proofs of the Theorems are too long, boring and (in my opinion) they can be simplified and shortened if a different approach is considered (see below). In my opinion, the details of the proof of Theorem 3 can be omitted.
The paper, properly shortened, is suitable to be published as brief paper.

5) Suggestions for improving the paper:

The proof of Theorem 1 is long, articulated and difficult to follow because too many cases are considered. The proof can be shortened (for example) taking into account the solutions of the discrete time equation (1) with different initial conditions.

$$x(k+1) = a\,x - \Delta\,\mathrm{sgn}(ax) \tag{1}$$

The fact that (see points 2 and 3 of Theorem 1) when $|a| < 1$ the set $\{-\Delta_a,\ \Delta_a\}$, where $\Delta_a = \frac{\Delta}{1+|a|}$, is a global attractor and the two points $-\Delta_a$ and Δ_a are 2-periodic points when $0 \le a < 1$ and 1-periodic points when $-1 < a < 0$ can be easily proved as follows. Let $x(0) = x_0$ be the initial condition and let us suppose that $x(k)$ satisfies the relation: $\mathrm{sgn}(a\,x(k)) = \mathrm{sgn}(a\,x_0)(-\mathrm{sgn}(a))^k$. With these assumptions, equation (1) transforms as follows:

$$x(k+1) = a\,x - \Delta\,\mathrm{sgn}(a\,x_0)(-\mathrm{sgn}(a))^k \tag{2}$$

Applying the $\mathcal{Z}$-transformation to equation (2) one obtains:

$$z\,X(z) - z\,x_0 = a\,X(z) - \Delta\,\mathrm{sgn}(a\,x_0)\frac{z}{z+\mathrm{sgn}(a)}$$

Solving with respect to $X(z)$ one obtains:

$$\begin{aligned} X(z) &= \frac{z}{(z-a)}\,x_0 - \Delta\,\mathrm{sgn}(a\,x_0)\frac{z}{(z-a)(z+\mathrm{sgn}(a))} \\ &= \frac{z}{(z-a)}\,x_0 - \frac{\Delta\,\mathrm{sgn}(x_0)}{(1+|a|)}\left[\frac{z}{(z-a)} - \frac{z}{(z+\mathrm{sgn}(a))}\right] \end{aligned}$$

from which

$$x(k) = x_0\,a^k - \Delta_a\,\mathrm{sgn}(x_0)\left[a^k - (-\mathrm{sgn}(a))^k\right] \tag{3}$$

Since $|a| < 1$, the system (1) is asymptotically stable and therefore stable bounded input - bounded output. This means that for all the initial conditions $x_0 \in \mathcal{D}$ for which $\mathrm{sgn}(a\,x(k)) = \mathrm{sgn}(a\,x_0)(-\mathrm{sgn}(a))^k$, when $k \to \infty$ the trajectory $x(k)$ tends to

$$x(k) \simeq \Delta_a\,\mathrm{sgn}(x_0)(-\mathrm{sgn}(a))^k$$

that is the set $\{-\Delta_a,\ \Delta_a\}$ is an global attractor for the domain $\mathcal{D}$. Moreover, from (3) it follows that when $x_0 = \pm\Delta_a$:

$$x(k) = \pm\Delta_a\, a^k \mp \Delta_a \left[a^k - (-\operatorname{sgn}(a))^k\right] = \pm\Delta_a(-\operatorname{sgn}(a))^k$$

This proofs that the two points $-\Delta_a$ and Δ_a are 2-periodic points when $0 \leq a < 1$, and 1-periodic points when $-1 < a < 0$ (the point 3 of Theorem 1). Domain $\mathcal{D}$ is the set of all the initial conditions x_0 which satisfy relation $\operatorname{sgn}(x(k)) = \operatorname{sgn}(x_0)(-\operatorname{sgn}(a))^k$, that is, see eq. (3),

$$\operatorname{sgn}\left(x_0\, a^k - \Delta_a \operatorname{sgn}(x_0)[a^k - (-\operatorname{sgn}(a))^k]\right) = \operatorname{sgn}(x_0)(-\operatorname{sgn}(a))^k$$

from which

$$\operatorname{sgn}\left((|x_0| - \Delta_a)a^k + \Delta_a(-\operatorname{sgn}(a))^k\right) = (-\operatorname{sgn}(a))^k$$

Since $\Delta_a > 0$ it follows that:

$$\operatorname{sgn}\left((\frac{|x_0|}{\Delta_a} - 1)a^k + (-\operatorname{sgn}(a))^k\right) = (-\operatorname{sgn}(a))^k$$

and finally

$$\operatorname{sgn}\left((\frac{|x_0|}{\Delta_a} - 1)|a|^k + (-1)^k\right) = (-1)^k \tag{4}$$

When $|a| < 1$ the equation (4) is satisfied if

$$\left|\frac{|x_0|}{\Delta_a} - 1\right| < 1 \qquad \rightarrow \qquad |x_0| < 2\,\Delta_a$$

that is for $x_0 \in \mathcal{D} =]-2\,\Delta_a,\ 2\,\Delta_a[$. Let now consider the case $x_0 \notin \mathcal{D}$ and let us suppose that $x(k) \notin \mathcal{D}$ satisfies the relation: $\operatorname{sgn}(a\, x(k)) = \operatorname{sgn}(a\, x_0)(\operatorname{sgn}(a))^k$. With these assumptions, equation (1) transforms as follows:

$$x(k+1) = a\, x - \Delta \operatorname{sgn}(a\, x_0)(\operatorname{sgn}(a))^k \tag{5}$$

Applying the $\mathcal{Z}$-transformation to equation (2), solving with respect to $X(z)$ and applying the $\mathcal{Z}$-antitransformation one obtains

$$x(k) = x_0\, a^k + \Delta_b \operatorname{sgn}(x_0) \left[a^k - (\operatorname{sgn}(a))^k\right] \tag{6}$$

where $\Delta_b = \frac{\Delta}{1-|a|}$. The equation (6) holds for all $x_0 \notin \mathcal{D}$ and $x(k) \notin \mathcal{D}$ which satisfy relation $\operatorname{sgn}(x(k)) = \operatorname{sgn}(x_0)(\operatorname{sgn}(a))^k$, that is, see eq. (6),

$$\operatorname{sgn}\left(x_0\, a^k - \Delta_b \operatorname{sgn}(x_0)[a^k - (\operatorname{sgn}(a))^k]\right) = \operatorname{sgn}(x_0)(\operatorname{sgn}(a))^k$$

from which one obtains

$$\operatorname{sgn}\left((\frac{|x_0|}{\Delta_a} - 1)|a|^k + 1\right) = 1 \tag{7}$$

When $|a| < 1$ the equation (4) is satisfied if

$$\frac{|x_0|}{\Delta_a} - 1 > -1 < 1 \qquad \rightarrow \qquad |x_0| > 0$$

that is also $\forall x_0 \notin \mathcal{D} =]-2\,\Delta_a,\ 2\,\Delta_a[$. Since $|a| < 1$ and $\Delta_b < 1$, in a finite time the trajectory $x(k)$ reaches the domain $\mathcal{D}$, that is, $\forall x_0 \notin \mathcal{D}$ it exists a finite instant k such that $x(k) \in \mathcal{D}$.

Similar considerations can be done to prove point 1 of Theorem 1.

6) Small errors and typing errors:

- page 3, eq. (4): "$\ldots 0_{(p-m)\times(p-m)} \ldots$" $\rightarrow$ "$\ldots 0_{(p-m)\times(n-p)} \ldots$"
- page 5, eq. (13): "$\ldots + \int_0^h d^{A\tau} \mathrm{d}\, \tau u_k \ldots$" $\rightarrow$ "$\ldots + \int_0^h e^{A\tau} b\, \mathrm{d}\tau\, u_k \ldots$"