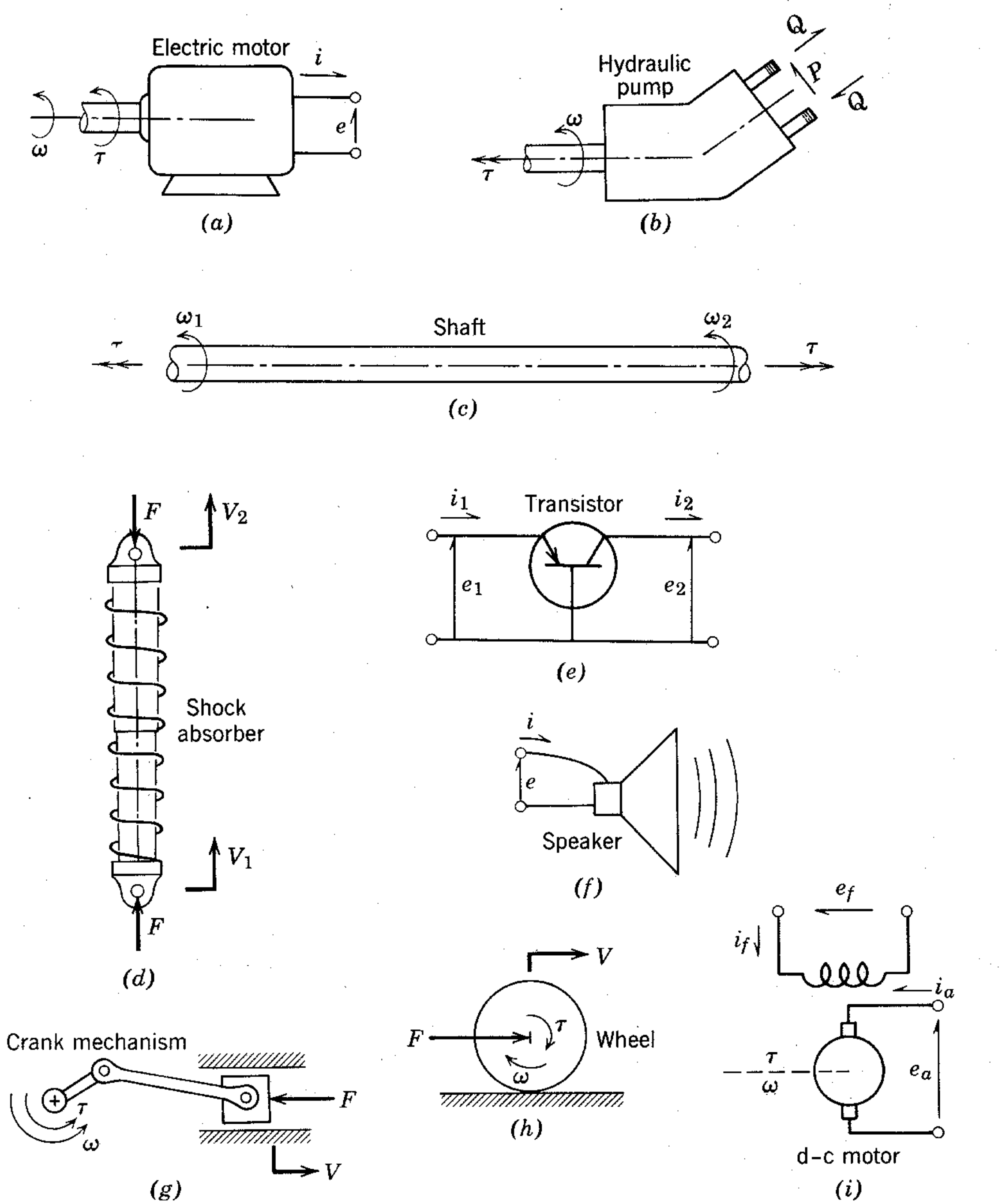
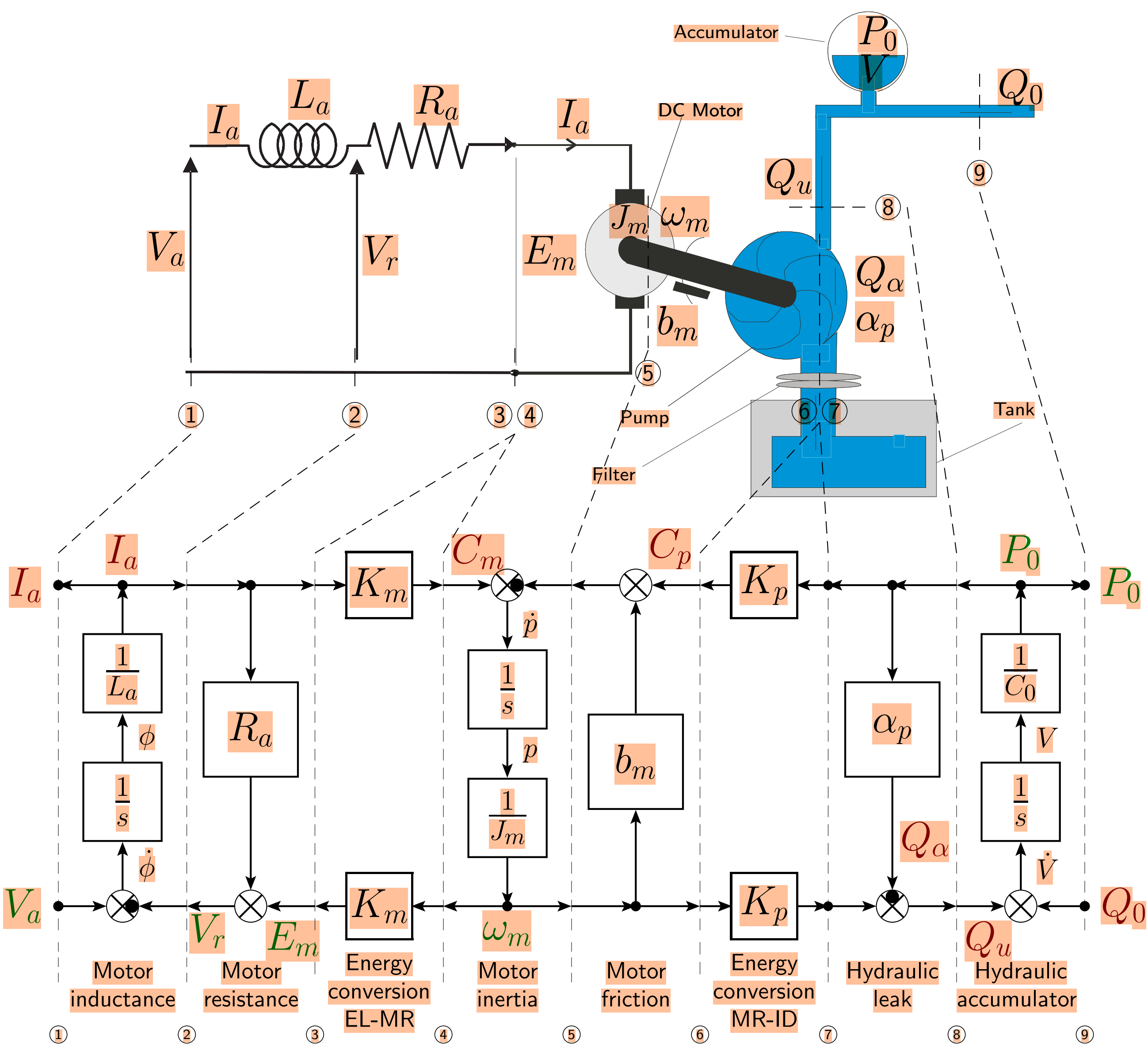
The Power-Oriented Graphs Modeling Technique

- Complex physical systems can always be decomposed in basic physical elements which interact with each other by means of "energetic ports" and "power flows".
- Examples of elementary physical systems:



- **The Power-Oriented Graphs** (POG):
  - is a graphical modeling techniques that uses an "*energetic approach*" for modeling physical systems.
  - use the "*power*" and "*energy*" variables as basic concepts for modeling physical systems.
  - the POG block schemes are easy to use, easy to understand and can be directly implemented in Simulink.
  - is based on the same energetic concept of the Bond Graph modeling technique. See: Karnopp, Margolis, Rosemberg, "*System Dynamics - A unified approach*", John Wiley & Sons.
- ***Example***. A DC electric motor that moves an hydraulic pump. The physical system and the corresponding POG block scheme:

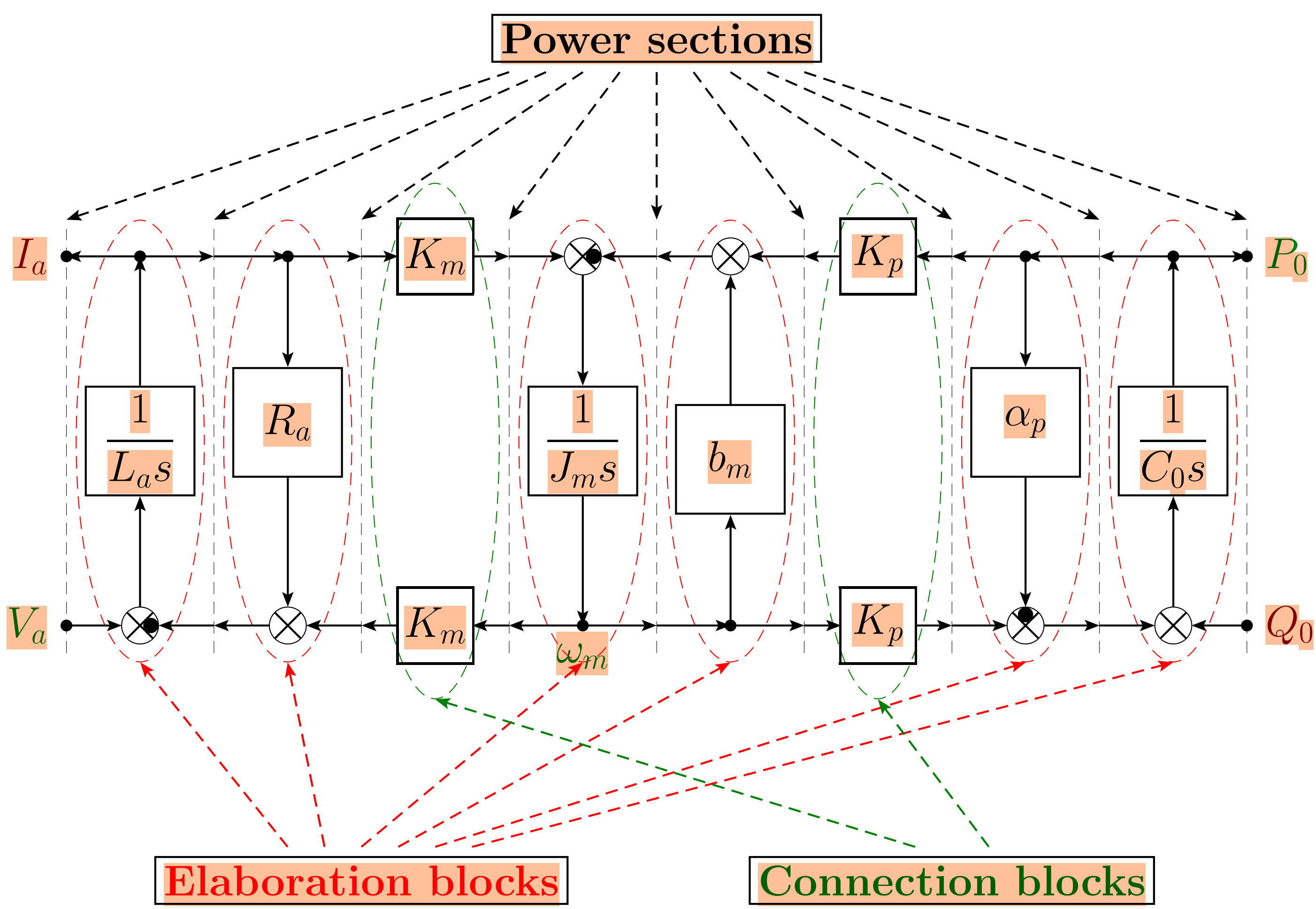


- The "*energetic approach*" is useful for modeling because the physical systems are always characterized by the following properties:

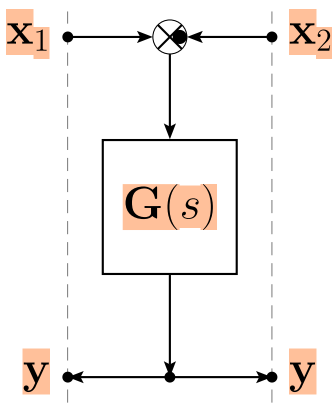
  1) a physical system "*stores and/or dissipates energy*";
  2) the dynamic model of a physical system describes "*how the energy moves*" within the system,
  3) the energy moves from point to point within the system only by means of two "*power variables*".

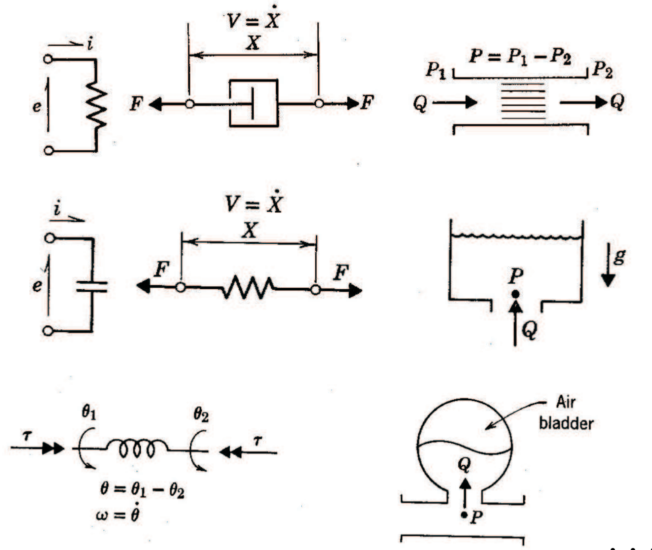
- ***Power sections.*** The "dashed lines" of the POG schemes represent the "power sections" of the system. The inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ of the two "power variables" $\mathbf{x}$ and $\mathbf{y}$ touched by the dashed line has the physical meaning of "*power flowing through the section*".

- ***POG blocks.*** The POG technique uses **two blocks** for modeling the physical systems: the ***Elaboration block*** and the ***Connection block***.

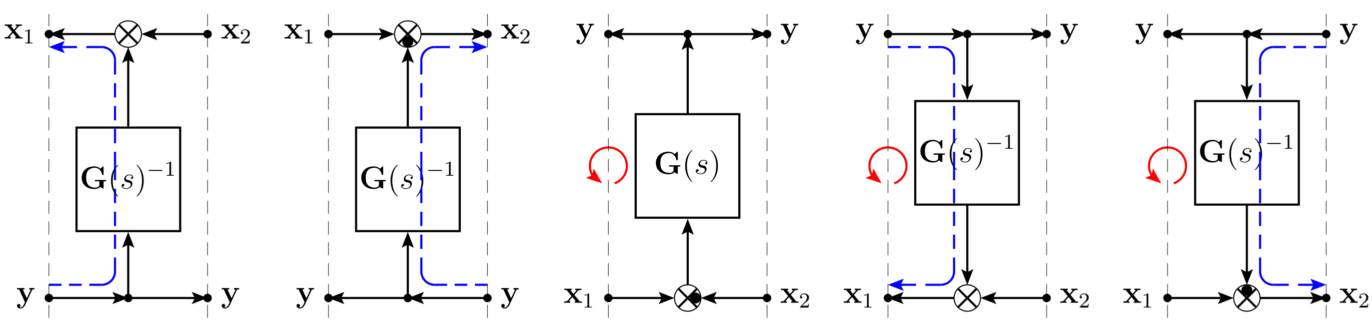


- The ***Elaboration block*** is used for modeling the physical elements that store and/or dissipate energy (i.e. springs, masses, dampers, capacities, inductances, resistances, hydraulic inductances, etc.).





- Equivalent ways of representing the elaboration block:



Input path inversion | Output path inversion | Upside down | Input path inversion + upside down | Output path inversion + upside down

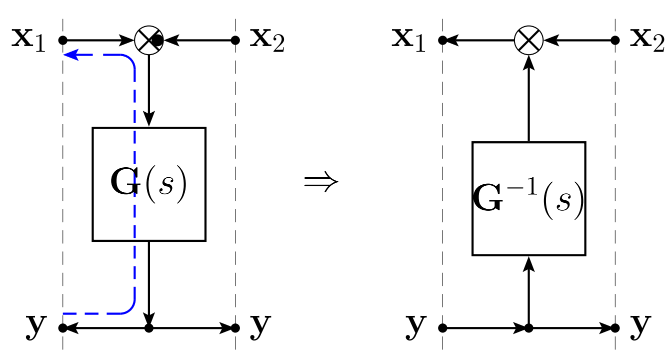
- The **black spot** within the summation element represents, when it is present, a **minus sign** that multiplies the entering variable.

- **Rules for inverting a path**:

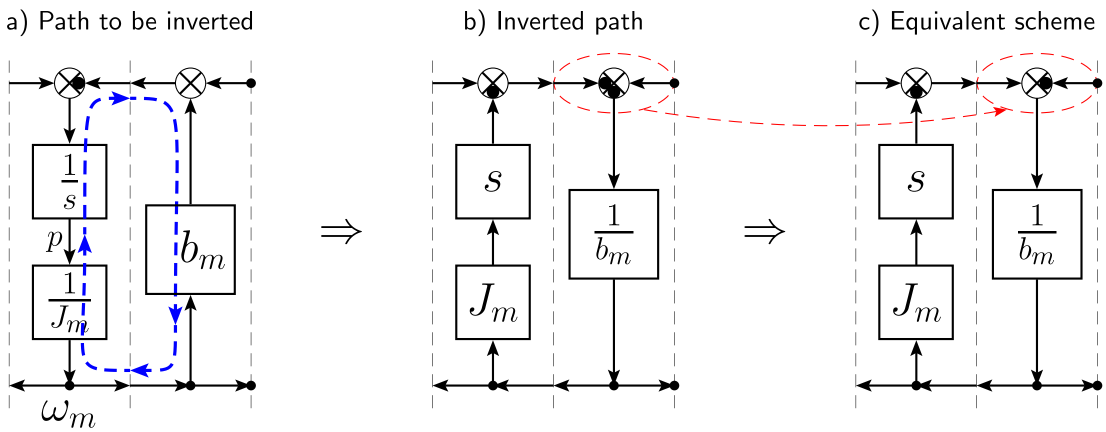
  1) Invert each line of the path;

  2) Invert each block of the path;

  3) In the summation blocks invert the sign of the variables which belong to the path;

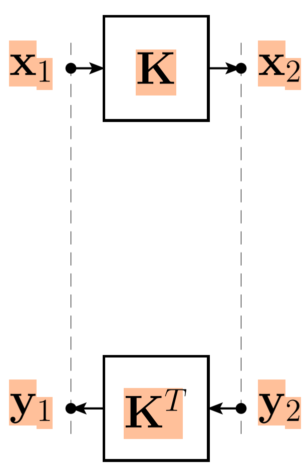


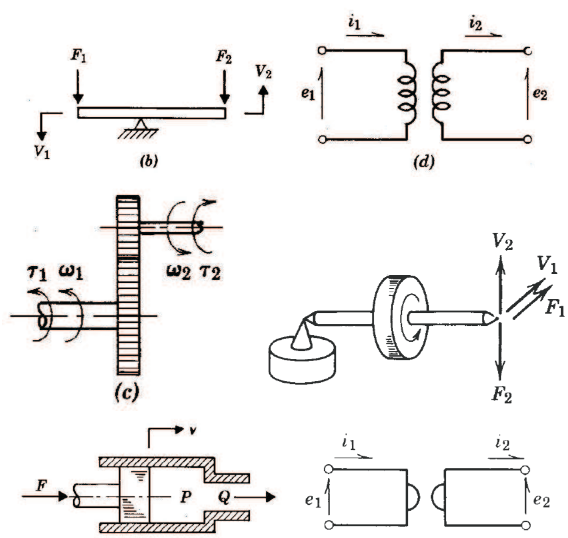
- **Example of path inversion**:



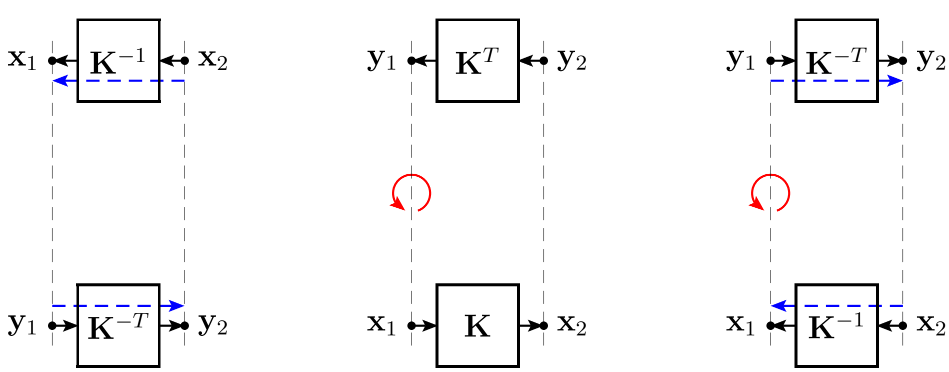
The POG block scheme does not change if all the signs of a summation block are switched to the opposite value.

- The ***Connection block*** is used for modeling the physical elements that "transform the power without losses" (i.e. *neutral elements* such as levers, gear reductions, transformers, etc.).





- Equivalent ways of representing the connection block:



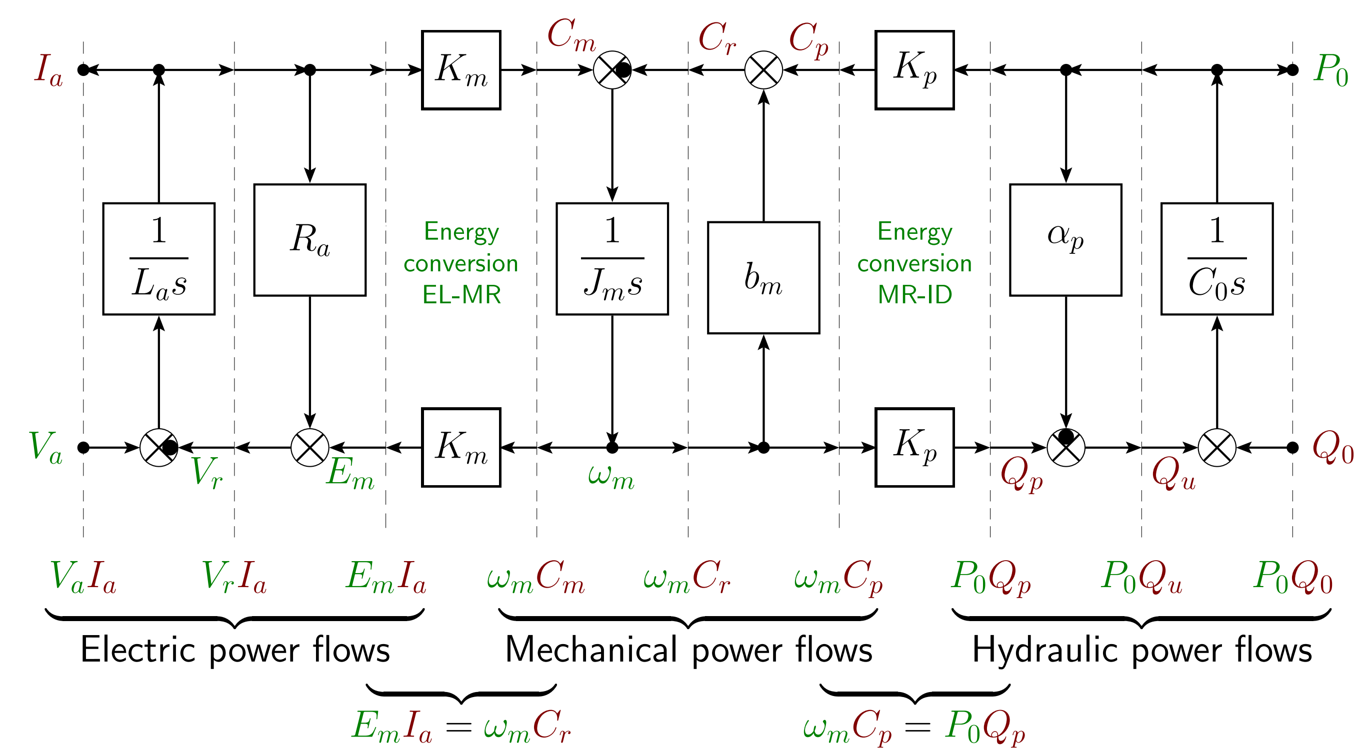
Inverted path | Upside down | Inverted path + upside down

- Matrix $\mathbf{K}$ can also be rectangular or time varying.

- The main ***Energetic domains*** in modeling physical systems are: *Electrical, Mechanical (Translational and Rotational) and Hydraulic*.
- Each energetic domain is characterized by two ***power variables***:

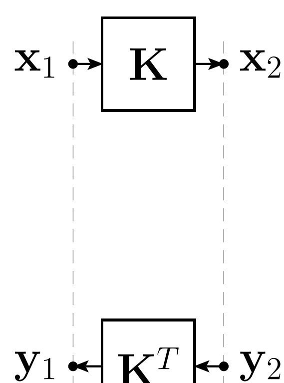
POG variables	Electrical	Mech. Tras.	Mech. Rot.	Hydraulic
Across-var.: v_e	V Voltage	$\dot{x}$ Velocity	ω Angular vel.	P Pressure
Through-var.: v_f	I Current	F Force	τ Torque	Q Volume flow rate

- Note: in the following, the **Across**-variables will be also called **Effort**-variables, and the **Through**-variables will be also called **Flow**-variables.
- In each dashed line of a POG schemes the product $P = v_e v_f$ of two power variables v_e and v_f has the physical meaning of "*power* P *flowing through that particular power section*".



- The connection blocks convert the power without generating nor dissipating energy.
- The input power flow $\mathbf{x}_1^T \mathbf{y}_1$ is always equal to the output power flow $\mathbf{x}_2^T \mathbf{y}_2$:

$$\begin{aligned}
\mathbf{x}_1^T \mathbf{y}_1 &= < \mathbf{x}_1^T, \mathbf{y}_1 > = < \mathbf{x}_1^T, \mathbf{K}^T \mathbf{y}_2 > \\
&= < (\mathbf{K}\mathbf{x}_1)^T, \mathbf{y}_2 > = < \mathbf{x}_2^T, \mathbf{y}_2 > \\
&= \mathbf{x}_2^T \mathbf{y}_2
\end{aligned}$$



- The ***power variables*** can be divided in two groups:

  1) the ***Effort/Across-variables*** (voltage V, velocity $\dot{\mathbf{x}}$, angular velocity ω and pressure P) are defined *"between two points* of the space:

  2) the ***Flow/Through-variables*** (current I, force F, torque τ and volume flow rate Q) are defined *"in each point"* of the space:

Dynamic structure of the Energetic domains.

- Each Energetic domain is characterized by only 3 different types of physical elements:

  2 dynamic elements "$\mathcal{D}_e$" and "$\mathcal{D}_f$" which store the energy (i.e. capacitors, inductors, masses, springs, etc.);

  1 static element "$\mathcal{R}$" which dissipates (or generates) the energy (i.e. resistors, frictions, etc.);

- The dynamics of a physical system can be described using 4 variables:

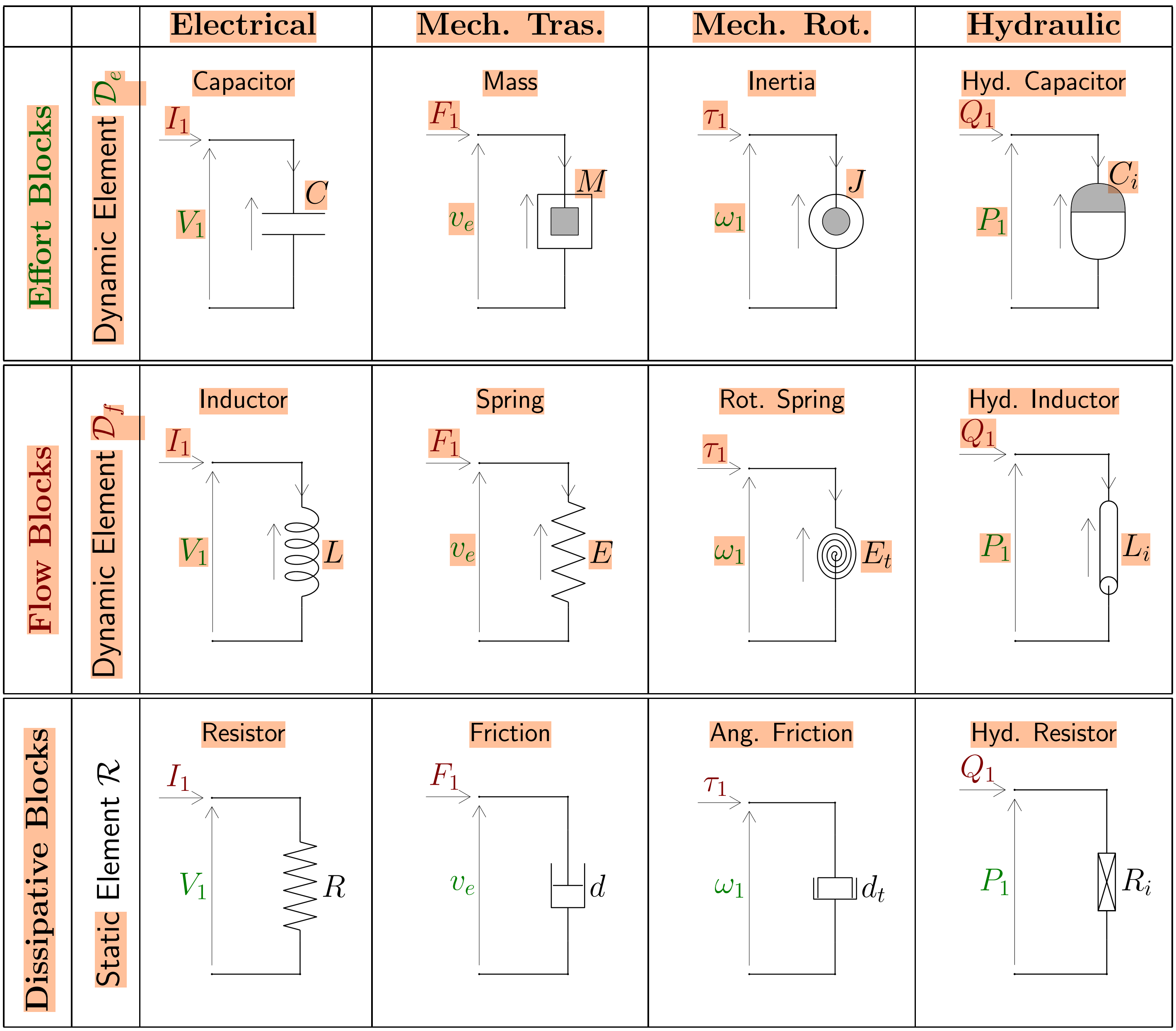
  2 energy variables q_e and q_f which define *how much energy is stored* within the *dynamic elements*;

  2 power variables v_e and v_f which describe *how the energy moves* within the system.

- ***Dynamic structure of the Energetic domains:***

		Electrical		**Mech. Tras.**		**Mech. Rot.**		**Hydraulic**
$\mathcal{D}_e$	C	Capacitor	M	Mass	J	Inertia	C_I	Hyd. Capacitor
q_e	Q	Charge	p	Momentum	p	Ang. Momentum	V	Volume
v_e	V	Voltage	$\dot{x}$	Velocity	ω	Ang. Velocity	P	Pressure
$\mathcal{D}_f$	L	Inductor	E	Spring	E	Spring	L_I	Hyd. Inductor
q_f	ϕ	Flux	x	Displacement	θ	Ang. Displacement	ϕ_I	Hyd. Flux
v_f	I	Current	F	Force	τ	Torque	Q	Volume flow rate
$\mathcal{R}$	R	Resistor	b	Friction	b	Ang. Friction	R_I	Hyd. Resistor

- Graphical representations of the physical elements (POG Modeler):

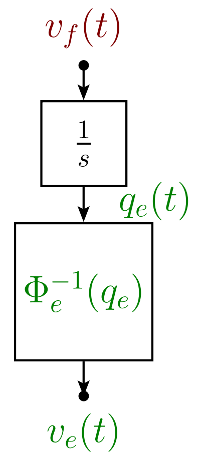


- The ***Dynamic Element*** D_e (Effort Block) is characterized by:

  1) an internal *energy variable* $q_e(t)$;
  2) a *Flow/through-variable* $v_f(t)$ as input variable;
  3) an *Effort/across-variable* $v_e(t)$ as output variable;
  4) a *constitutive relation* $q_e = \Phi_e(v_e)$ which links the internal energy variable $q_e(t)$ to the output power variable $v_e(t)$;
  5) a *differential equation*

  $$\dot{q}_e(t) = v_f(t)$$

  which links the internal energy variable $q_e(t)$ to the input power variable $v_f(t)$;



Element D_e

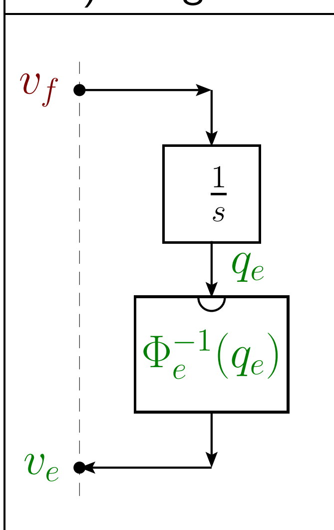
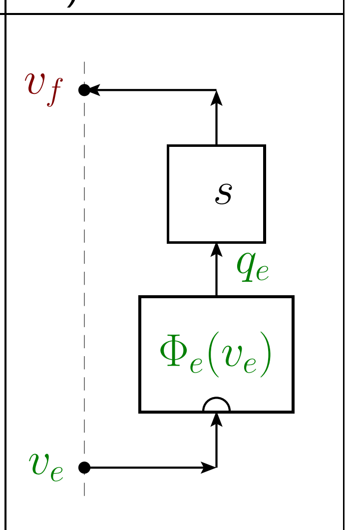
  6) the energy E_e stored in the *dynamic element* D_e is function only of the internal *energy variable* q_e:

  $$E_e = \int_0^t v_e(t)\, v_f(t)\, dt = \int_0^{q_e} \Phi_e^{-1}(q_e)\, dq_e = E_e(q_e).$$

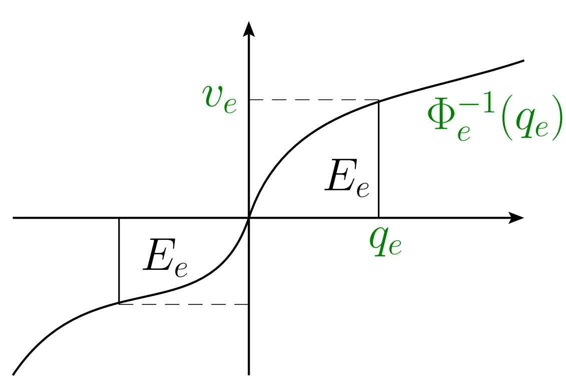
  where the following substitutions have been used:

  $$v_e(t) = \Phi_e^{-1}(q_e) \qquad\qquad dq_e = v_f(t)\, dt$$

- Dynamic orientation and stored energy:

1) Integral	2) Derivative
	
TO BE USED	DO NOT USE

Stored energy E_e

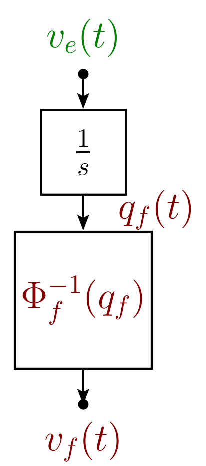


- The ***Dynamic Element*** D_f (Flow Block) has a structure which is *"dual"* respect to the structure of *dynamic element* D_e.

  1) an internal *energy variable* $q_f(t)$;
  2) an *Effort/across-variable* $v_e(t)$ as input variable;
  3) a *Flow/through-variable* $v_f(t)$ as output variable;
  4) a *constitutive relation* $q_f = \Phi_f(v_f)$ which links the internal energy variable $q_f(t)$ to the output power variable $v_f(t)$;
  5) a *differential equation*

$$\dot{q}_f(t) = v_e(t)$$

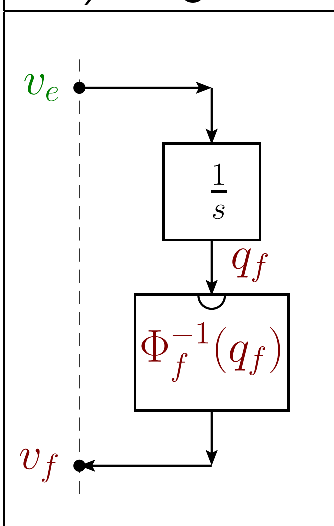
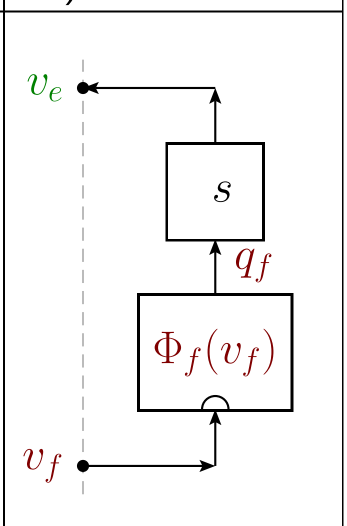
  which links the internal energy variable $q_f(t)$ to the input power variable $v_e(t)$;



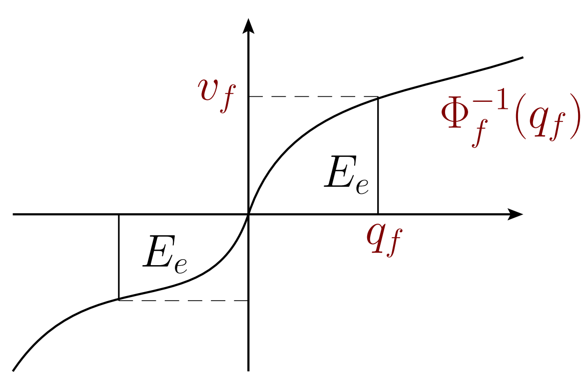
Element D_f

The dual structure can be easily obtained performing the following substitutions: $q_e(t) \rightarrow q_f(t)$, $v_f(t) \leftrightarrow v_e(t)$ and $\Phi_e(v_e) \rightarrow \Phi_f(v_f)$.

- Dynamic orientation and stored energy:

1) Integral	2) Derivative
	
TO BE USED	DO NOT USE

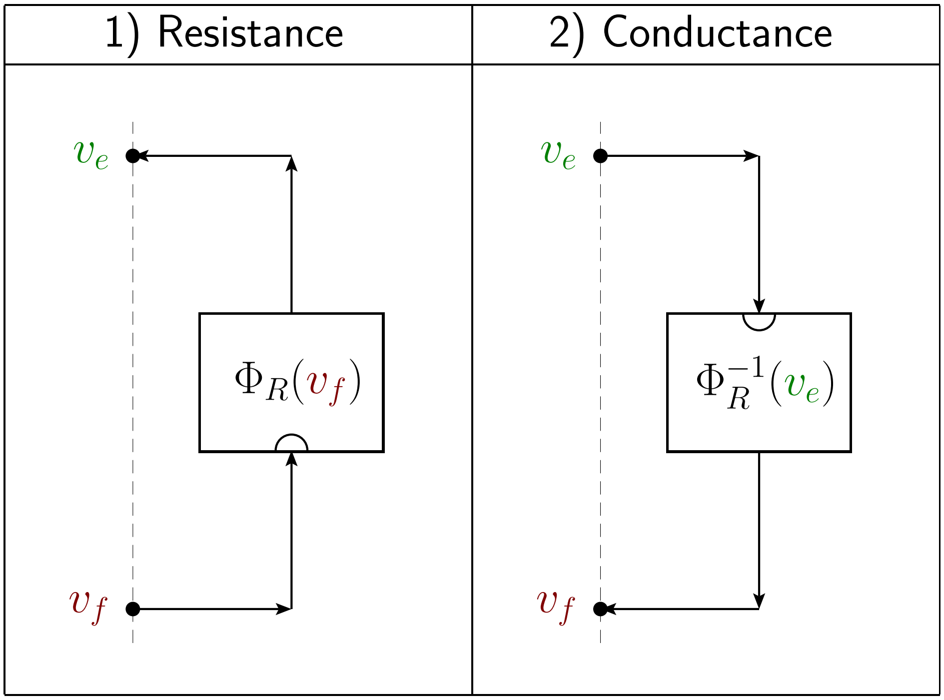
Stored energy E_e



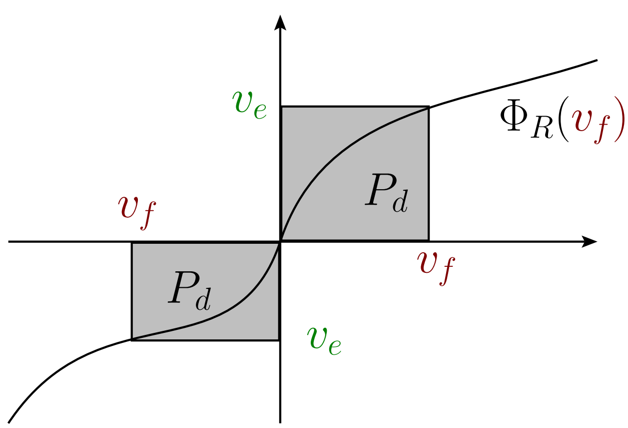
- <u>Note</u>: **the energy variables** q_e and q_f **are the** ***integral*** **of the input power variables** $v_f(t)$ and $v_e(t)$:

$$q_e = \int_0^t v_f(t)\, dt, \qquad q_f = \int_0^t v_e(t)\, dt.$$

- The *static element* $\mathcal{R}$ (Dissipative Block) is completely characterized by a static function $v_e = \Phi_R(v_f)$ which links the input variable v_f to the output variable v_e.



- Dissipated power P_d of the *static element* $\mathcal{R}$:



- The differential equation of a physical element can be obtained imposing the time-derivative of the energy variable equal to the input power variable:

1) For $\mathcal{D}_f$ elements: $\dot{q}_f(t) = v_e(t) \quad \Leftrightarrow \quad \dfrac{d\,\dot{q}_f(t)}{dt} = v_e(t)$

2) For $\mathcal{D}_e$ elements: $\dot{q}_e(t) = v_f(t) \quad \Leftrightarrow \quad \dfrac{d\,\dot{q}_e(t)}{dt} = v_f(t)$

- **Electromagnetic domain**:

	Name	Constitutive Rel.	Linear case	Differential Eq.
$\mathcal{D}_e$ q_e v_e	C Capacitor Q Charge V Voltage	$Q = \Phi_C(V)$	$Q = C\,V$	$\dfrac{d\,Q}{dt} = I$
$\mathcal{D}_f$ q_f v_f	L Inductor ϕ Flux I Current	$\phi = \Phi_L(I)$	$\phi = L\,I$	$\dfrac{d\,\phi}{dt} = V$
$\mathcal{R}$	R Resistance	$V = \Phi_R(I)$	$V = R\,I$	

- **Mechanic Translational domain:**

	Name	Constitutive Rel.	Linear case	Differential Eq.
$\mathcal{D}_e$ q_e v_e	M Mass P Momentum $\dot{x}$ Velocity	$P = \Phi_M(\dot{x})$	$P = M\,\dot{x}$	$\dfrac{d\,P}{dt} = F$
$\mathcal{D}_f$ q_f v_f	E String x Displacement F Force	$x = \Phi_E(F)$	$x = E\,F$	$\dfrac{d\,x}{dt} = \dot{x}$
$\mathcal{R}$	b Friction	$F = \Phi_b(\dot{x})$	$F = b\,\dot{x}$	

For the spring can use the stiffness K instead of the elasticity E:

$$K = \frac{1}{E}, \qquad x = E\,F \;\leftrightarrow\; \underbrace{F = K\,x}_{\text{Hook law}}$$

- **Mechanic Rotational domain**:

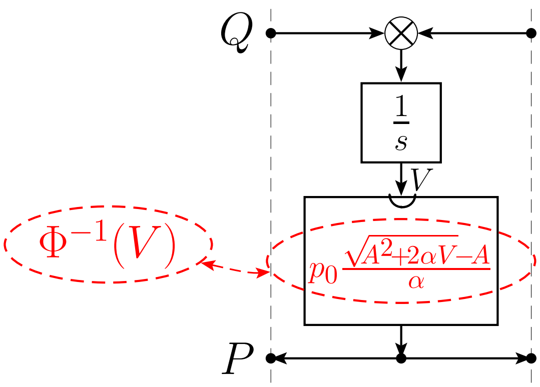
		Name	Constitutive Rel.	Linear case	Differential Eq.
$\mathcal{D}_e$	J	Inertia			
q_e	P	Ang. Momentum	$P = \Phi_J(\omega)$	$P = J\,\omega$	$\dfrac{d\,P}{dt} = \tau$
v_e	ω	Ang. Velocity			
$\mathcal{D}_f$	E	Rot. Spring			
q_f	θ	Ang. Displacement	$\theta = \Phi_E(\tau)$	$\theta = E\,\tau$	$\dfrac{d\,\theta}{dt} = \omega$
v_f	τ	Torque			
$\mathcal{R}$	b	Rot. Friction	$\tau = \Phi_b(\omega)$	$\tau = b\,\omega$	

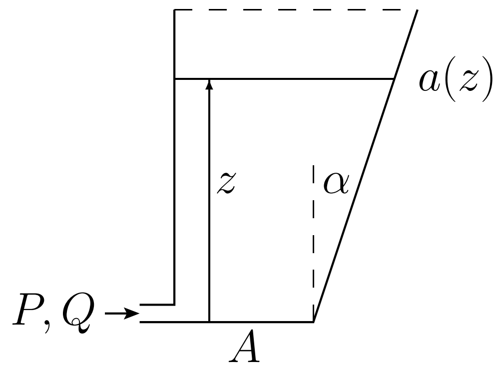
- **Hydraulic domain**:

		Name	Constitutive Rel.	Linear case	Differential Eq.
$\mathcal{D}_e$	C_I	Hyd. Capacitor			
q_e	V	Volume	$V = \Phi_C(P)$	$V = C_I\,P$	$\dfrac{d\,V}{dt} = Q$
v_e	P	Pressure			
$\mathcal{D}_f$	L_I	Hyd. Inductor			
q_f	ϕ_I	Hyd. Flux	$\phi_I = \Phi_L(Q)$	$\phi_I = L_I\,Q$	$\dfrac{d\,\phi_I}{dt} = P$
v_f	Q	Volume flow rate			
$\mathcal{R}$	R	Hyd. Resistor	$P = \Phi_R(Q)$	$P = R_I\,Q$	

Nonlinear dynamic elements: examples

- Example. Tank with variable section.





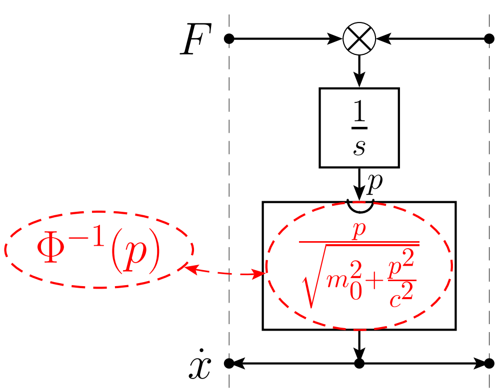
The area of the liquid $a(z)$ at height z and the volume of liquid V are:

$$a(z) = A + \alpha z, \qquad V = \int_0^z a(z)dz = A\,z + \frac{\alpha\, z^2}{2}$$

Since $z = \frac{P}{p_0}$, functions $V = \Phi(P)$ and $P = \Phi^{-1}(V)$ are defined as follows:

$$V = \Phi(P) = A\frac{P}{p_0} + \frac{\alpha\, P^2}{2p_0^2}, \qquad P = \Phi^{-1}(V) = p_0\frac{\sqrt{A^2 + 2\alpha\, V} - A}{\alpha}.$$

- Example. A translating mass.



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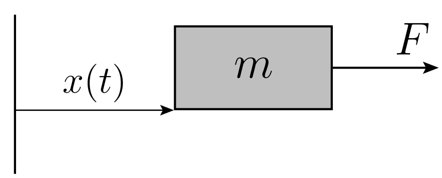
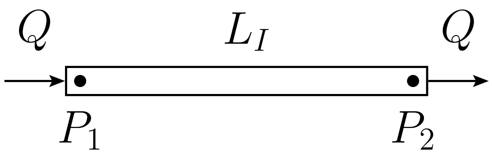


- Taking into account the relativistic effects, mass m and $p = \Phi(\dot{x})$ are:

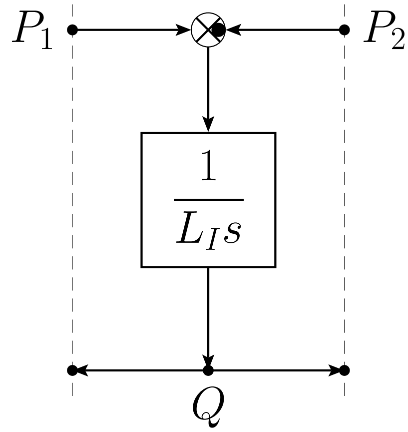
$$m = \frac{m_0}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}, \qquad p = \Phi(\dot{x}) = \frac{m_0\,\dot{x}}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} \qquad \dot{x} = \Phi^{-1}(p) = \frac{p}{\sqrt{m_0^2 + \frac{p^2}{c^2}}}$$

The basic elements of the Hydraulic domain

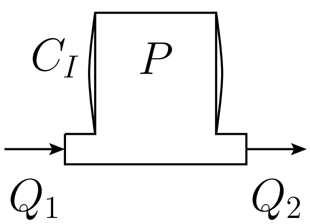
- Hydraulic Inductor:



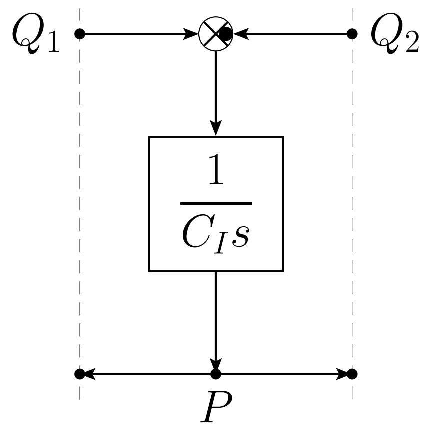
$$\frac{d}{dt}[L_I Q] = P_1 - P_2$$



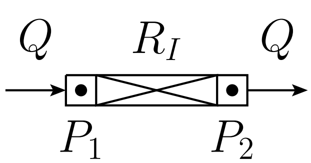
- Hydraulic Capacitor:



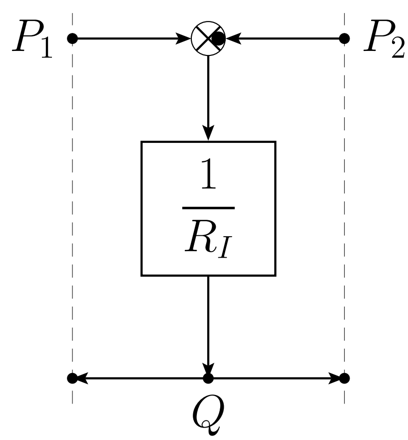
$$\frac{d}{dt}[C_I P] = Q_1 - Q_2$$



- Hydraulic Resistance:

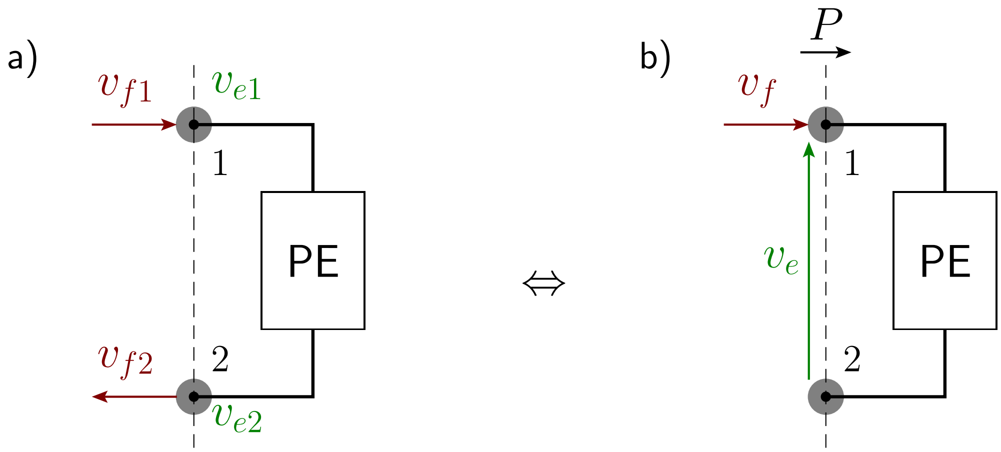


$$P_1 - P_2 = R_I Q$$



Connection of physical elements

- ***Physical Elements***. The physical systems are composed by **physical elements** (PE) (i.e. *dynamic elements* D_e and D_f or *static element* $\mathcal{R}$) which interact with the external world by means of two terminals:



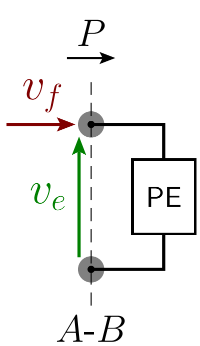
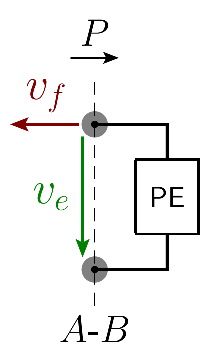
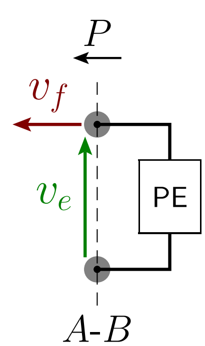
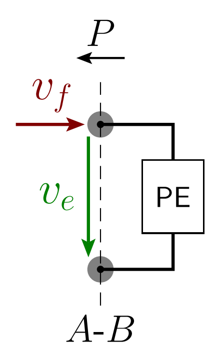
Each terminal, see case a), **is characterized by two power variables** (v_{e1}, v_{f1}) and (v_{e2}, v_{f2}). Choosing $v_e = v_{e1} - v_{e2}$ and $v_f = v_{f1} = v_{f2}$ as new power variables, the power interaction of the Physical Element PE with the external world can be described using the power section P in case b).

- The value of the power P flowing through the section is the product of the two power variables $v_e(t)$ and $v_f(t)$:

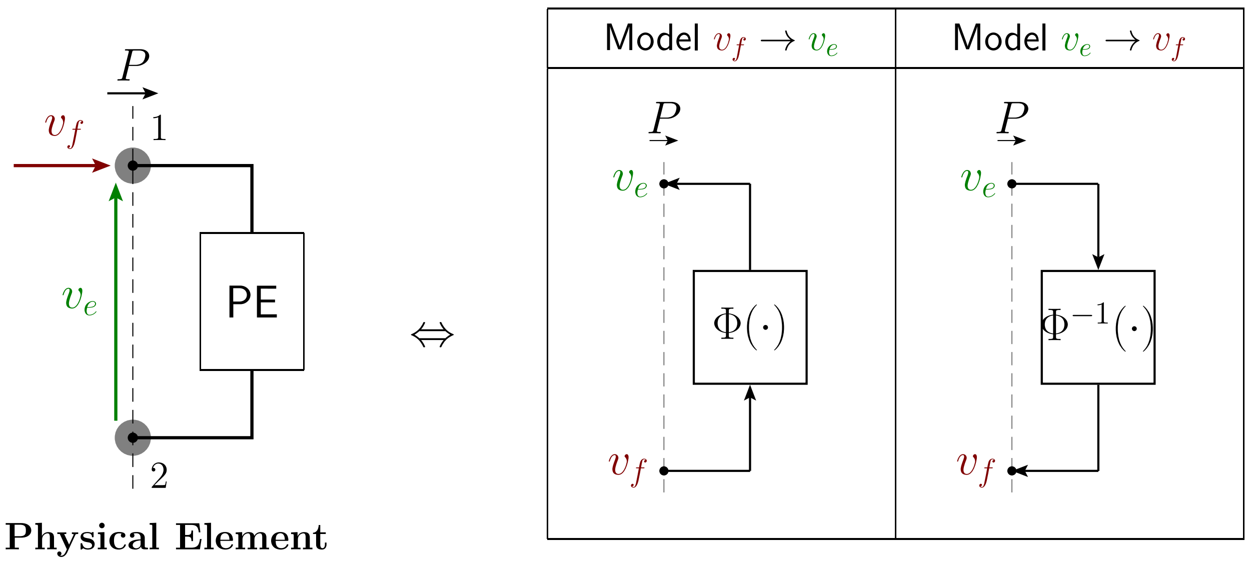
$$P(t) = v_e(t)\, v_f(t)$$

The sign and the direction of power $P(t)$ depend on the sign and the positive reference direction chosen for the variables $v_e(t)$ and $v_f(t)$.

- The **signs of the power** P flowing through a physical section A-B are:

	a) Power P flows from A to B		b) Power P flows from B to A	
Power flows				

- ***Integral and derivative causality***. The POG dynamic model of a physical elements (PE), that is an element D_e, D_f or $\mathcal{R}$, can be graphically described by using two block schemes having different orientation:



The two possible "orientations" of the PE dynamic model are:

1) v_f as input and v_e as output: model $v_f \to v_e$
2) v_e as input and v_f as output: model $v_e \to v_f$.

The function $\Phi(\cdot)$ shown in the figures symbolically represents the dynamic or the static equation describing the physical element.

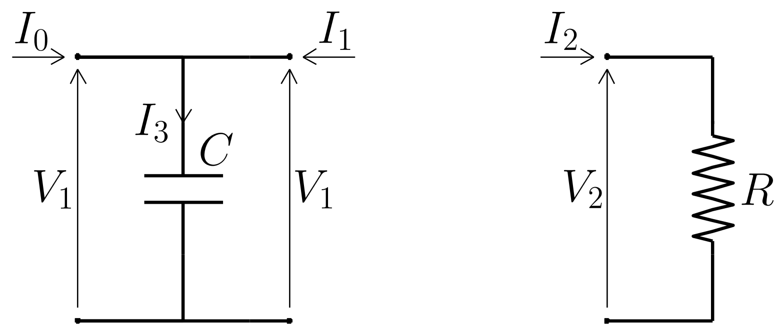
- If PE is a static element $\mathcal{R}$, the two diagrams are both suitable for describing the mathematical model of the physical element.

- If PE is a dynamic element D_e or D_f, the two diagrams represent the two possible **causality modes** of describing the physical element:

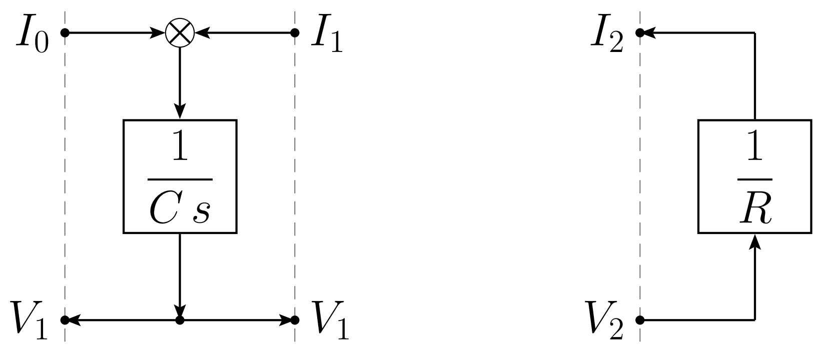
  1) the ***integral causality*** (TO BE USED) is physically realizable, useful in simulation and is the preferred dynamic model in the POG technique.

  2) the ***derivative causality*** (DO NOT USE) is still a correct mathematical model of the PE, but it is not used in the POG technique because it is not physically realizable and it is not useful in simulation.

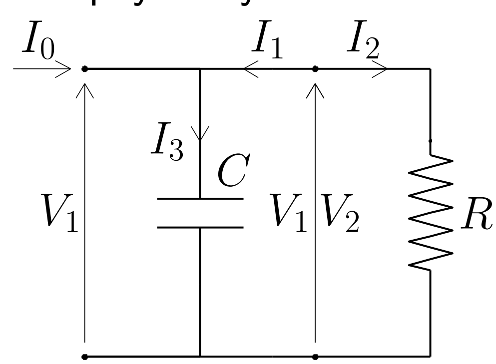
- Let us now consider the following two Physical Elements:



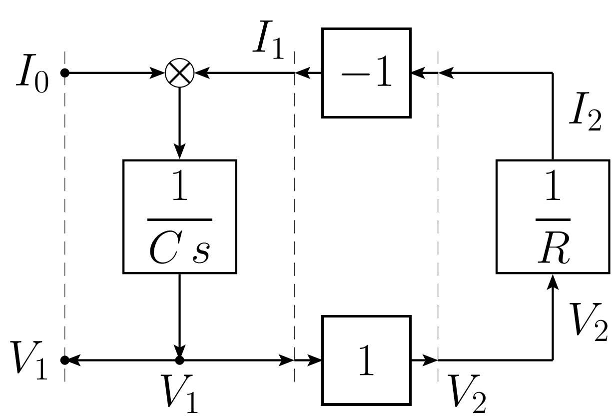
- The dynamic models of these two elements are:



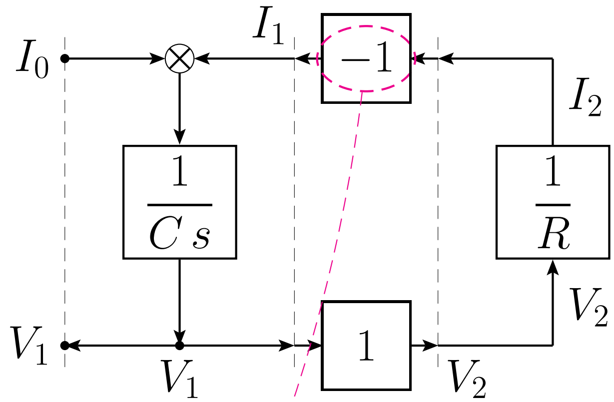
- When the two elements are physically connected together:



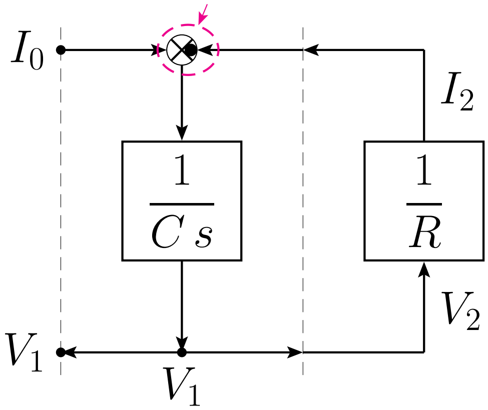
the corresponding dynamic models can be connected taking into account the constraints $V_2 = V_1$ and $I_1 = -I_2$ given by the positive directions chosen for the power variables:



- For graphical reasons the following block scheme:



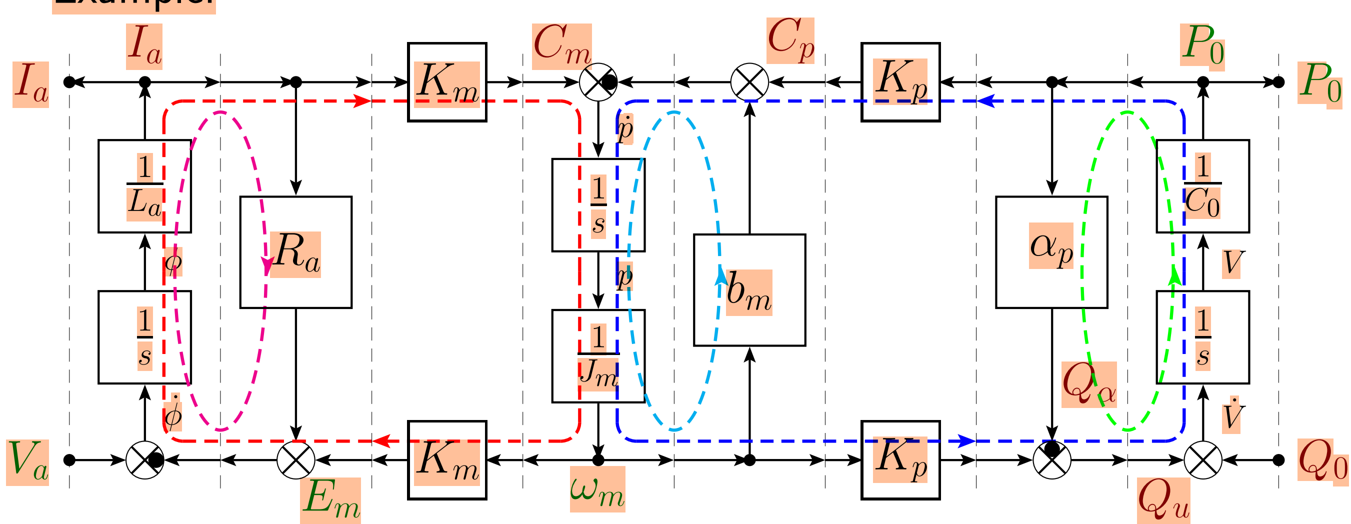
will be substituted by the following more compact scheme:



where the minus sign has been inserted within the summation element.

- ***Property***. ***All the loops*** of a POG scheme ***contains an odd number of minus signs*** (i.e. black spots in the summation elements).

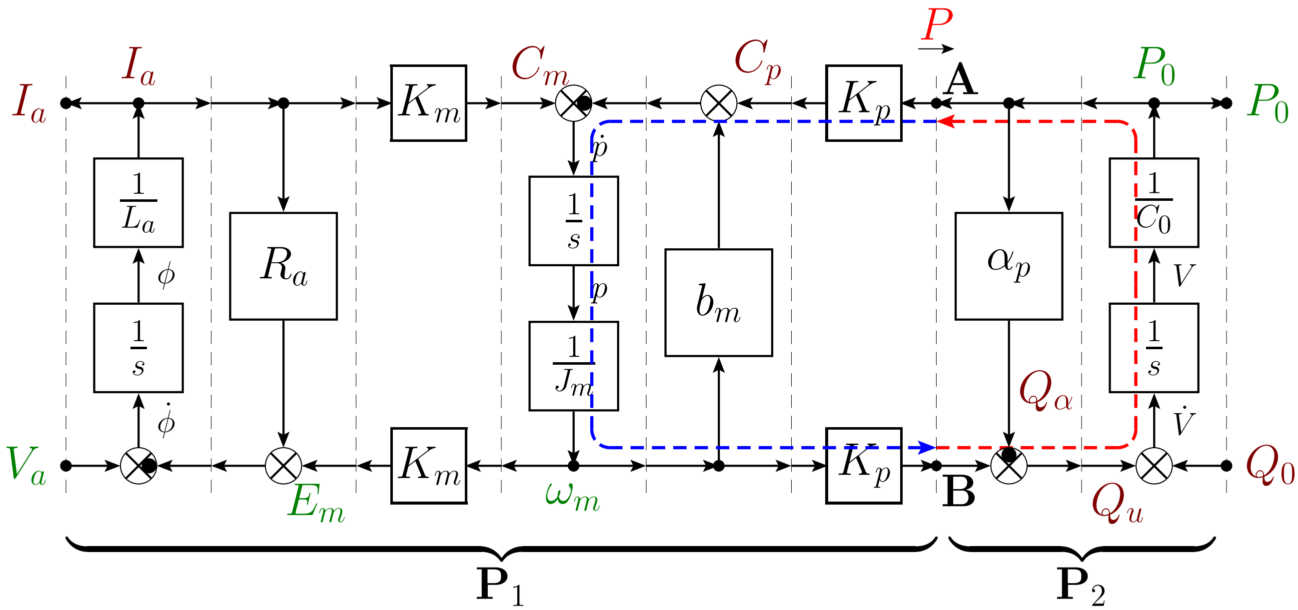
Example:



All the five loops of this block scheme contain "one" minus sign.

- This rule can be used to verify the "consistency" and the correctness of the considered POG block scheme.

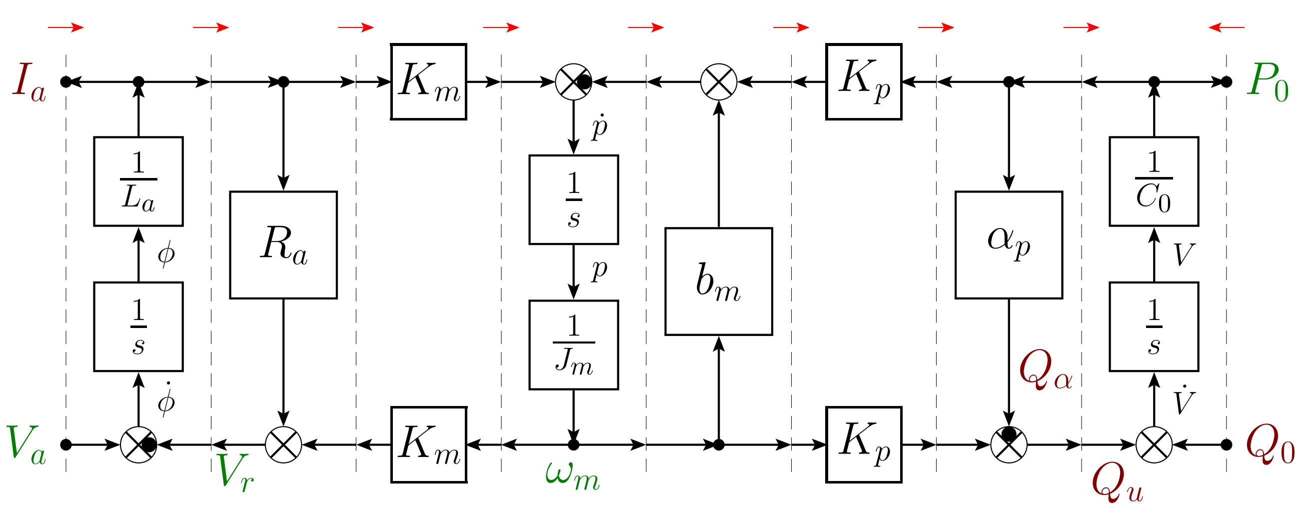
- **Property**. The ***direction*** of the power P flowing through a section of a POG block scheme ***is "positive"*** if an ***"even" number of minus signs*** is present along the paths which goes from the input to the output.



Let us consider, for example, the power section **A**-**B** which divides the block scheme in two subsystems: $\mathbf{P}_1$ and $\mathbf{P}_2$. In this case the power P flows from section $\mathbf{P}_1$ to section $\mathbf{P}_2$ because:

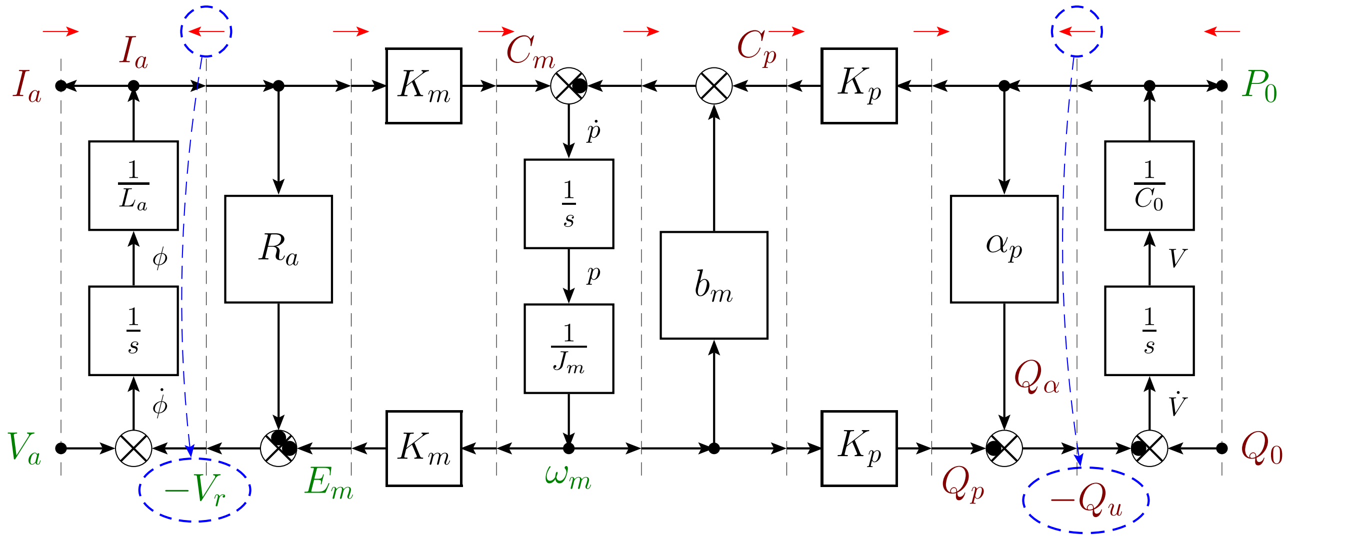
  - the red dashed path that goes from **B** to **A** within subsystem $\mathbf{P}_2$ contains "zero" minus signs (i.e. an even number);
  - the blue dashed path that goes from **A** to **B** within subsystem $\mathbf{P}_1$ contains "one" minus signs (i.e. an odd number);

- Using the previous rule it is possible to compute the positive direction of the poter flows in each power section of a POG block schemes:

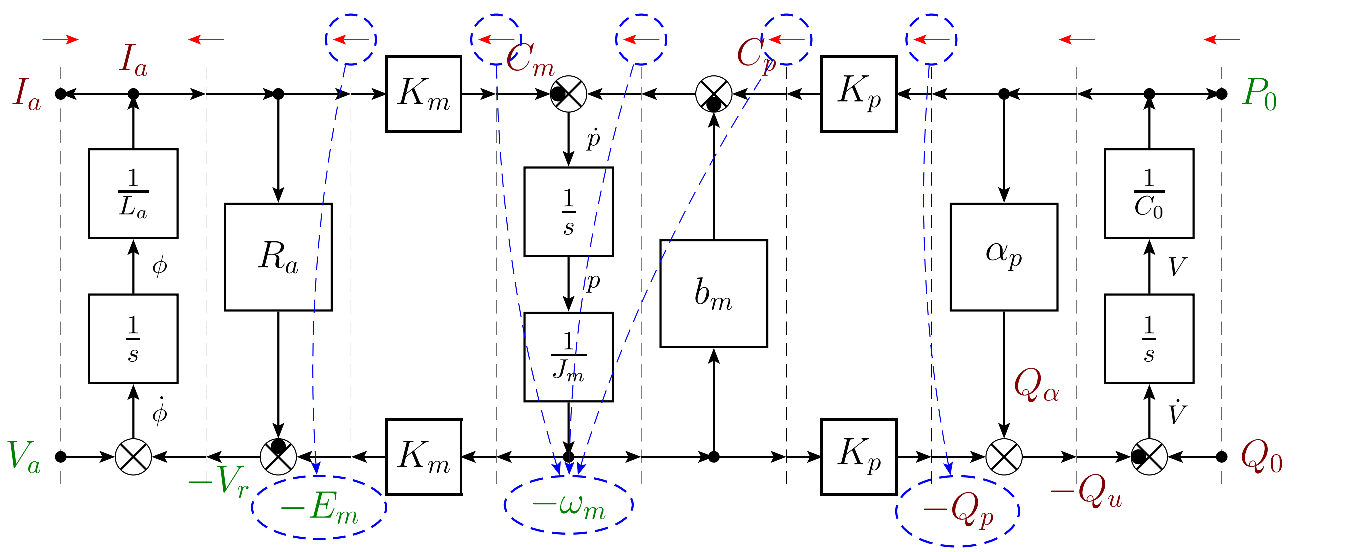


- The signs of the power flows depend on the sign of the power variables.

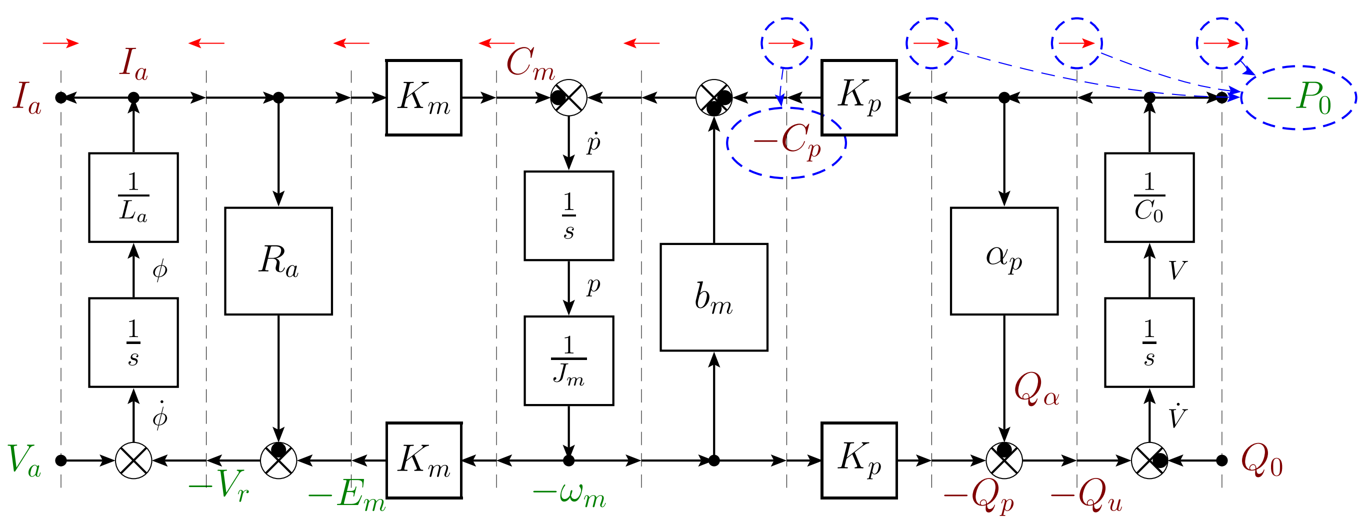
a) Changing the signs of variables V_r and Q_u one obtains:



b) Changing the signs of variables E_m, ω_m and Q_p one obtains:

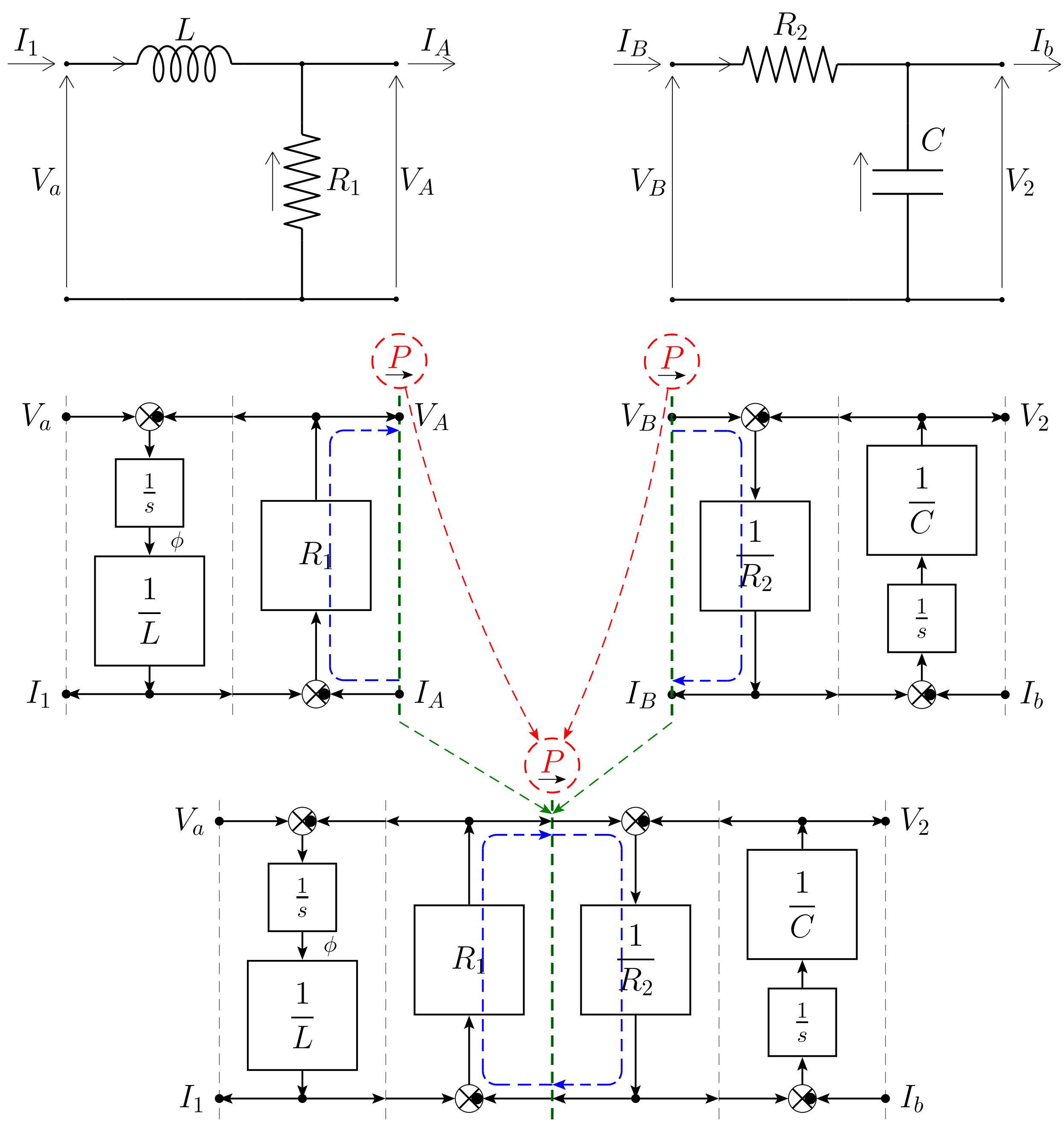


c) Changing the signs of variables C_p and P_0 one obtains:



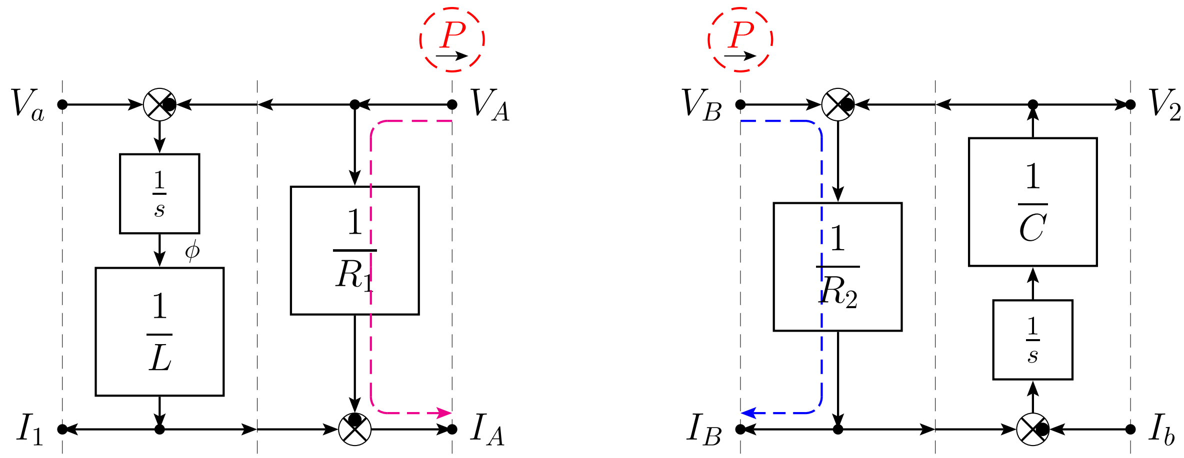
Connecting two POG block schemes

- Let us consider the following two electric circuits and POG schemes:

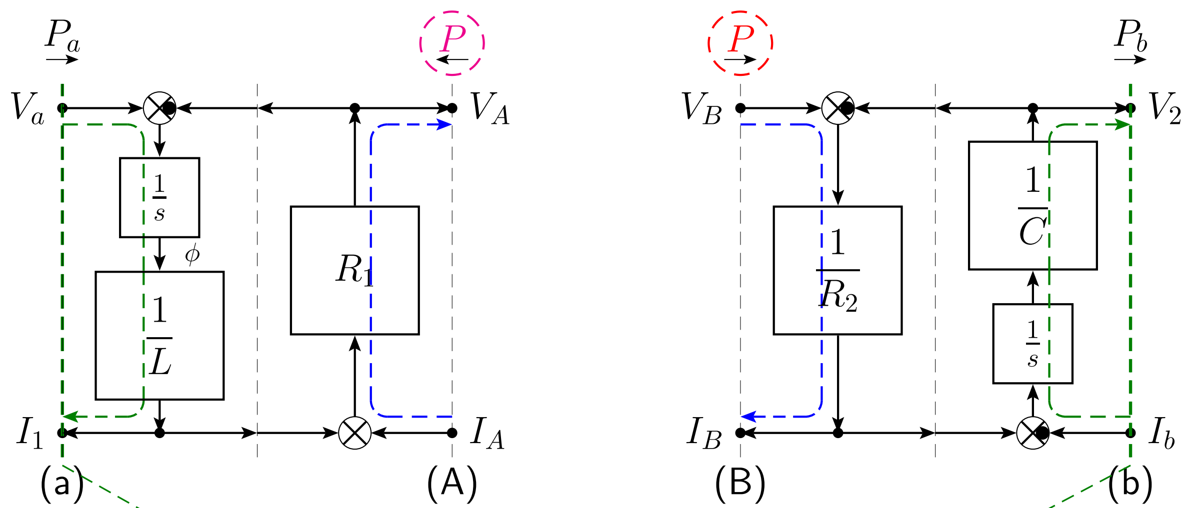


- Two POG block schemes can be "directly connected" only if:
  1) the two power sections "are oriented in the same way";
  2) the two power sections "share the same positive power flow".

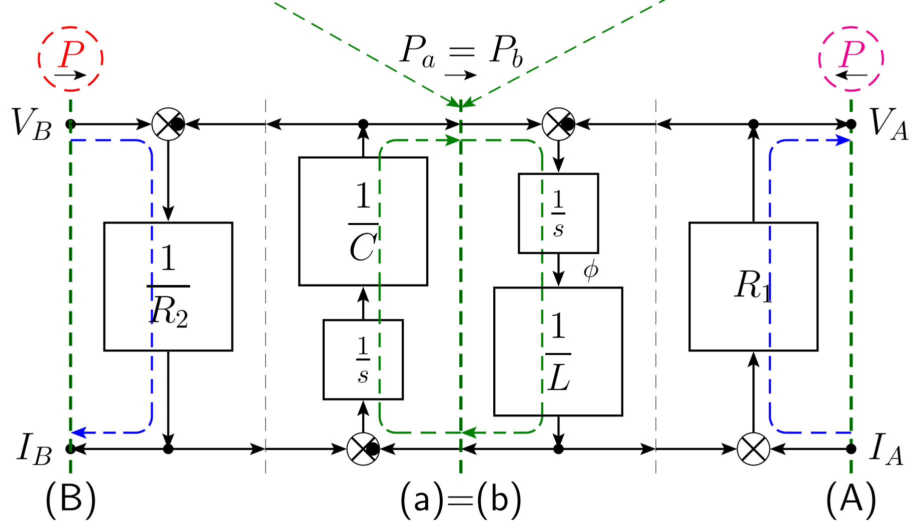
- The following two POG schemes "CANNOT be directly connected" because "they are NOT oriented in the same way":



- The following two POG schemes "CANNOT be directly connected" because "they do NOT share the same positive power flow":



- The two above POG schemes "CAN be directly connected" along sections (a)-(b) because "they share the same orientation and positive power flow":



Series and ***Parallel*** connections

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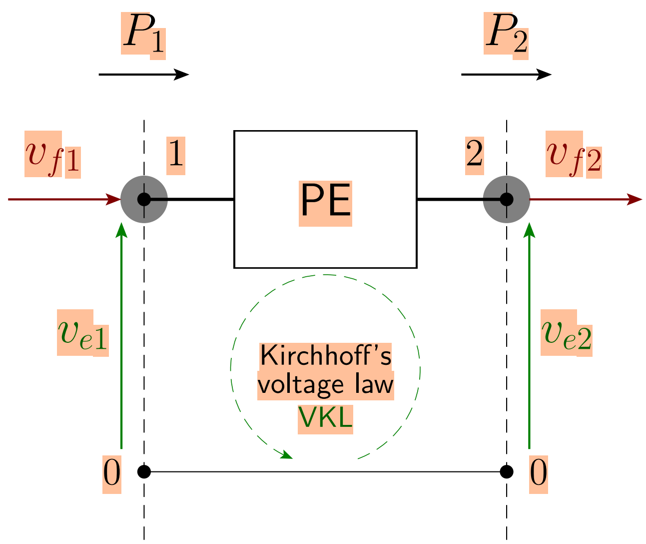
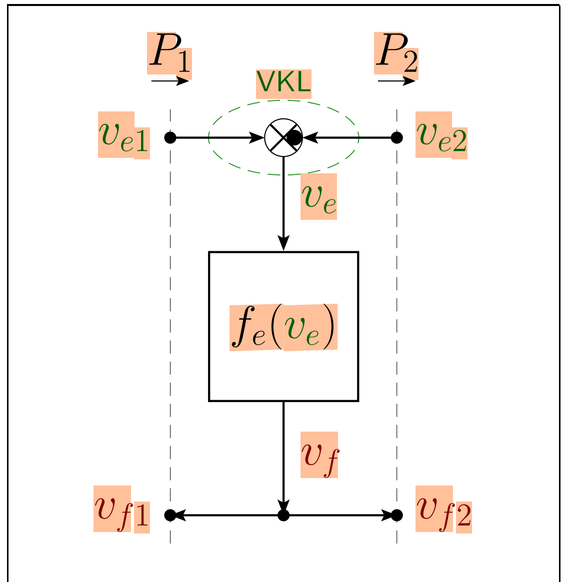
- ***Series***: a Physical Element PE is connected **in series** if its terminals share the same through-variable $v_f = v_{f1} = v_{f2}$:



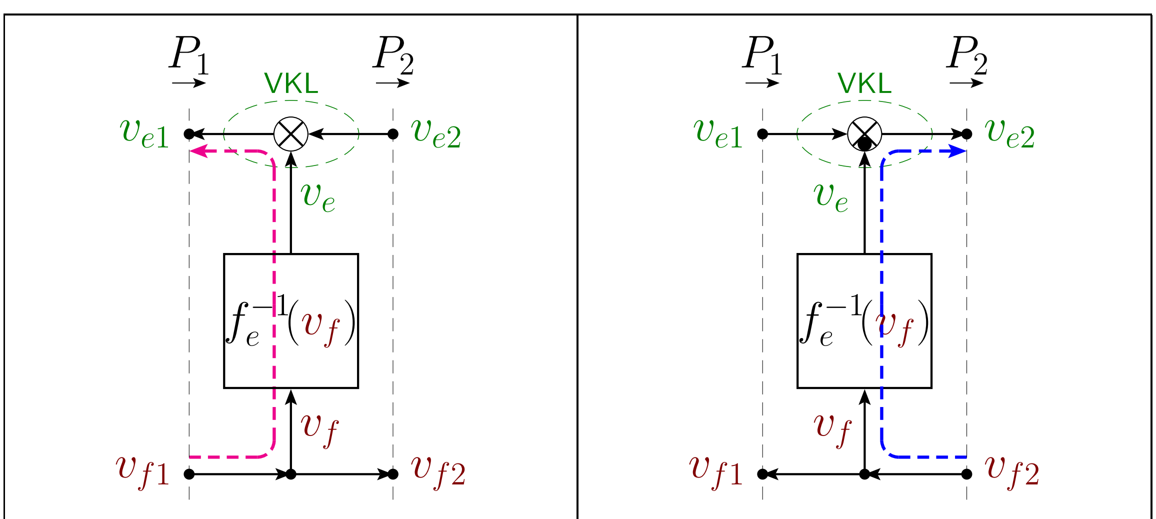
Physical Element connected in series

⇔



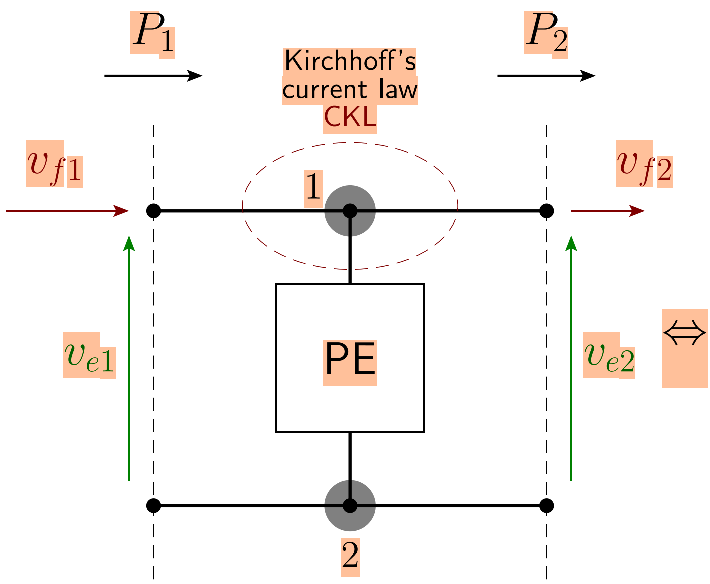
1) POG scheme with output v_f

- The summation element is a mathematical description of the **Voltage Kirchhoff's Law** (VKL) applied to a "closed" path which involves the *across variables* v_{e1}, v_{e2} and v_e
- Inverting the input and output paths of the POG block scheme 1) one obtains the following equivalent POG block schemes:

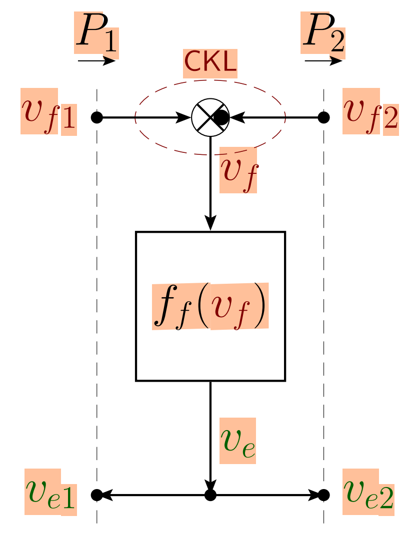


2) POG scheme with output v_{e1}. 3) POG scheme with output v_{e2}.

Parallel: a Physical Element PE is connected **in parallel** if its terminals share the same across-variable $v_e = v_{e1} = v_{e2}$:

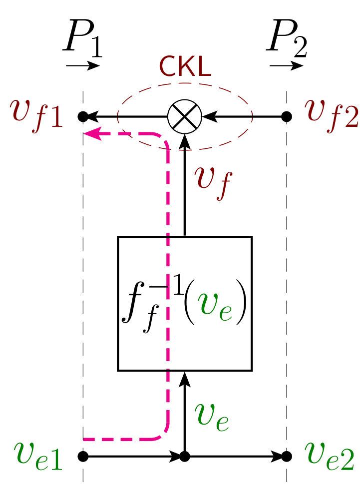


Physical Element connected in parallel

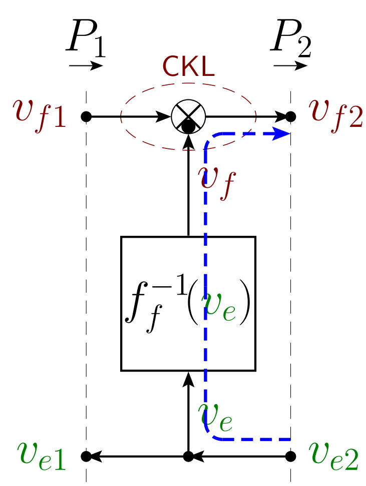


1) POG scheme with output v_e.

- The summation element is a mathematical description of the **Current Kirchhoff's Law** (CKL) applied to a node which involves the *through variables* v_{f1}, v_{f2} and v_f.
- Inverting the input and output paths of the POG block scheme 1) one obtains the following equivalent POG block schemes:

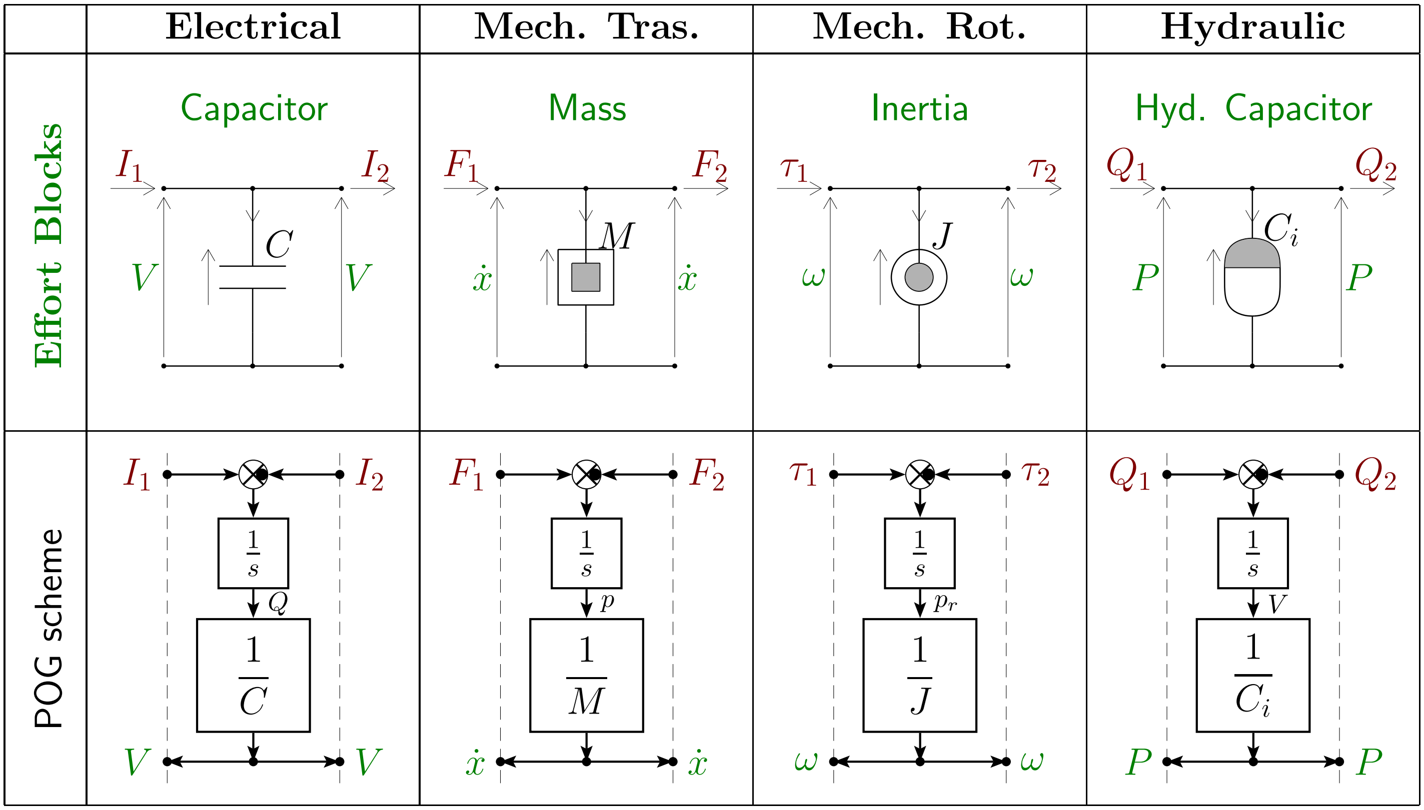


2) POG scheme with output v_{f1}.

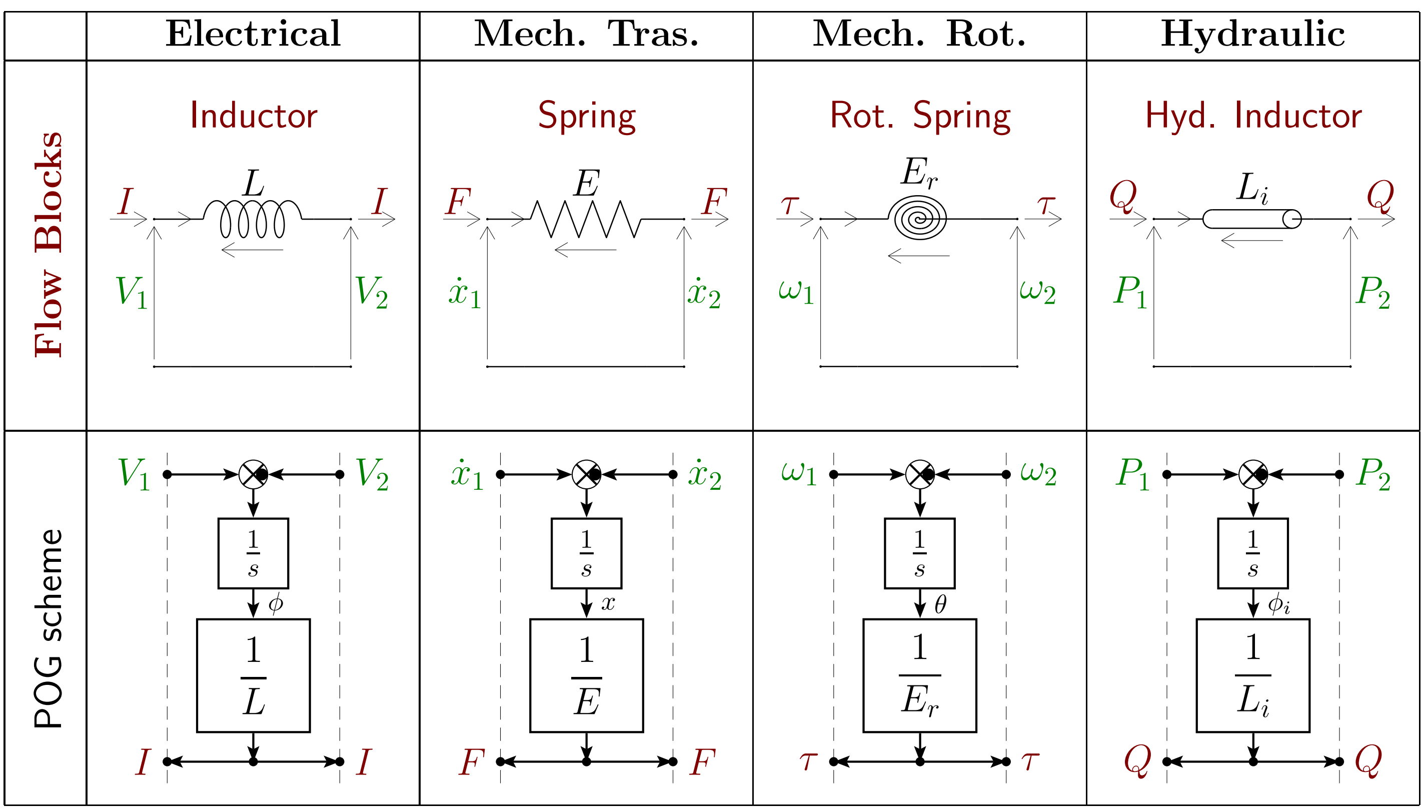


3) POG scheme with output v_{f2}.

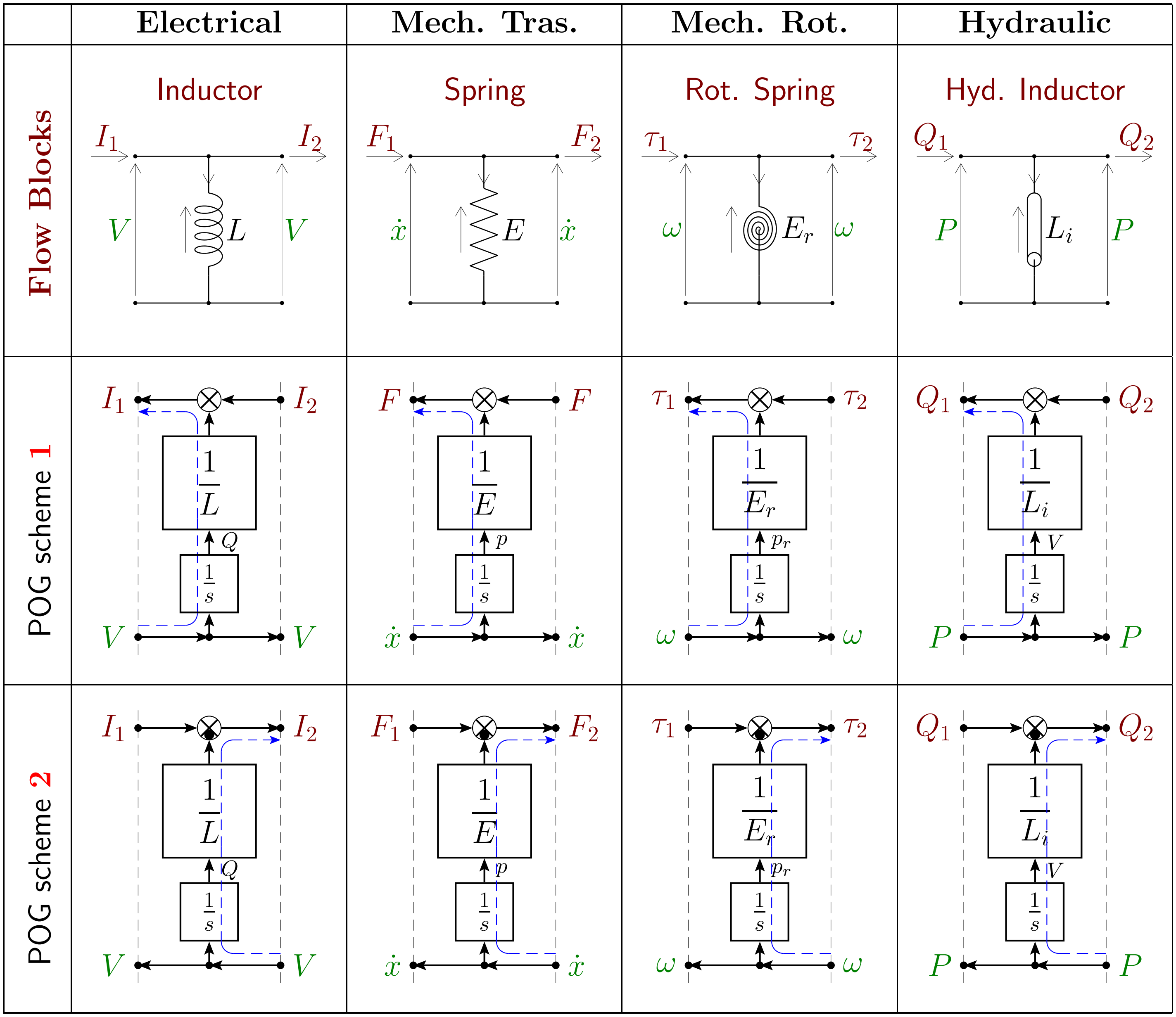
- The “Across/Effort” dynamic elements $\mathcal{D}_e$ connected “in parallel” can be graphically represented by using **only one** POG scheme:



- The “Through/Flow” dynamic elements $\mathcal{D}_f$ connected “in series” can be graphically represented by using **only one** POG scheme:



- The "Through/Flow" dynamic elements $\mathcal{D}_f$ connected "in parallel" can be graphically represented by using **two** POG schemes:

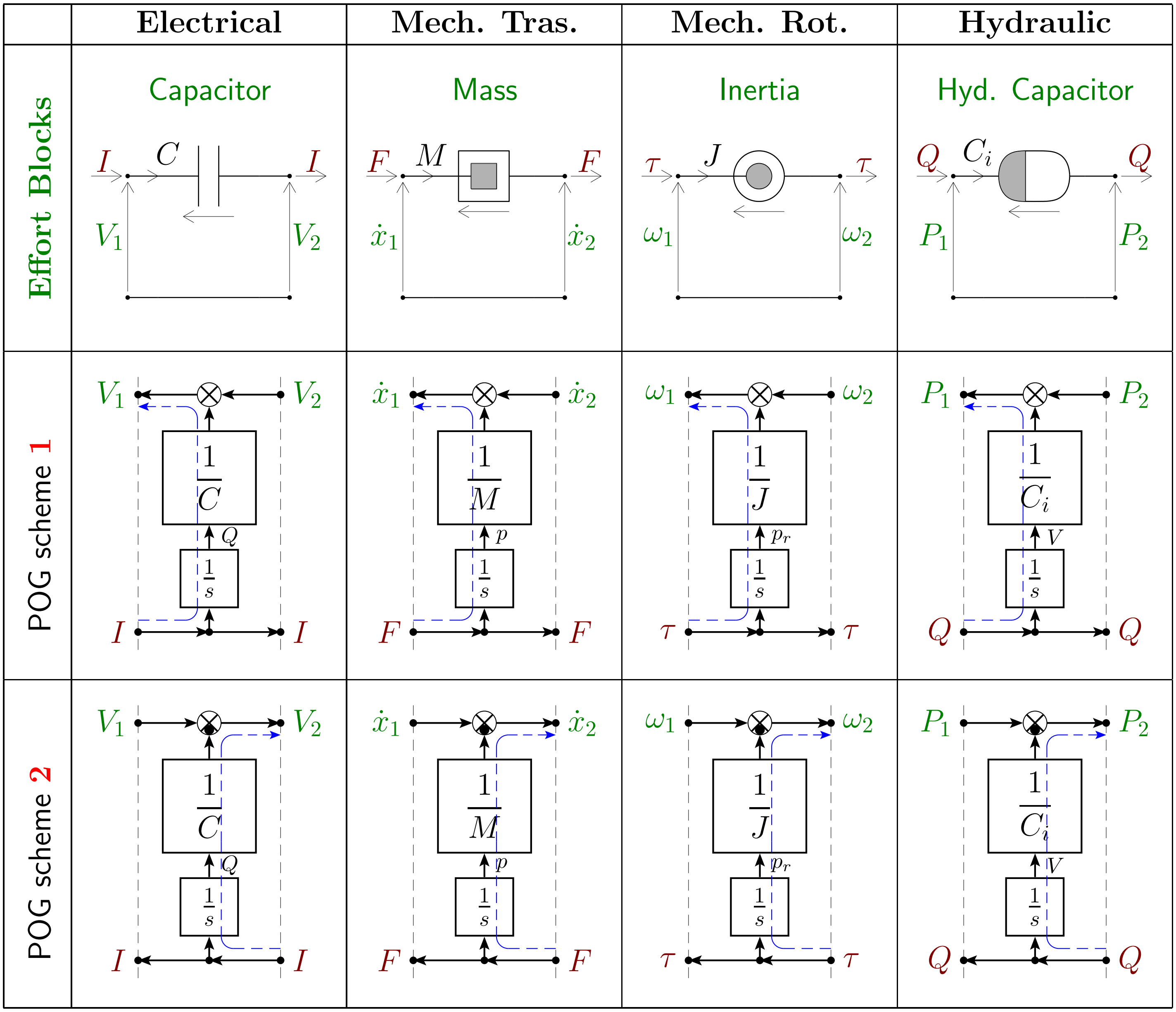


- Note: the two different POG block schemes have opposite direction of the input/output power variables.

- The choice between the two POG schemes depends on the orientation of the physical elements which are present before and after the block.

- The POG scheme **1** is used when the input Effort variable comes from the left. The POG scheme **2** is used when the input Effort variable comes from the right.

- The "Across/Effort" dynamic elements $\mathcal{D}_e$ connected "in series" can be graphically represented by using **two** POG schemes:

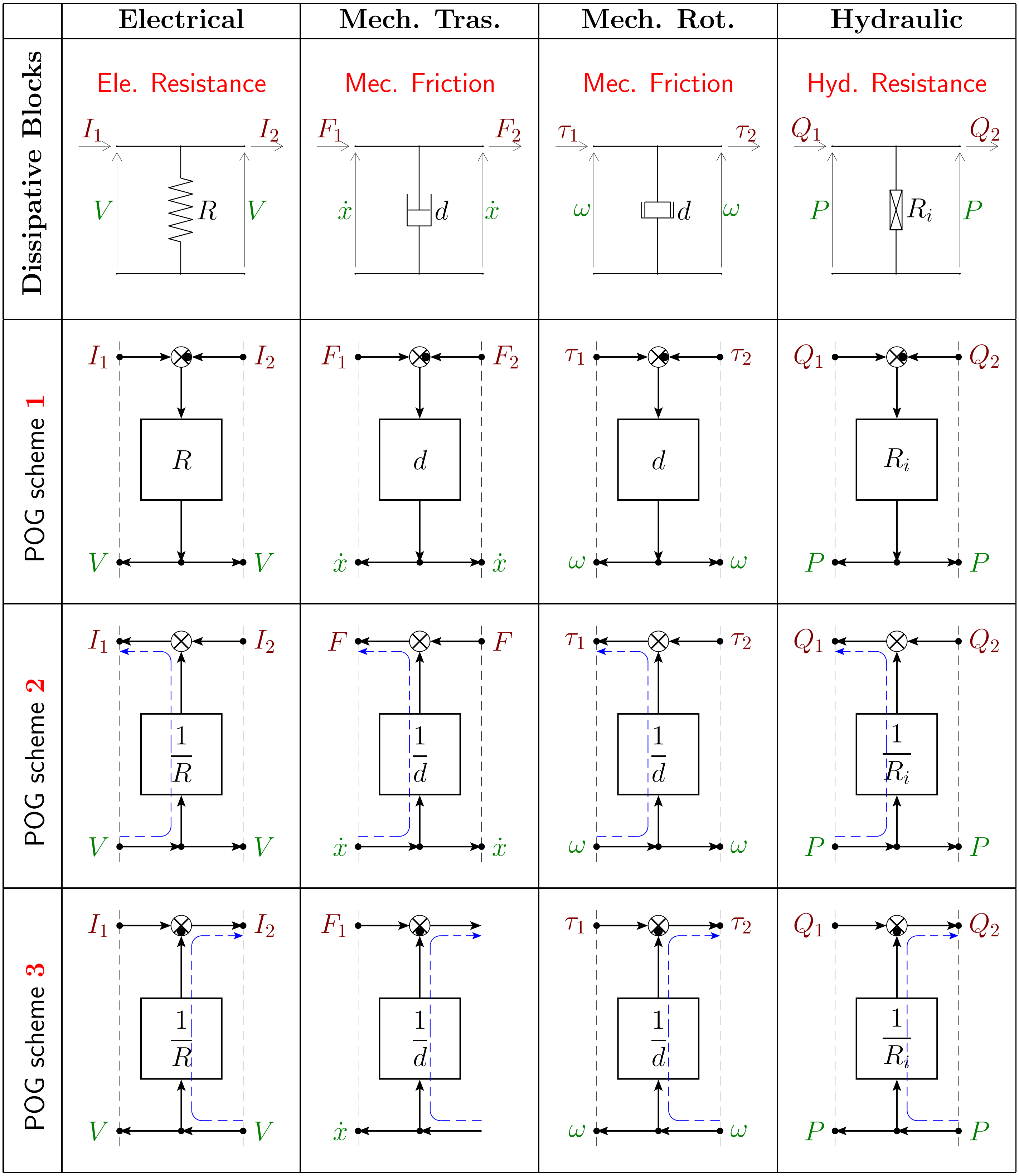


- Note: the two different POG block schemes have opposite direction of the input/output power variables.

- The choice between the two POG schemes depends on the orientation of the physical elements which are present before and after the block.

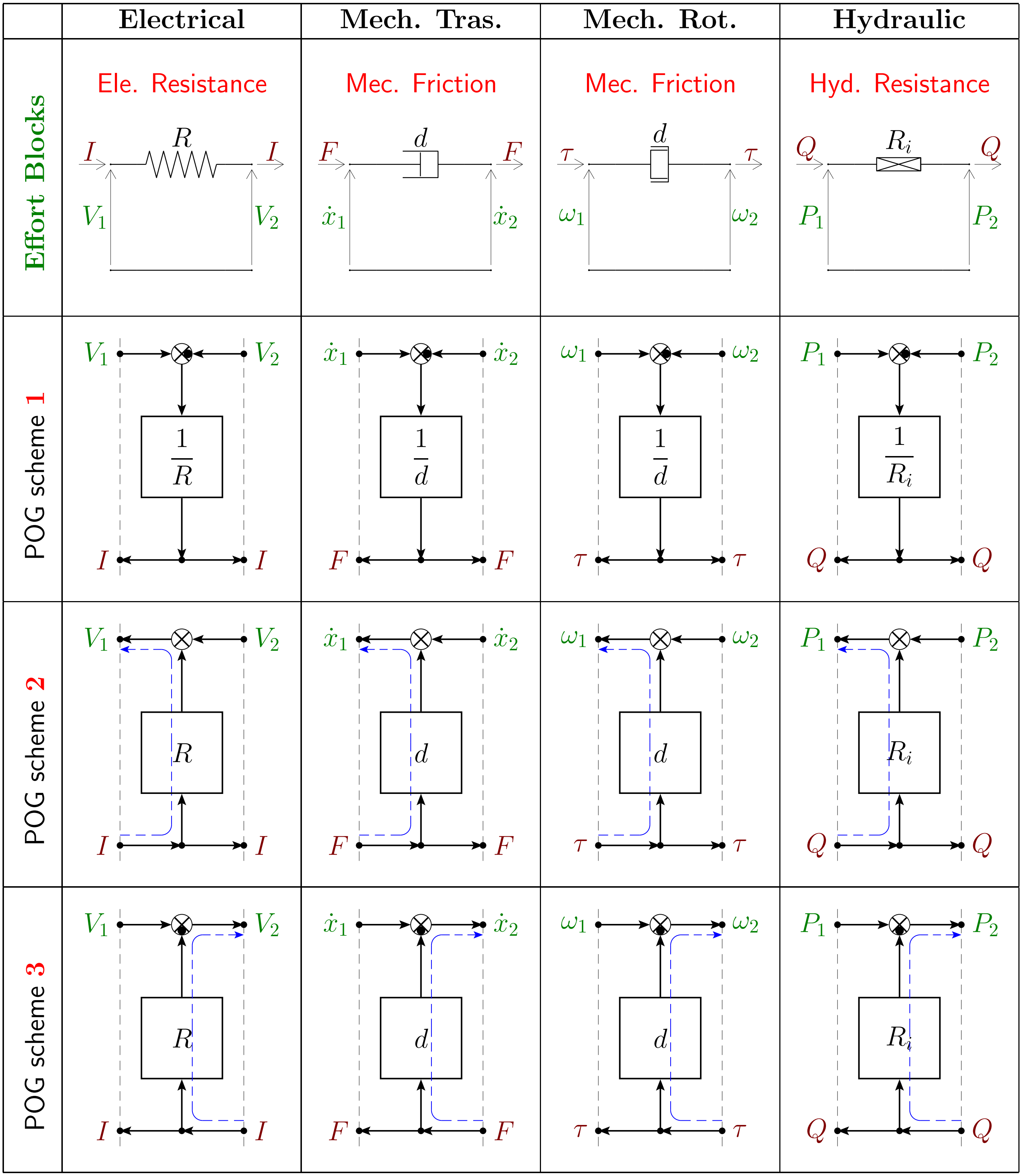
- The POG scheme **1** is used when the input Flow variable comes from the left. The POG scheme **2** is used when the input Flow variable comes from the right.

- The "**Dissipative**" static elements $\mathcal{R}$ connected "in parallel" can be graphically represented by using **three** different POG schemes:



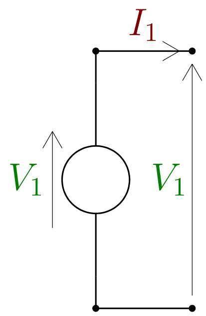
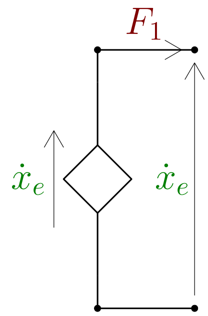
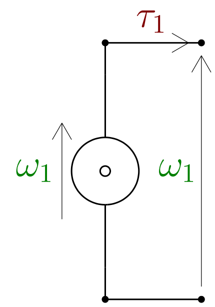
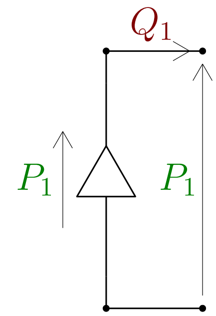
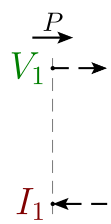
- The choice between the three POG schemes depends on the orientation of the physical elements which are present before and after the block.

- The "**Dissipative**" static elements $\mathcal{R}$ connected "in series" can be graphically represented by using **three** different POG schemes:



- The choice between the three POG schemes depends on the orientation of the physical elements which are present before and after the block.

- The “Effort/Across Generators” (POG Modeler):

<table>
<tr><th></th><th>Electrical</th><th>Mecc. Tras.</th><th>Mecc. Rot.</th><th>Hydraulic</th></tr>
<tr><td>Effort Generators</td>
<td>Voltage Gen.

</td>
<td>Velocity Gen.

</td>
<td>Ang. Velocity Gen.

</td>
<td>Pressure Gen.

</td></tr>
<tr><td>POG scheme</td>
<td>
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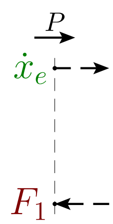
<td>
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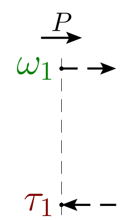
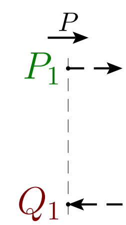
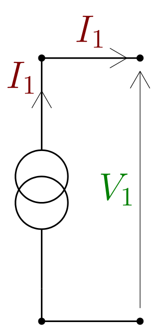
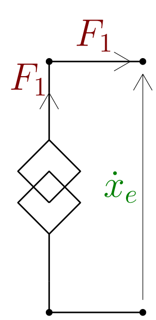
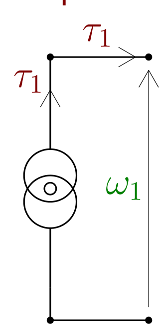
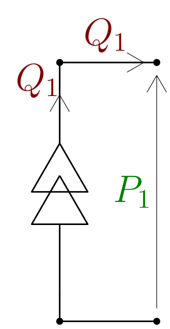
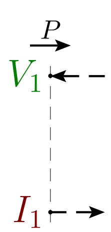
<td>
</td>
<td>
</td></tr>
</table>

- The “Flow/Through Generators” (POG Modeler):

<table>
<tr><th></th><th>Electrical</th><th>Mecc. Tras.</th><th>Mecc. Rot.</th><th>Hydraulic</th></tr>
<tr><td>Flow Generators</td>
<td>Current Gen.

</td>
<td>Force Gen.

</td>
<td>Torque Gen.

</td>
<td>Vol. Flow Rate Gen.

</td></tr>
<tr><td>POG scheme</td>
<td>
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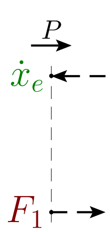
<td>
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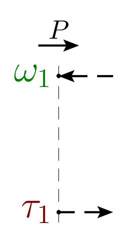
<td>
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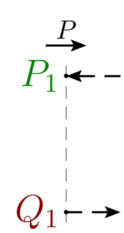
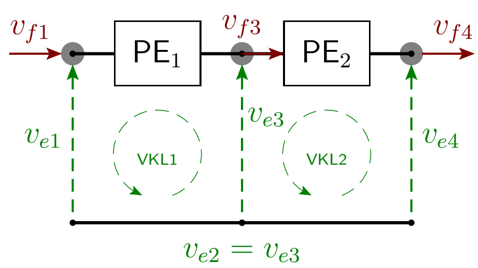
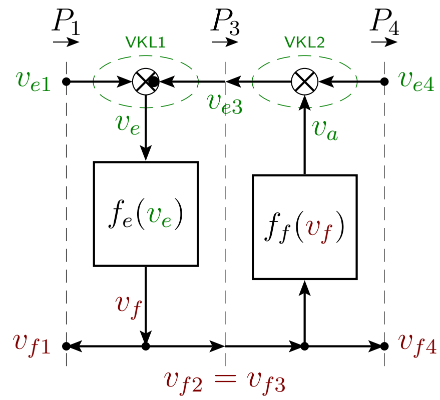
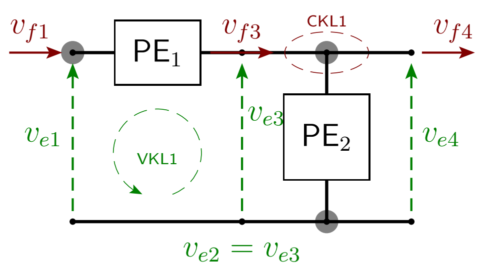
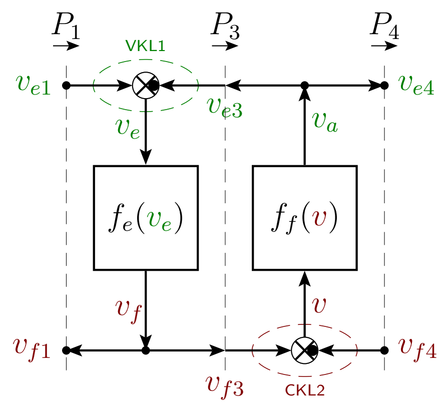
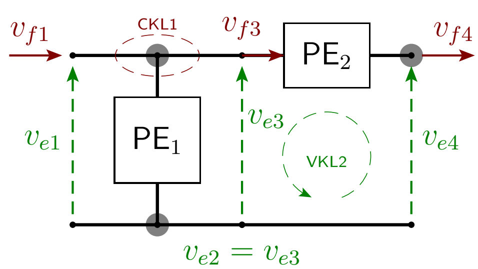
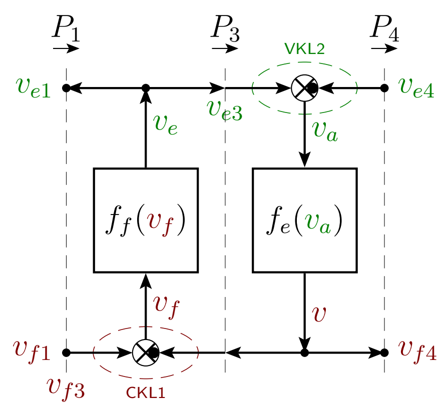
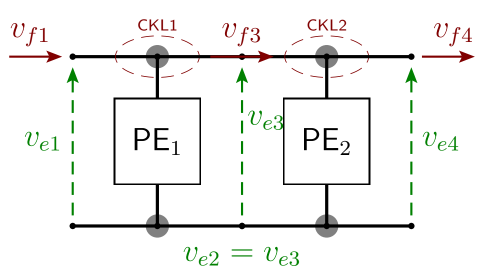
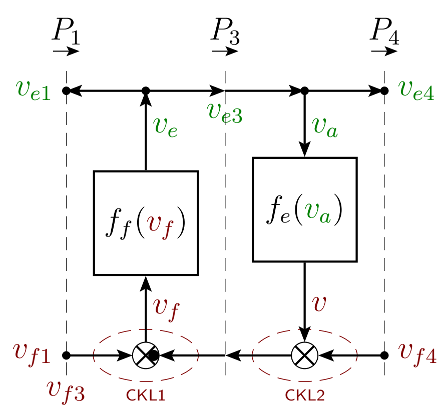
<td>
</td></tr>
</table>

- By default the positive direction of the power **exits** the generators.

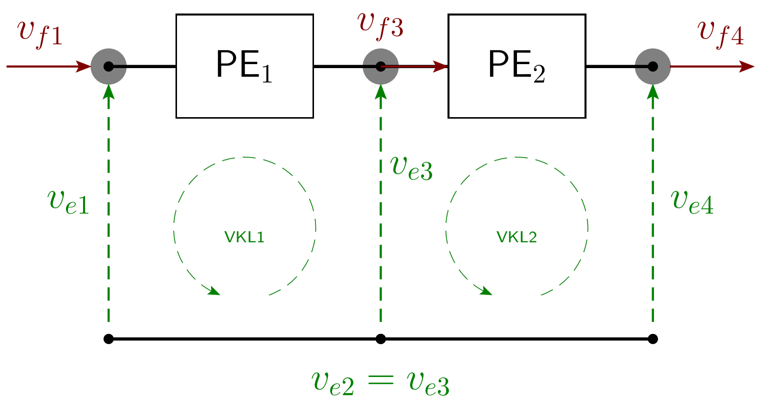
Connecting physical elements

- Two physical elements PE_1 and PE_2 can be connected as follows:

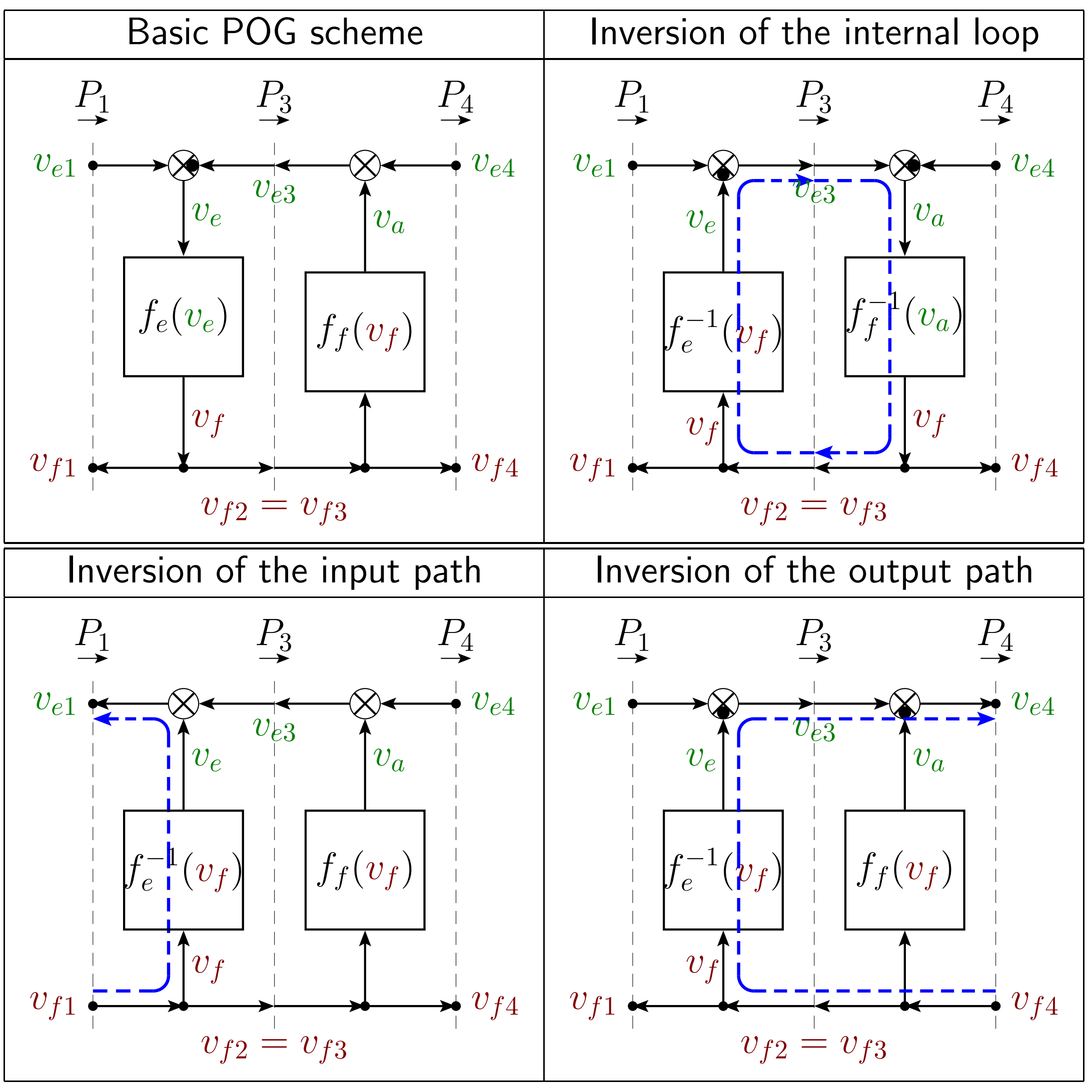
Physical connection	Basic POG scheme
a) *Series - Series* 	
b) *Series - Parallel* 	
c) *Parallel - Series* 	
d) *Parallel - Parallel* 	

- The basic POG scheme associated to a PE_1-PE_2 connection can be drawn in four different ways. For the "*Series - Series*" connection:

a) *Series - Series*

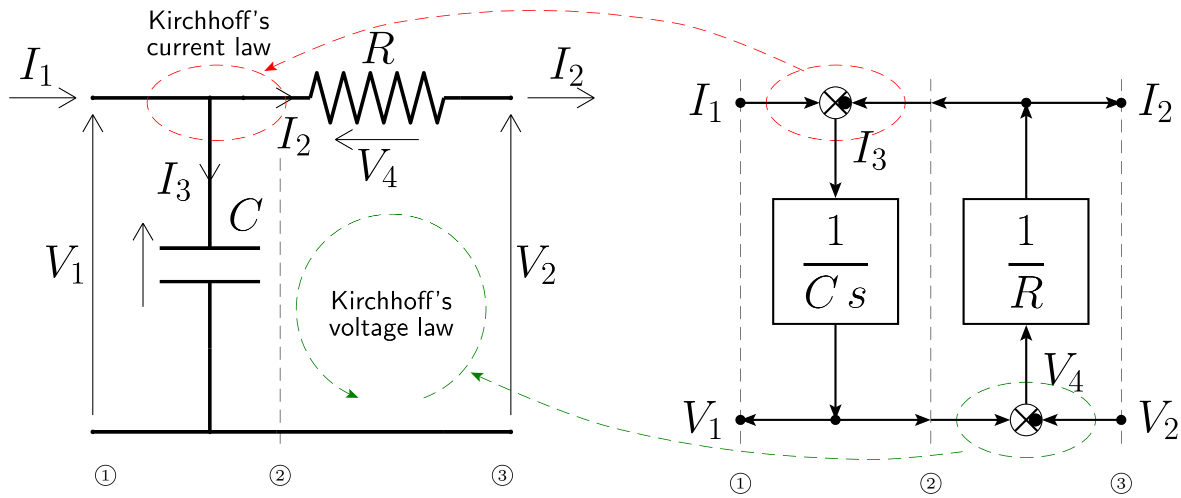


the following POG schemes can be used:



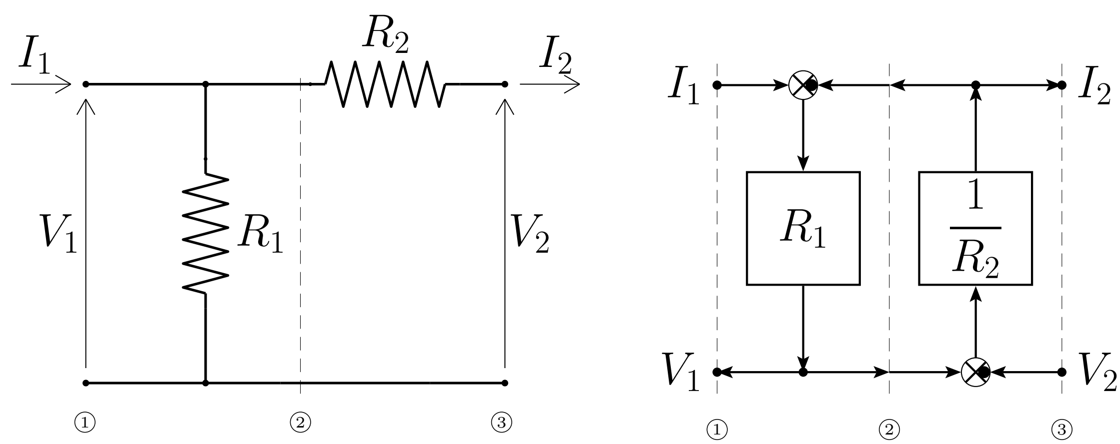
- Other four possible POG block schemes can be obtained considering the "upside down" versions of the above reported POG schemes.

- Example. A C-parallel and R-series connection:

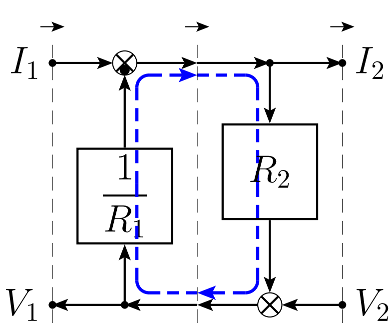
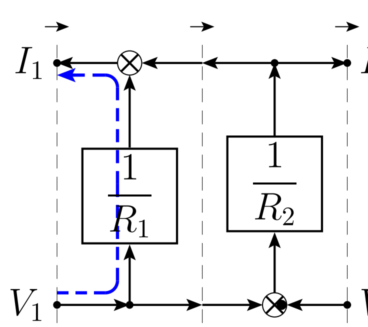
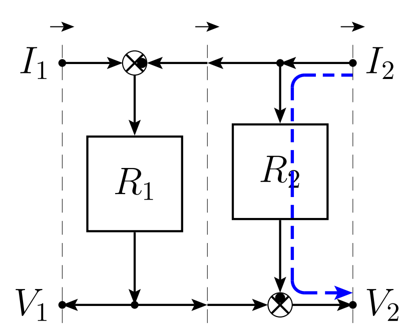


The internal loop and the input path of the POG scheme CANNOT be inverted because the capacitor must be described using its "integral causality" model. The output path can be inverted.

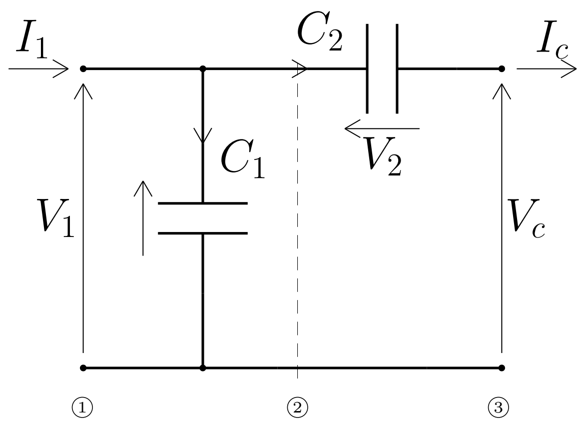
- Example. A R_1-parallel and R_2-series connection:

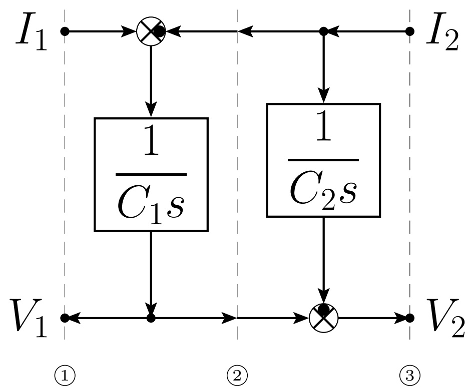


In this case, also the following equivalent POG block schemes can be used:

Inversion of the internal loop	Inversion of the input path	Inversion of the output path
		

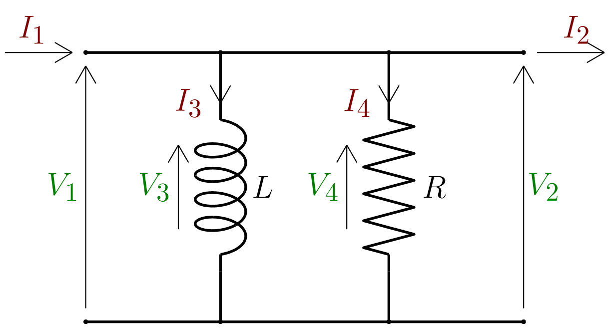
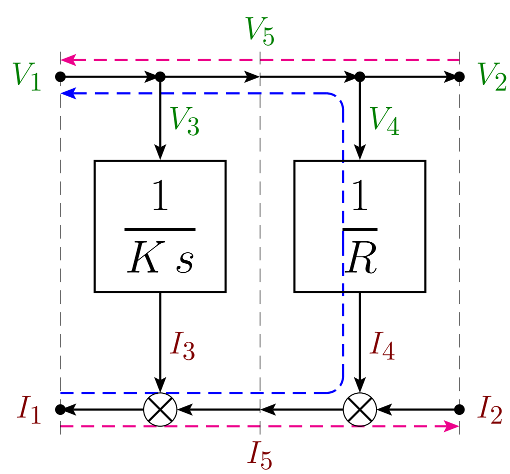
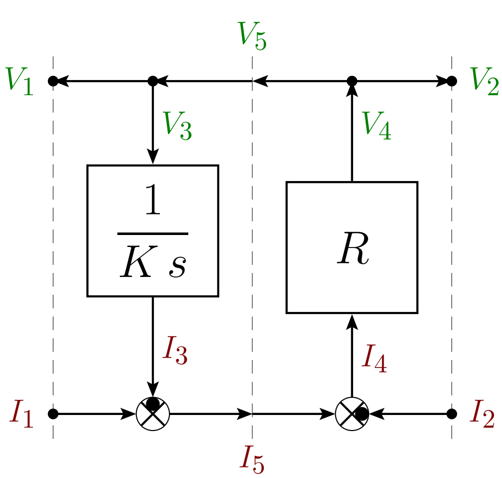
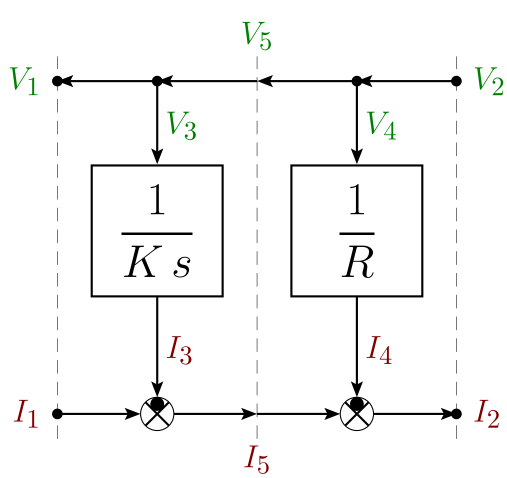
- Example. A C_1-parallel and C_2-series connection:



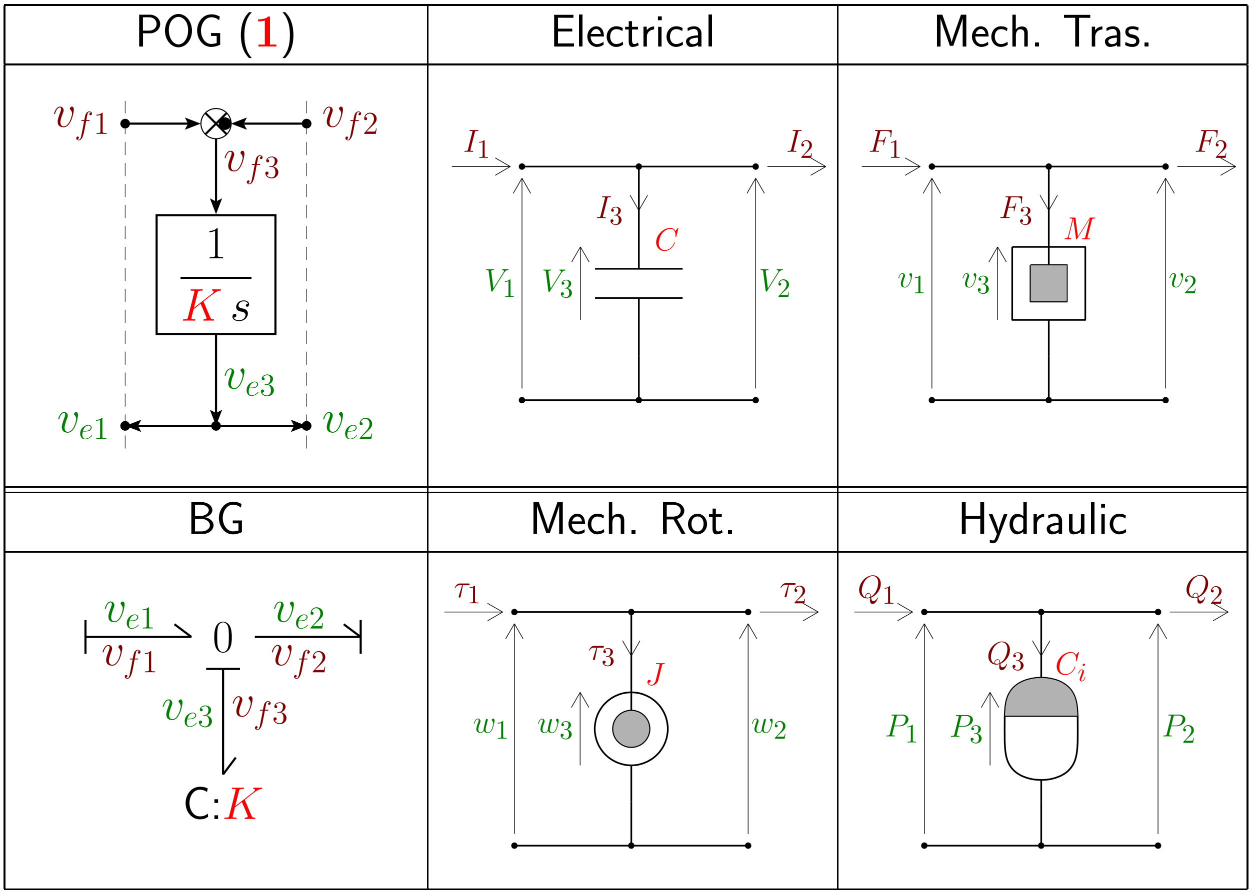


In this case there is only one POG block scheme that can be used for describing the given physical system.

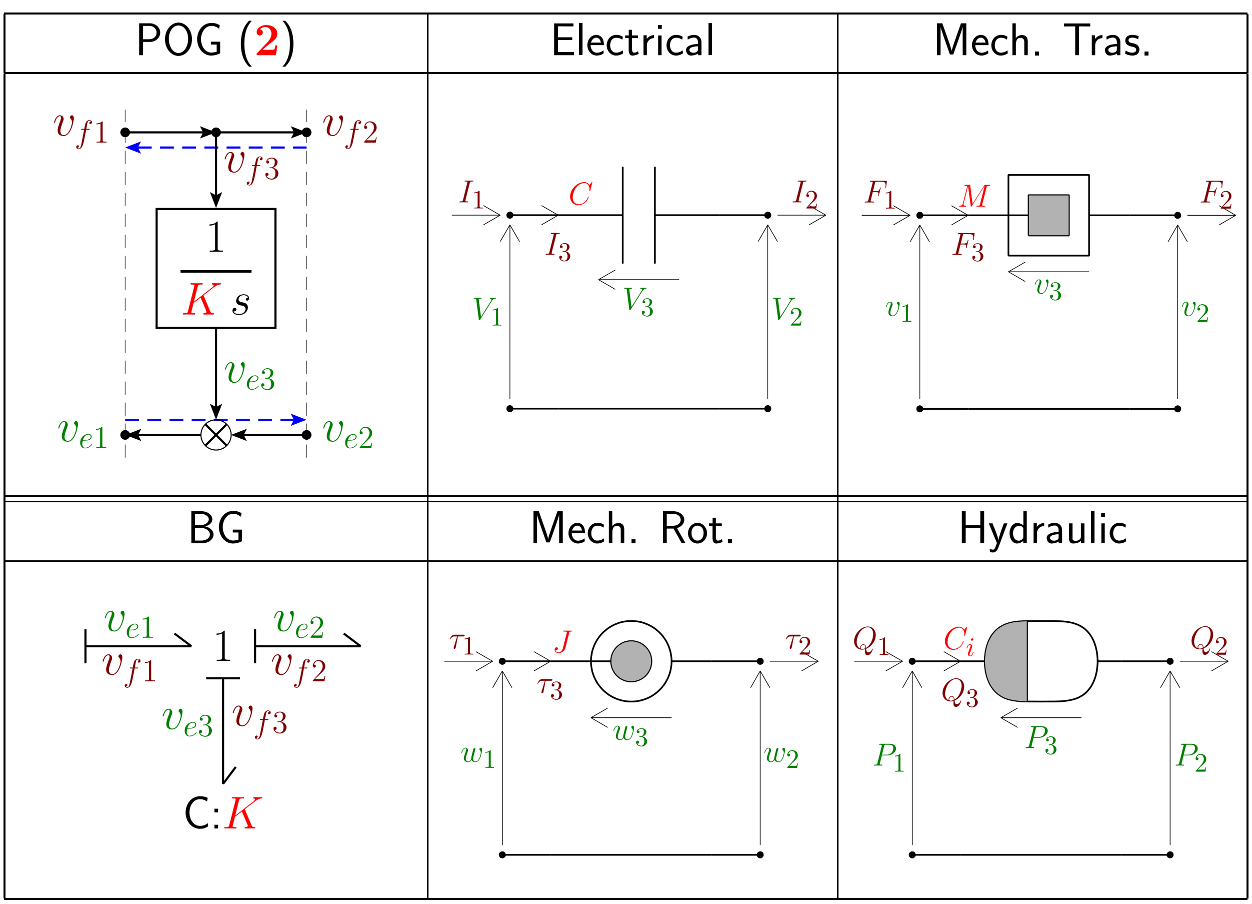
- An Electrical System that can be described by three POG block schemes:

Electrical System	POG scheme 1
	
POG scheme 2	**POG scheme 3**
	

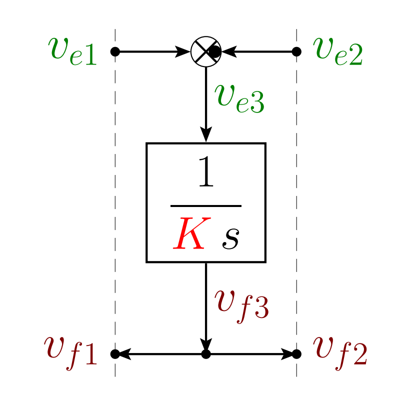
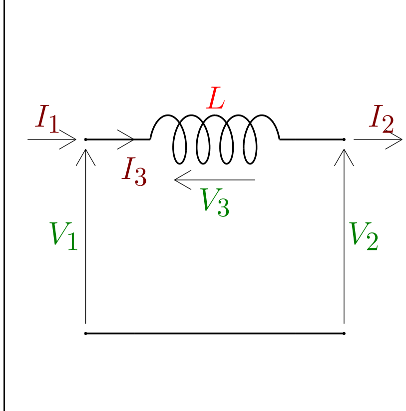
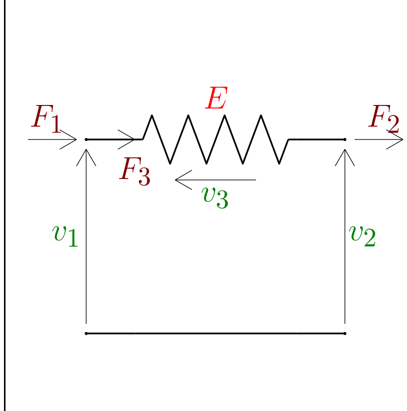
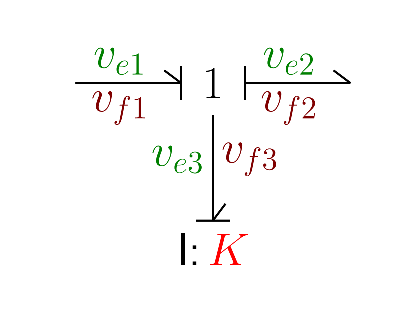
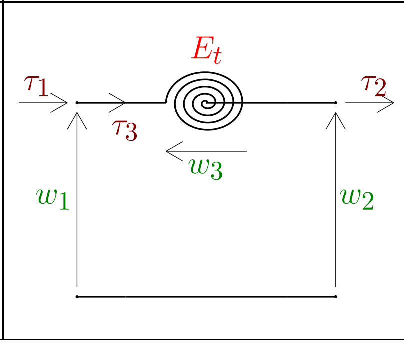
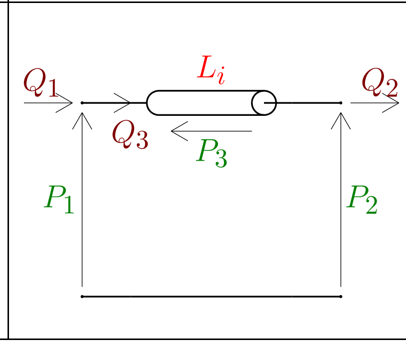
- **Effort Blocks** in **Parallel**:



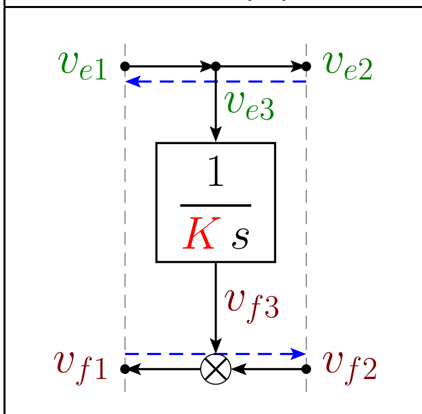
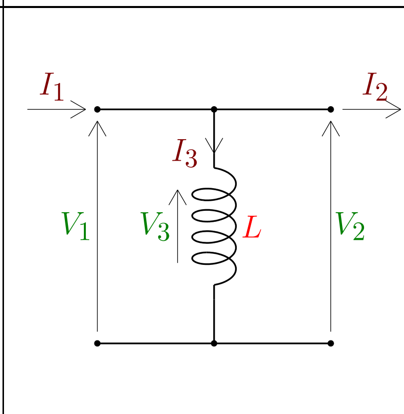
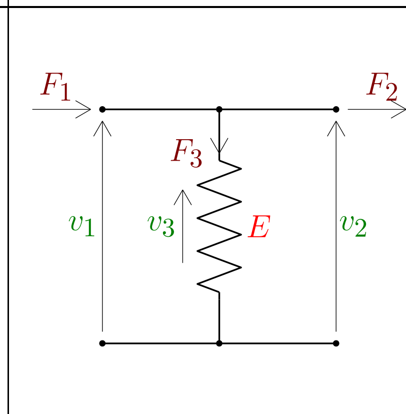
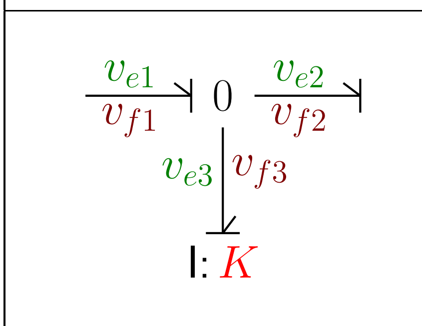
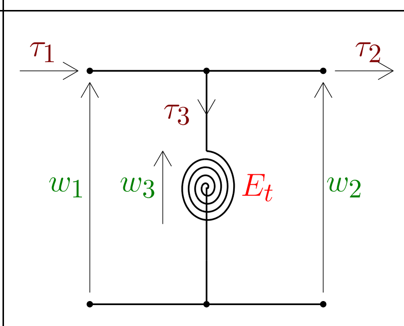
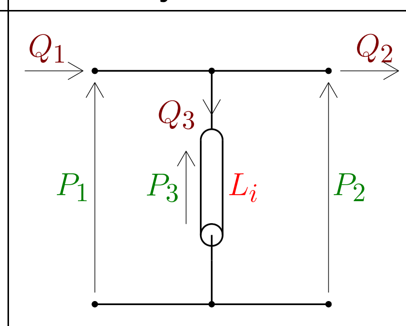
- **Effort Blocks** in **Series**:



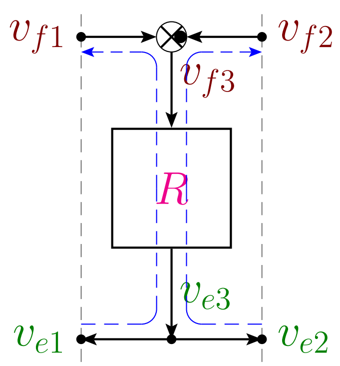
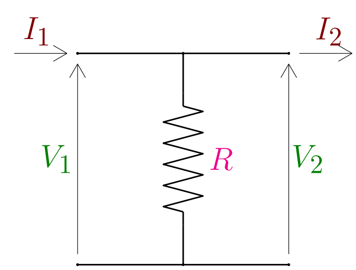
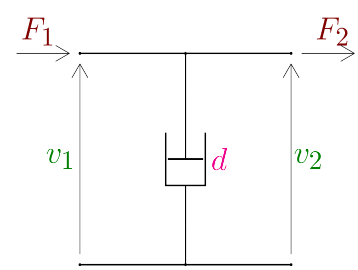
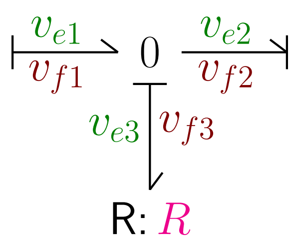
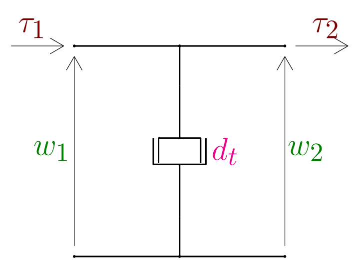
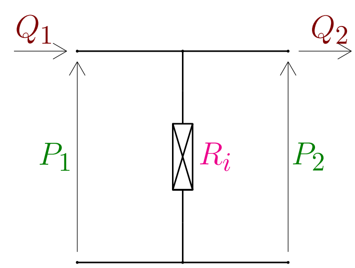
- **Flow Blocks** in **Series**:

POG (1)	Electrical	Mech. Tras.
		
BG	**Mech. Rot.**	**Hydraulic**
		

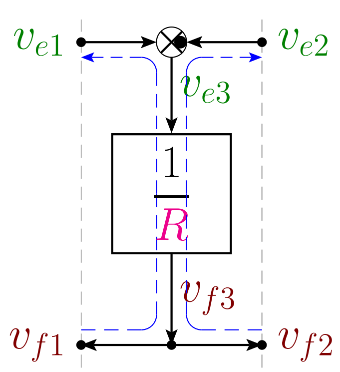
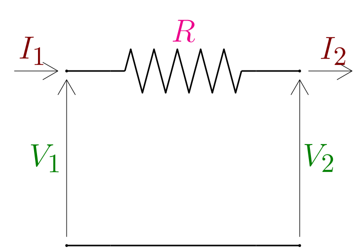
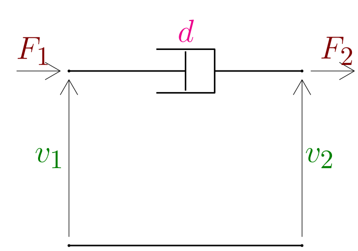
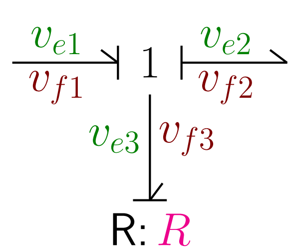
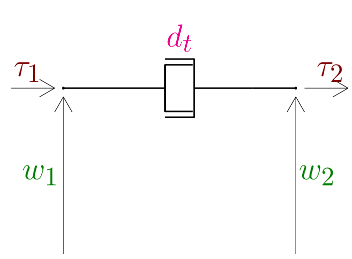
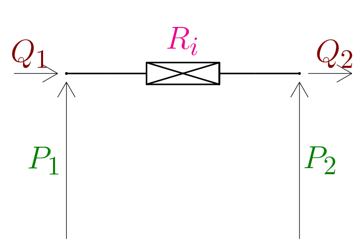
- **Flow Blocks** in **Parallel**:

POG (2)	Electrical	Mech. Tras.
		
BG	**Mech. Rot.**	**Hydraulic**
		

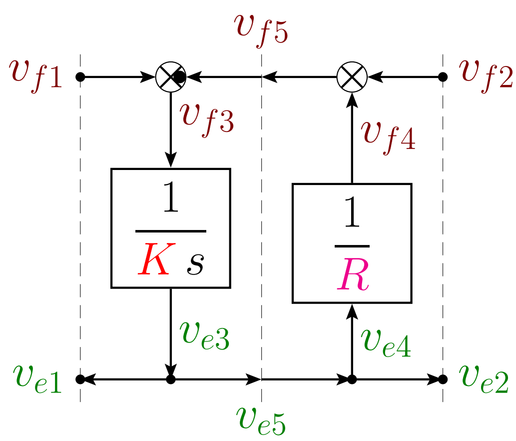
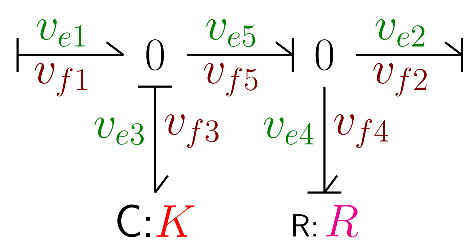
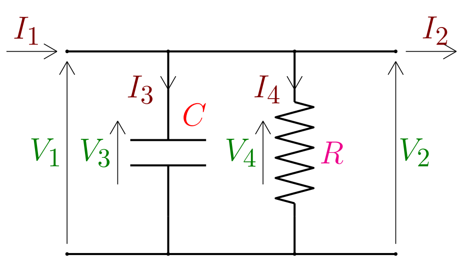
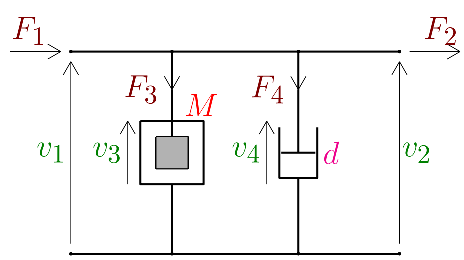
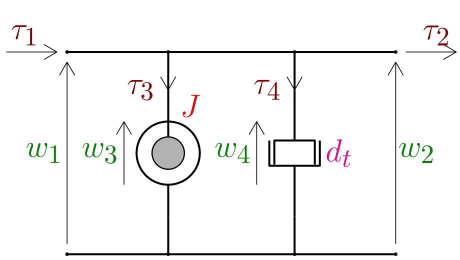
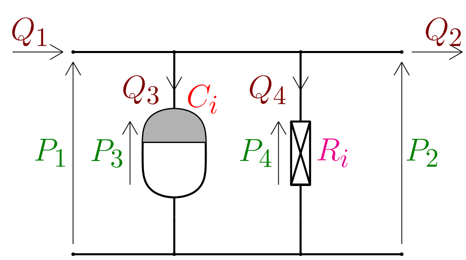
- **Dissipative Blocks** in **Parallel**:

POG (**3**)	Electrical	Mech. Tras.
		
BG	**Mech. Rot.**	**Hydraulic**
		

- **Dissipative Blocks** in **Series**:

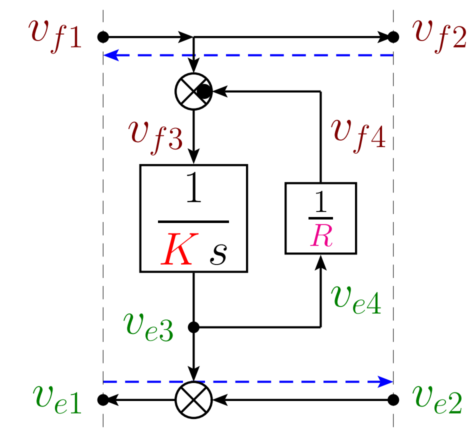
POG (**3**)	Electrical	Mech. Tras.
		
BG	**Mech. Rot.**	**Hydraulic**
		

- **Parallel** of an **Effort Block** in **Parallel** with a **Dissipative Block**:

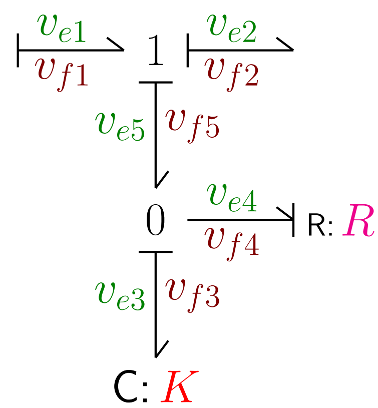
POG (1)	BG
	
Electrical	**Mech. Tras.**
	
Mech. Rot.	**Hydraulic**
	

- **Series** of an **Effort Block** in **Parallel** with a **Dissipative Block**:

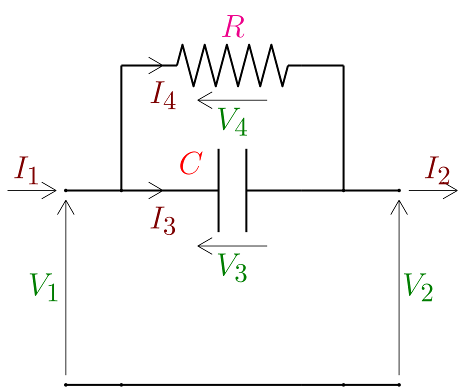
POG (2)



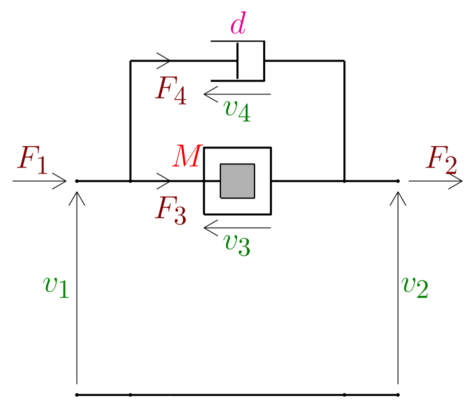
BG



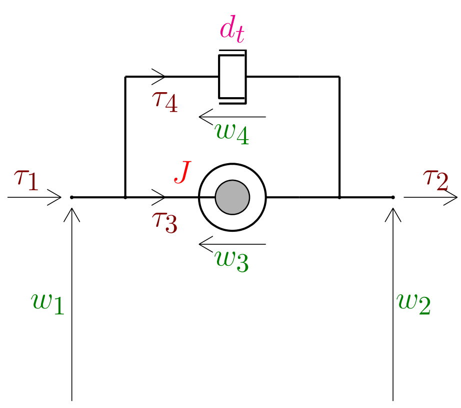
Electrical



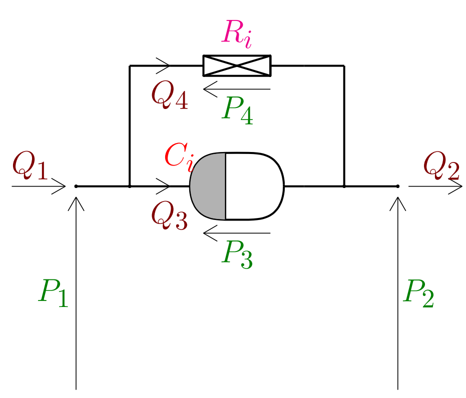
Mech. Tras.



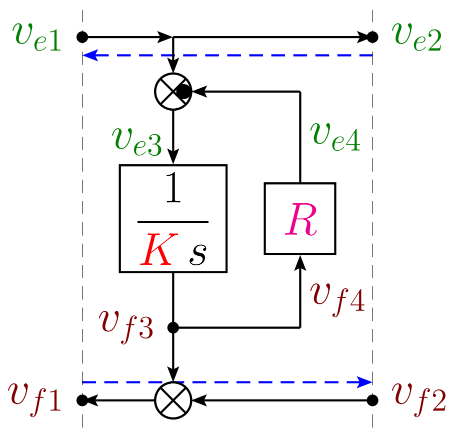
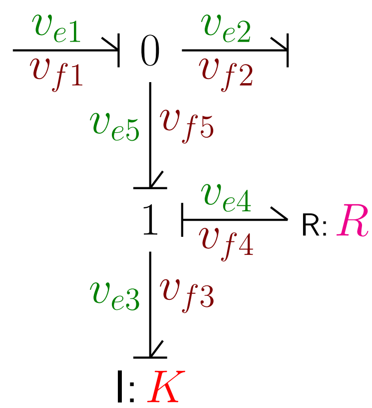
Mech. Rot.

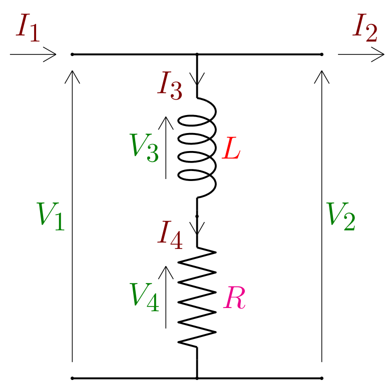
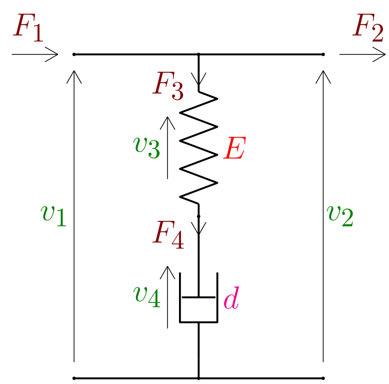


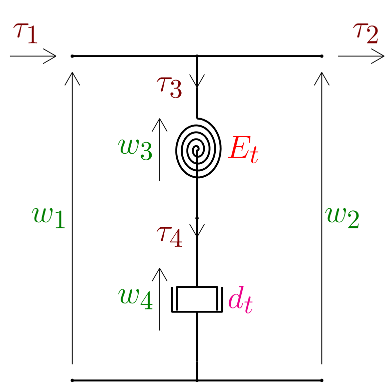
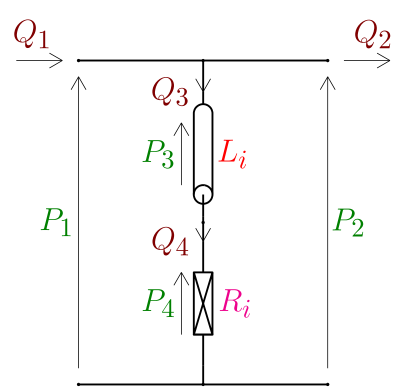
Hydraulic



- **Parallel** of a **Flow Block** in **Series** with a **Dissipative Block**:

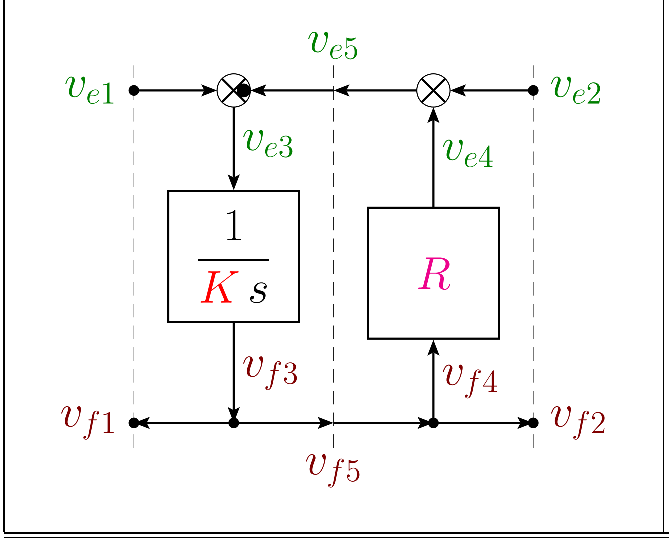
POG (**2**)	BG
	

Electrical	Mech. Tras.
	

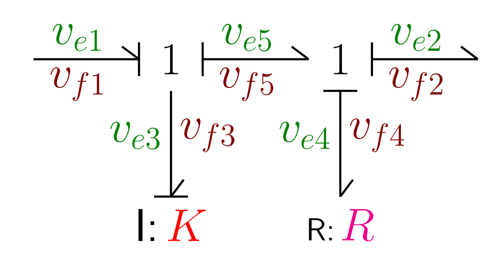
Mech. Rot.	Hydraulic
	

- **Series** of a **Flow Block** in **Series** with a **Dissipative Block**:

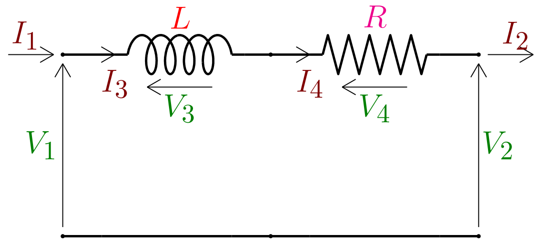
POG (1)



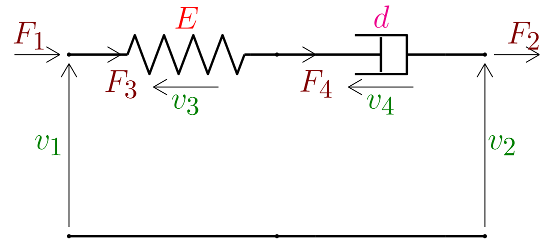
BG



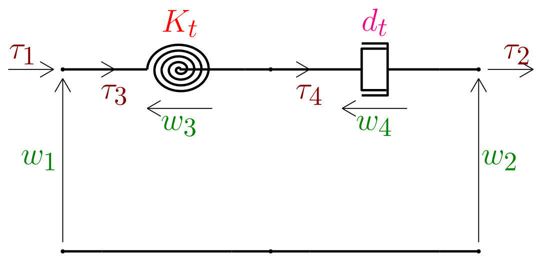
Electrical



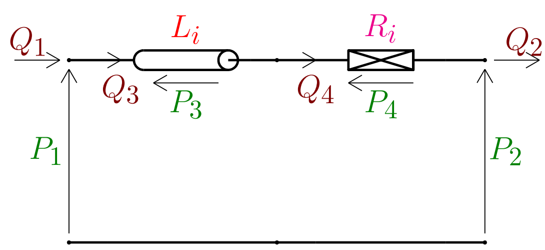
Mech. Tras.



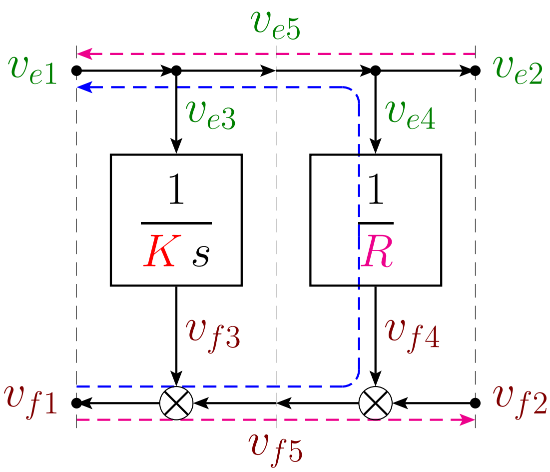
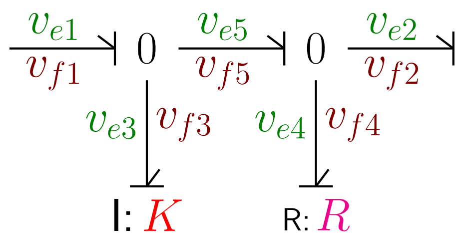
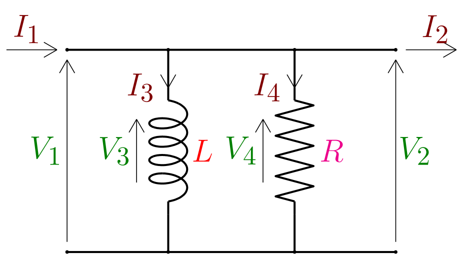
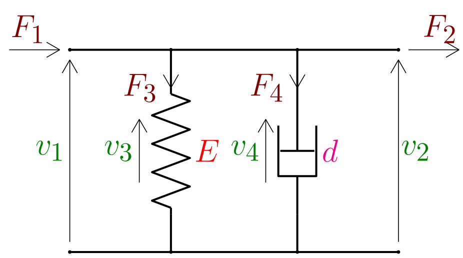
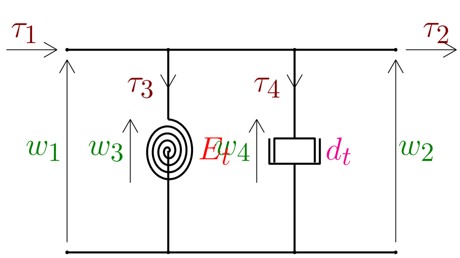
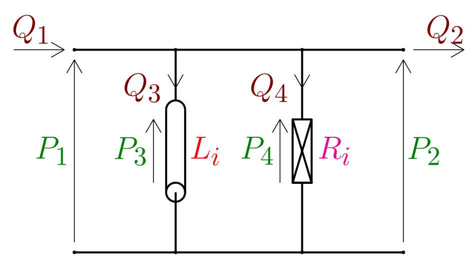
Mech. Rot.



Hydraulic

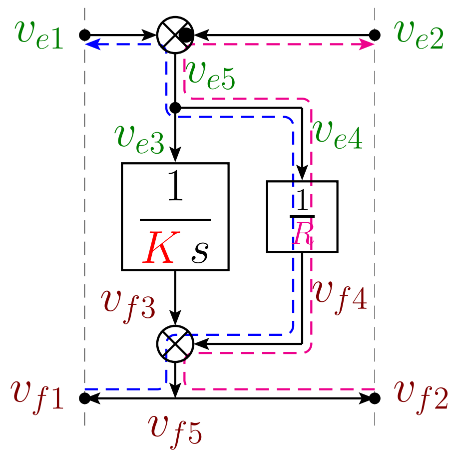


- **Parallel** of a **Flow Block** in **Parallel** with a **Dissipative Block**:

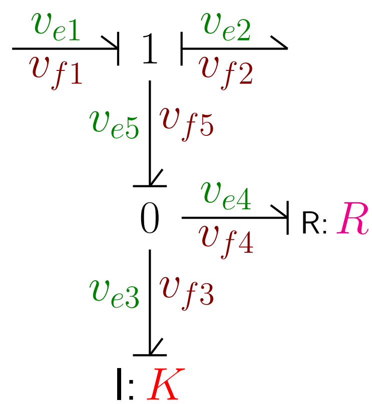
POG (**3**)	BG
	
Electrical	Mech. Tras.
	
Mech. Rot.	Hydraulic
	

- **Series** of a **Flow Block** in **Parallel** with a **Dissipative Block**:

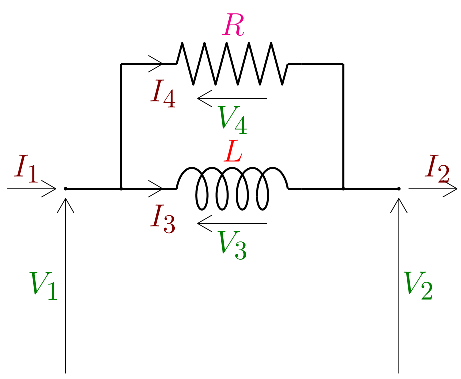
POG (**3**)



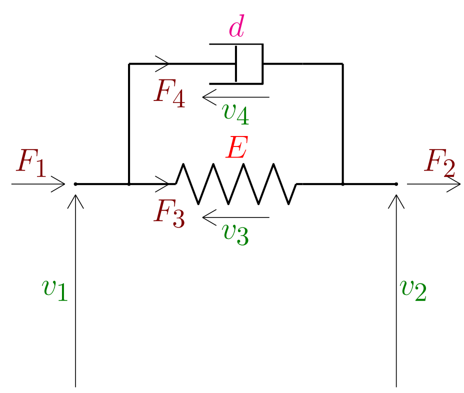
BG



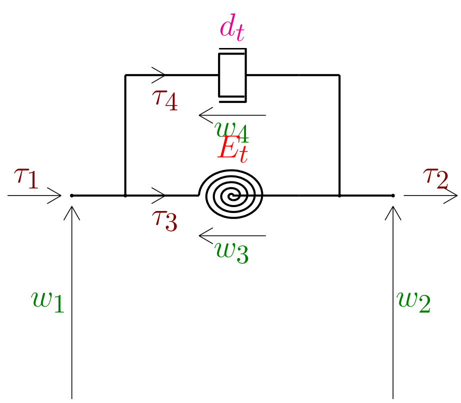
Electrical



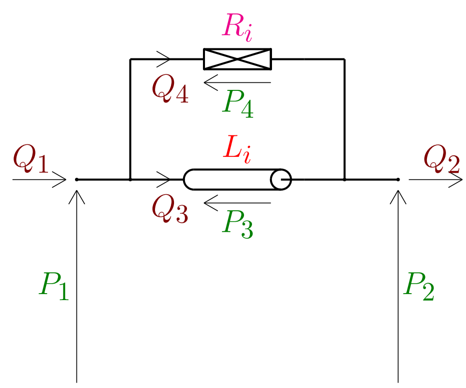
Mech. Tras.



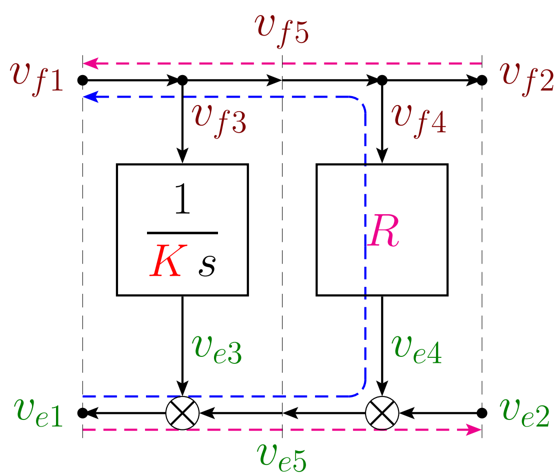
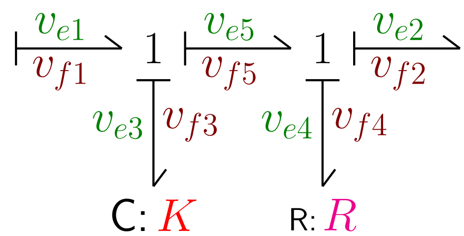
Mech. Rot.

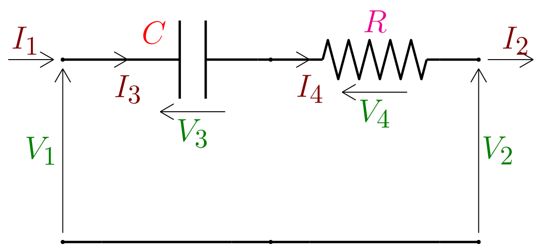
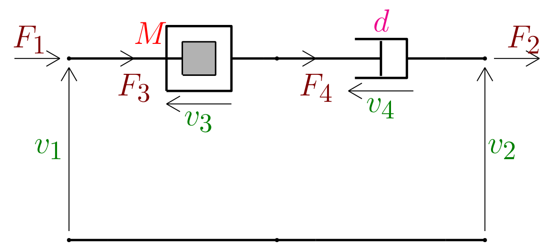


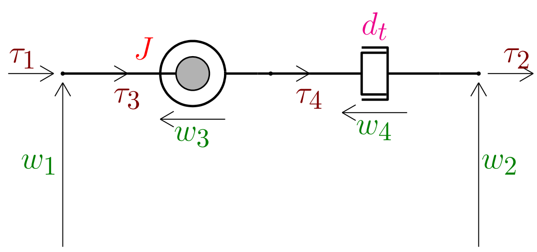
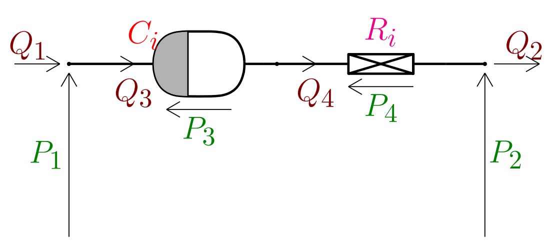
Hydraulic



- **Series** of an **Effort Block** in **Series** with a **Dissipative Block**:

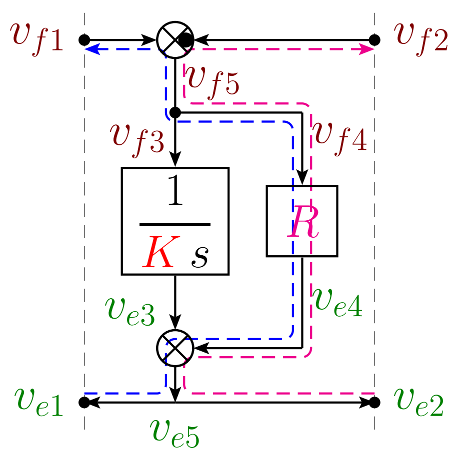
POG (3)	BG
	

Electrical	Mech. Tras.
	

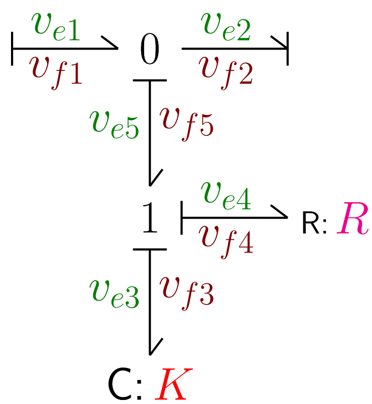
Mech. Rot.	Hydraulic
	

- **Parallel** of an **Effort Block** in **Series** with a **Dissipative Block**:

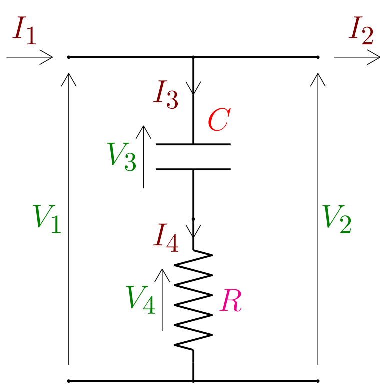
POG (**3**)



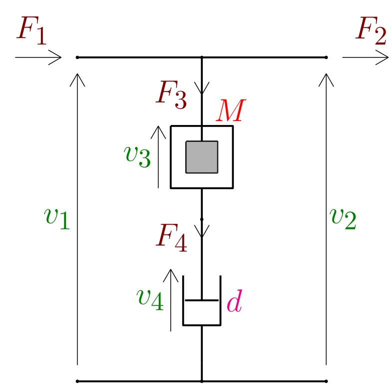
BG



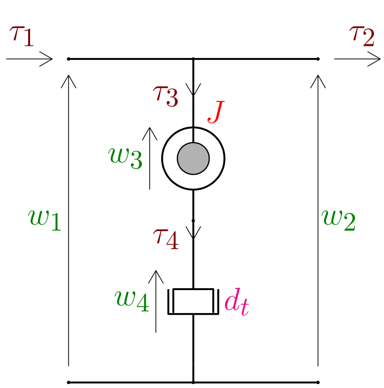
Electrical



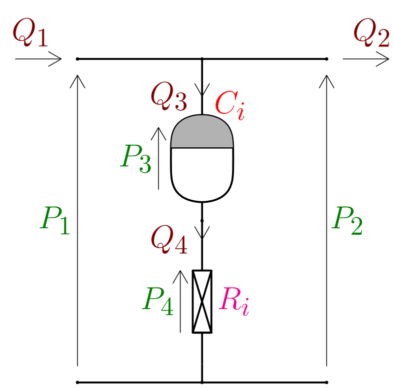
Mech. Tras.



Mech. Rot.



Hydraulic



POG modeling: simple examples

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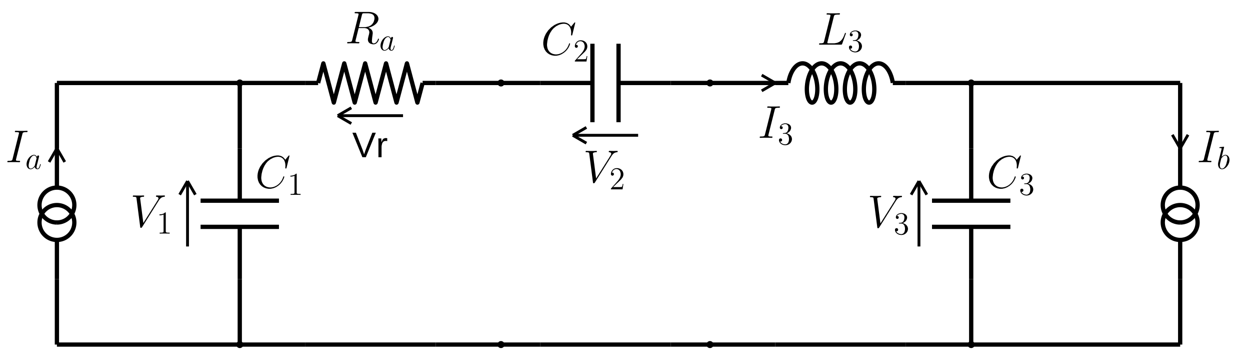
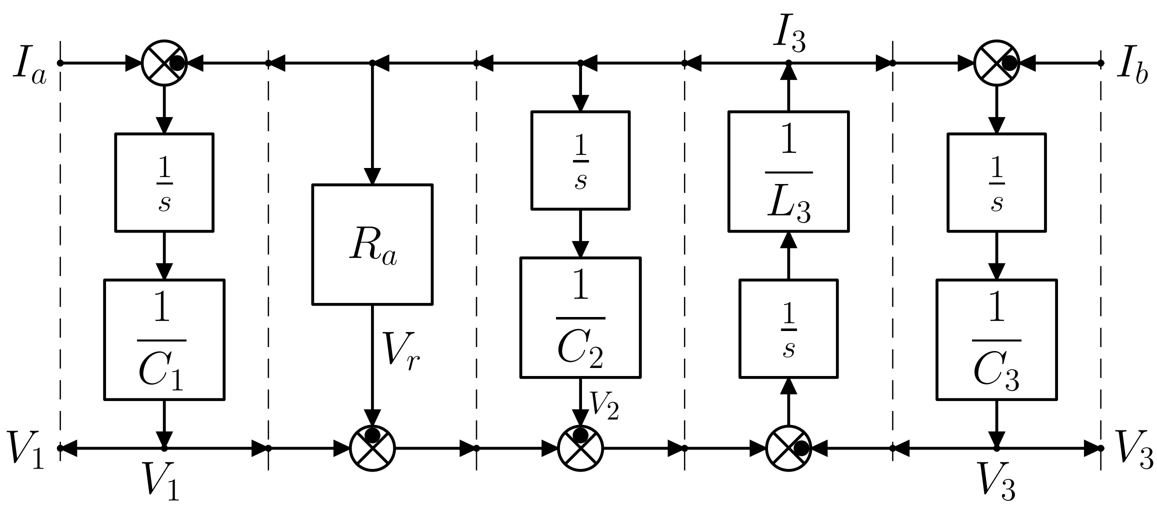
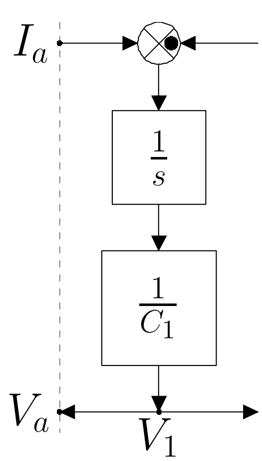
- Let us consider the following electric circuit:



- The corresponding POG block scheme is:



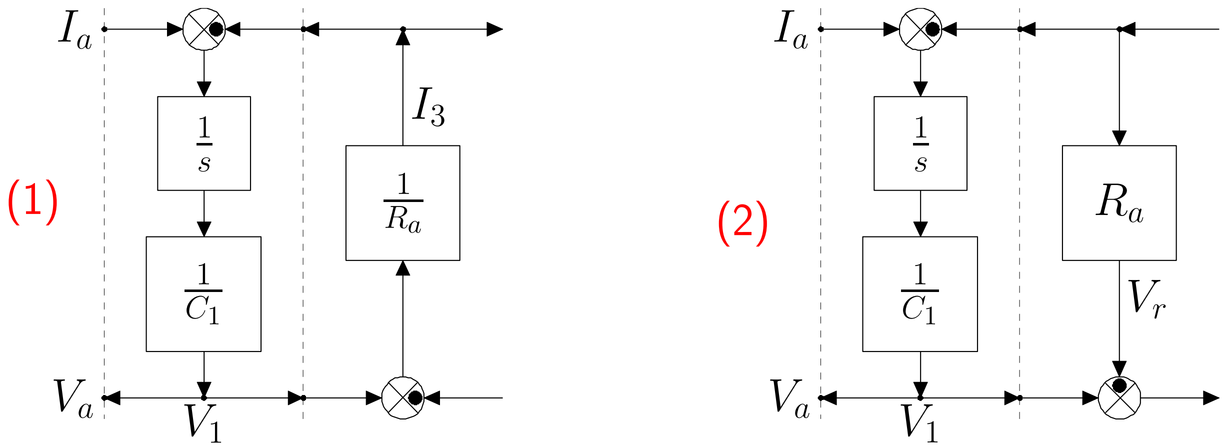
- The POG scheme can be obtained modeling the system from left ro right.



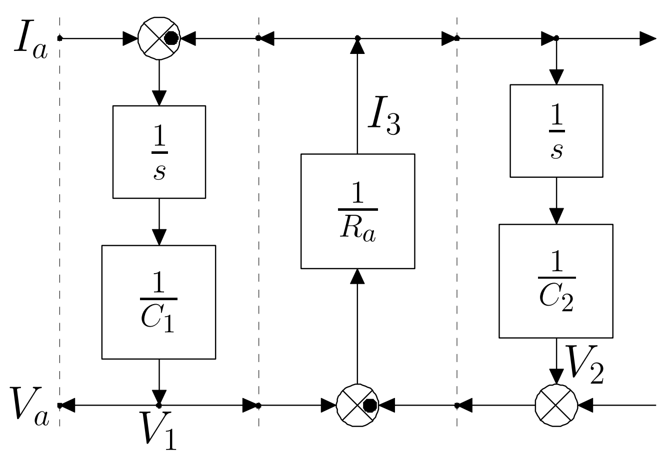
The input current I_a acts on a capacitor (Effort Block) in parallel connection, and therefore there is only one way of modeling the capacitor:

- The next element is the resistor R_a in series with an input voltage V_1 on the left. In this case the resistor can be modeled using two POG schemes.

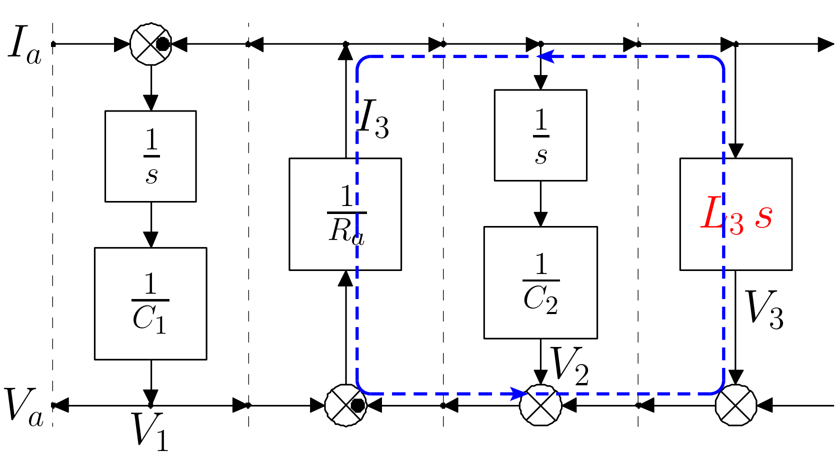
- The two possible POG schemes are:



- Let us consider the solution (1). The next element to be modeled is the capacitor C_2 in series with the input current I_3 on the left. In this case the capacitor C_2 can be modeled only as follows:

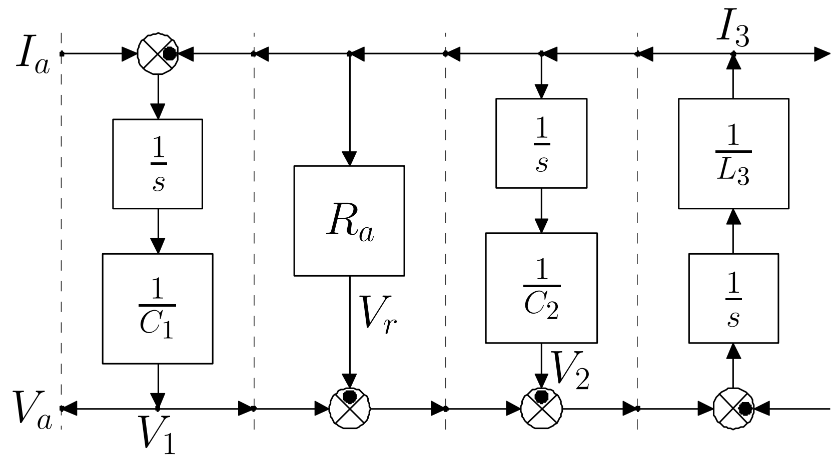


- The next element to be modeled is the inductor L_3 in series. Since the input current I_3 enters on the left, the inductor L_3 can be modeled only using the derivative causality:



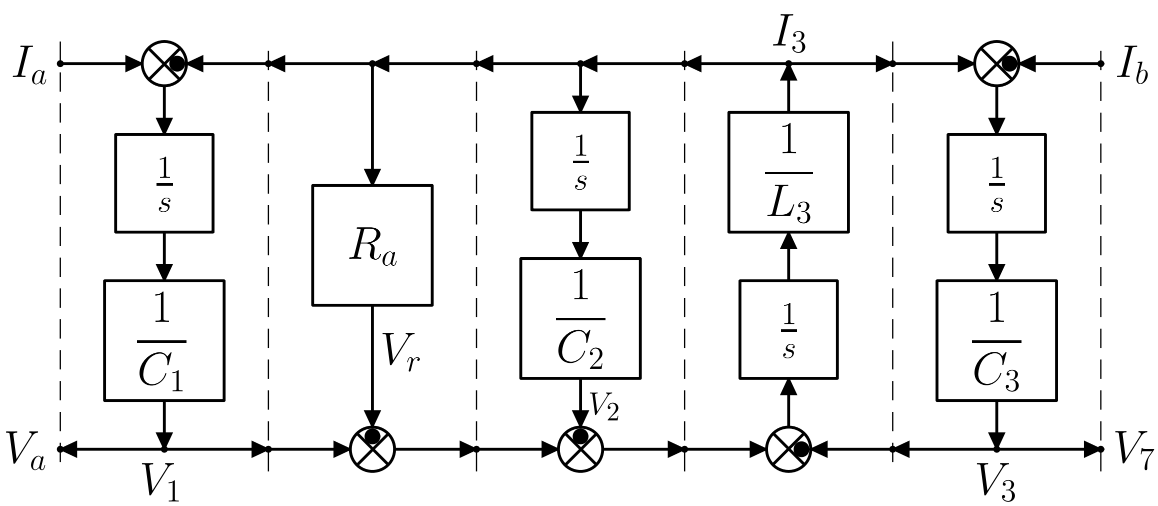
- In this case the causality problem can be solved inverting the blue dashed path shown in the picture.

- After inverting the path, the POG scheme has the following structure:



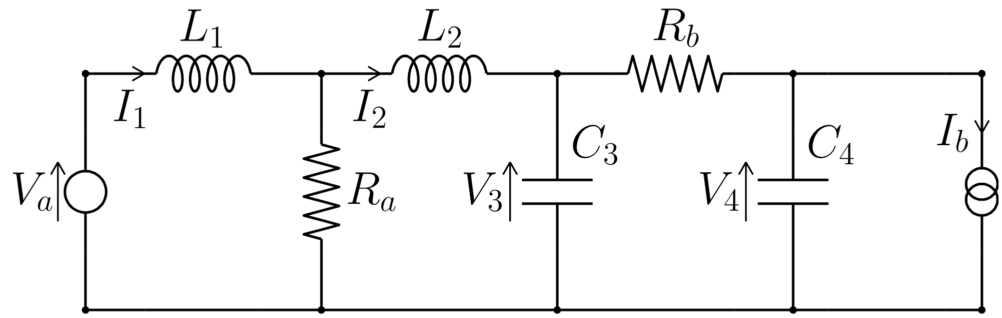
- This is exactly the POG scheme that would have been obtained choosing

:

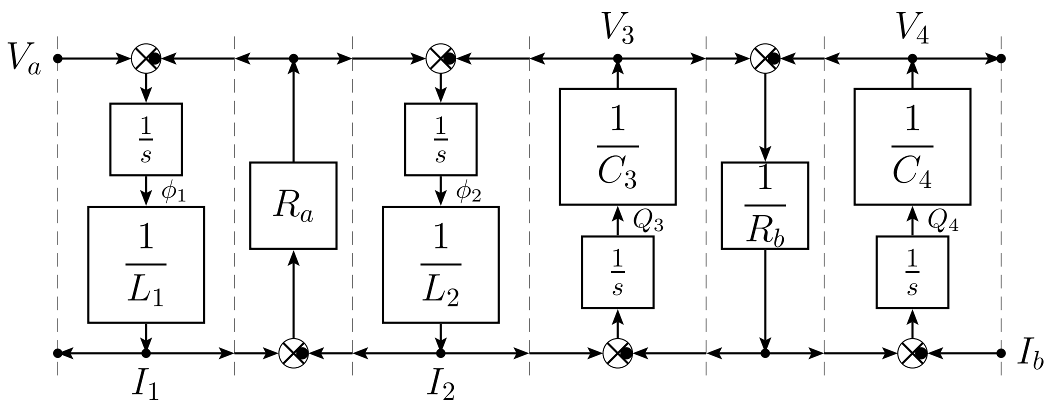


- In fact, the capacitor C_3 is connected in parallel and can be modeled only using the POG structure shown in the picture.

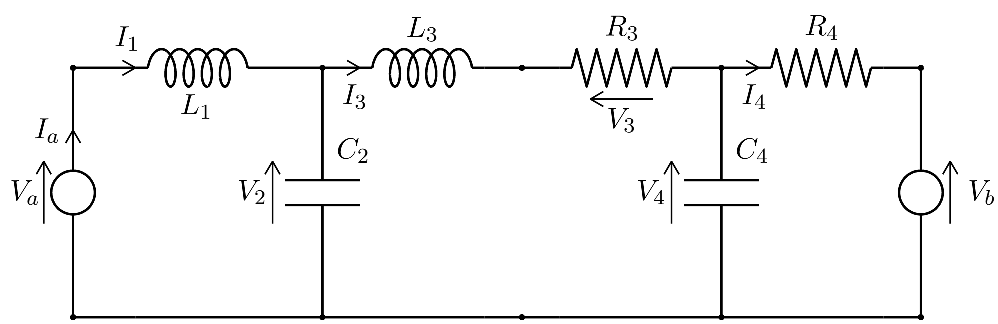
- Consider the following electric circuit:



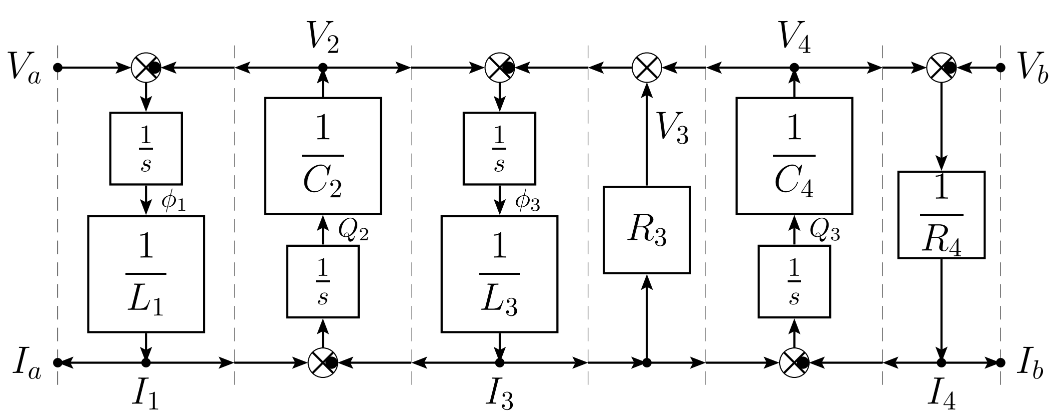
The corresponding POG block scheme is:



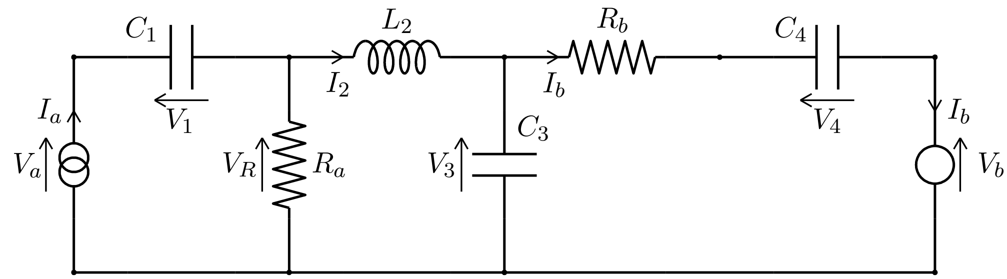
- Consider the following electric circuit:



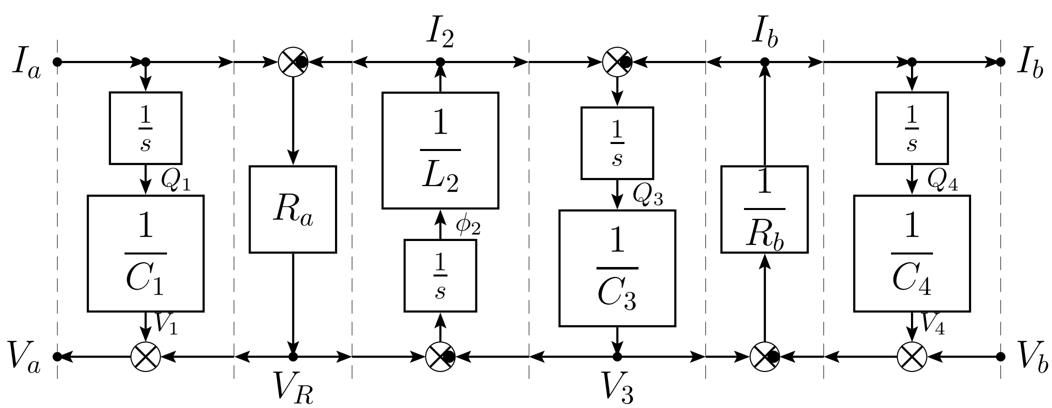
The corresponding POG block scheme is:



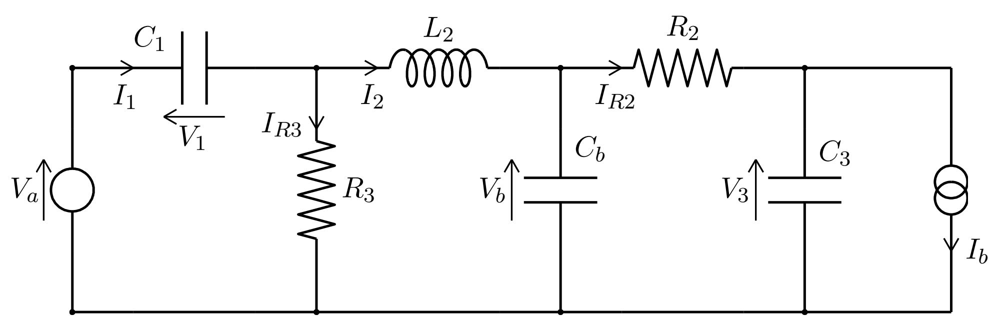
- Consider the following electric circuit:



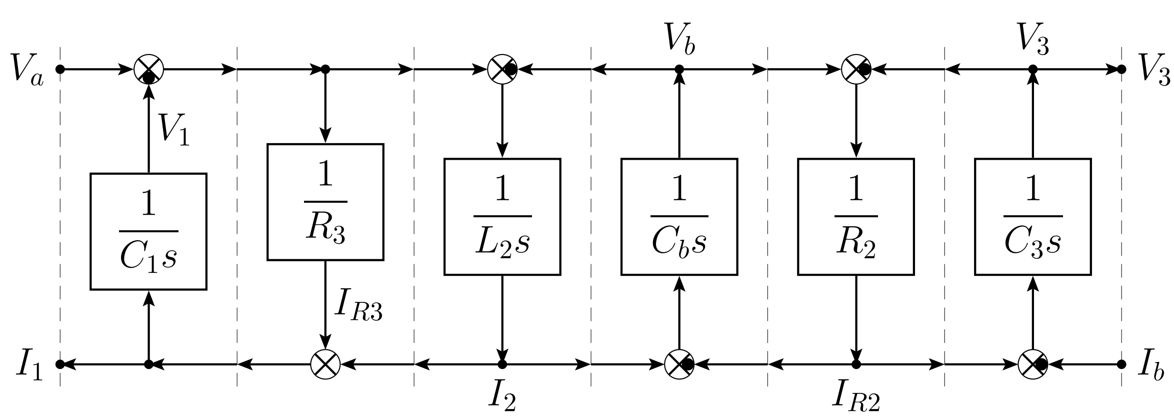
The corresponding POG block scheme is:



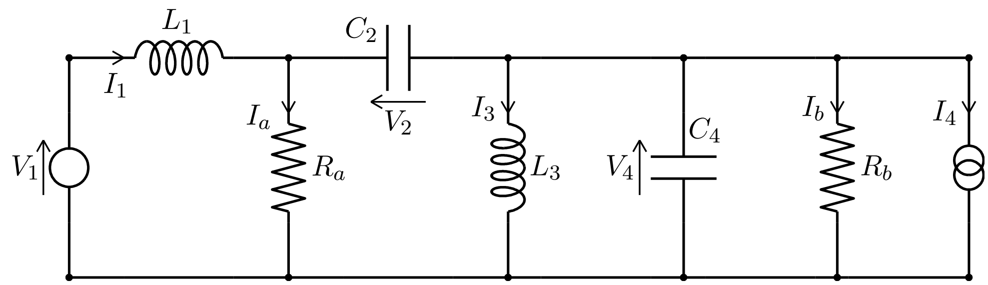
- Consider the following electric circuit:



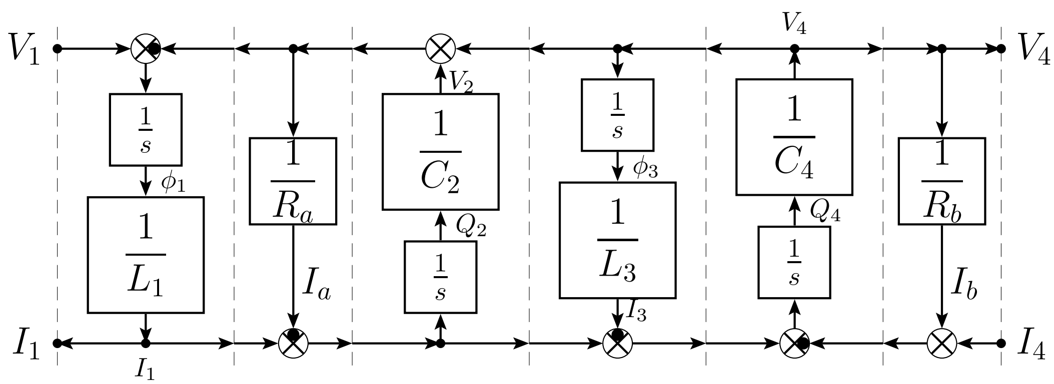
The corresponding POG block scheme is:



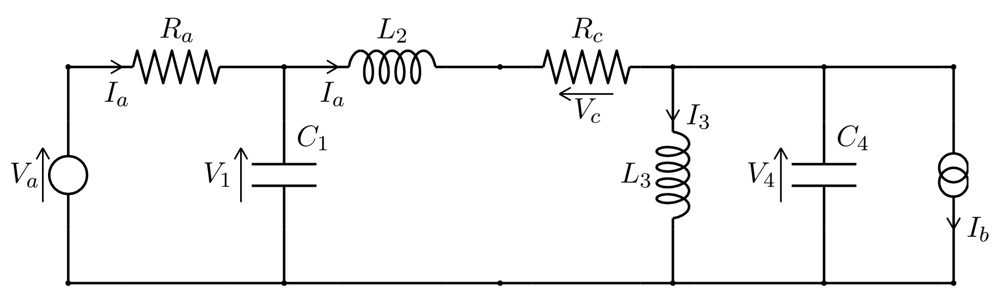
- Consider the following electric circuit:



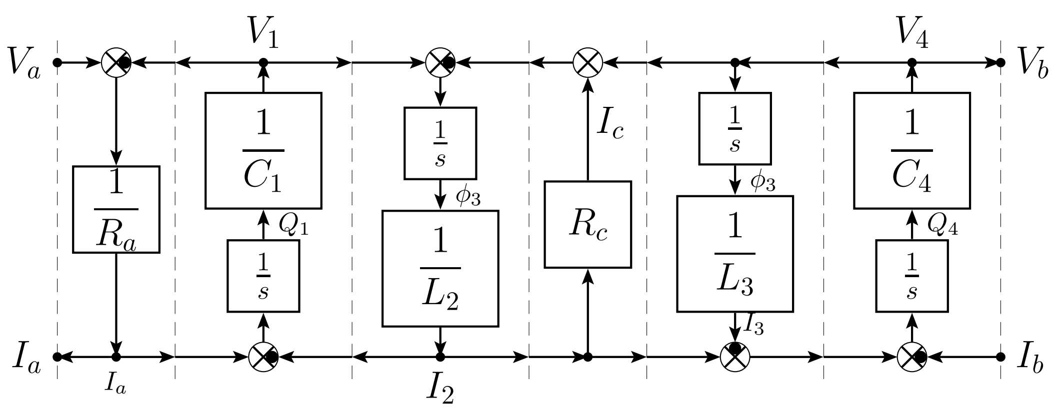
The corresponding POG block scheme is:



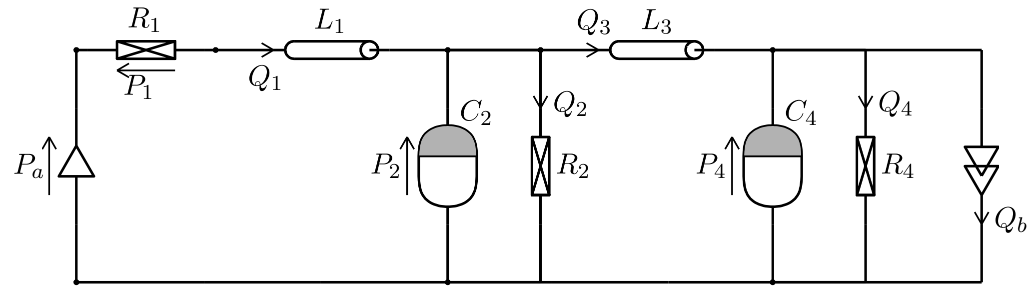
- Consider the following electric circuit:



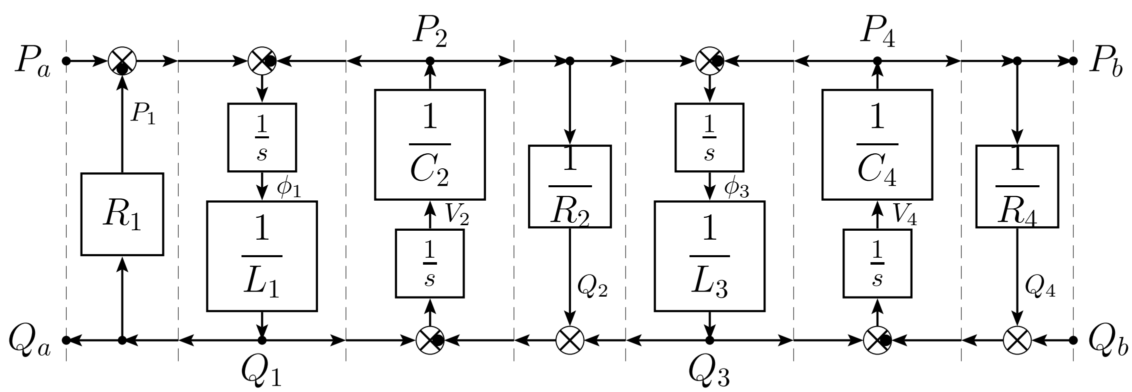
The corresponding POG block scheme is:



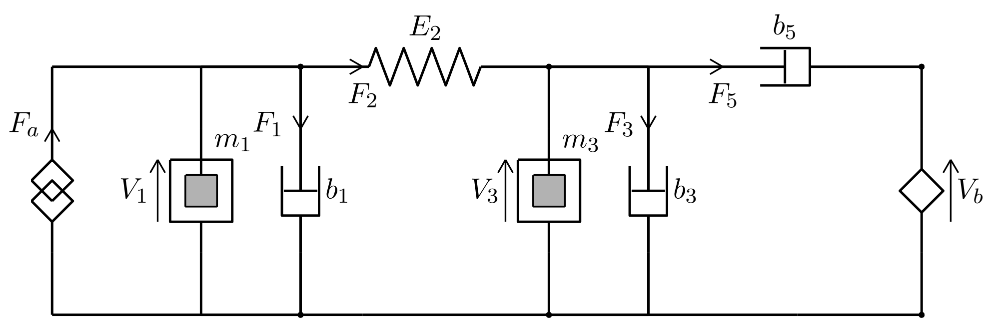
- Consider the following hydraulic circuit :



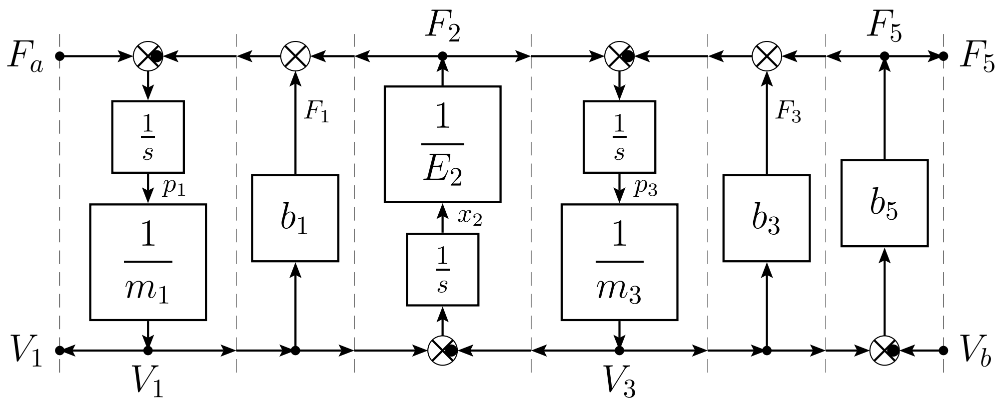
The corresponding POG block scheme is:



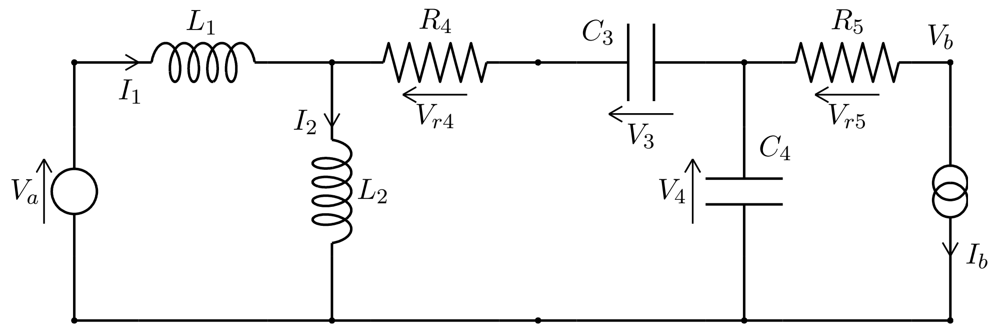
- Consider the following mechanical system:



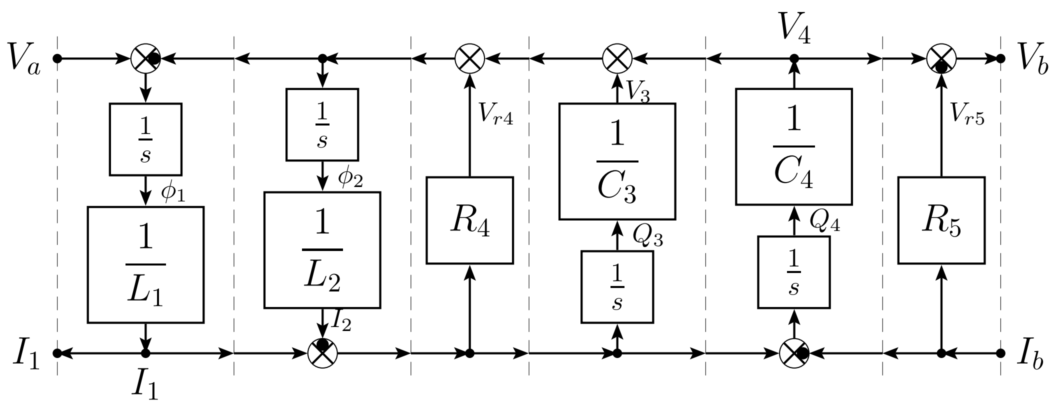
The corresponding POG block scheme is:



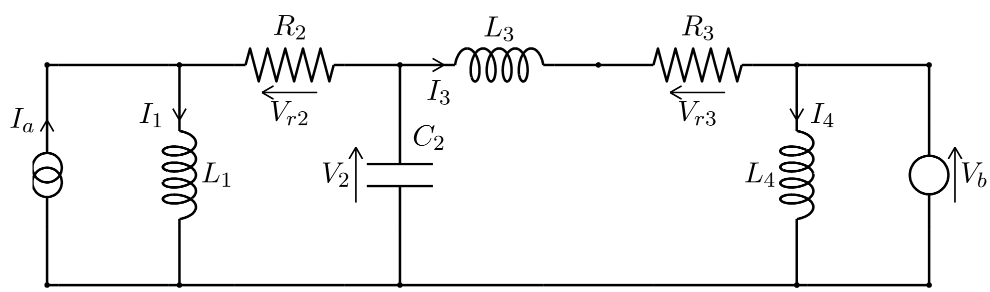
- Consider the following electric circuit:



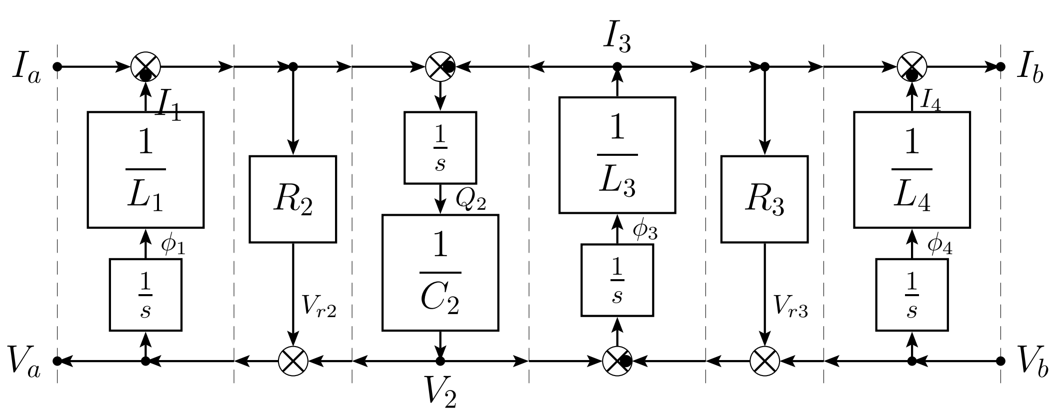
The corresponding POG block scheme is:



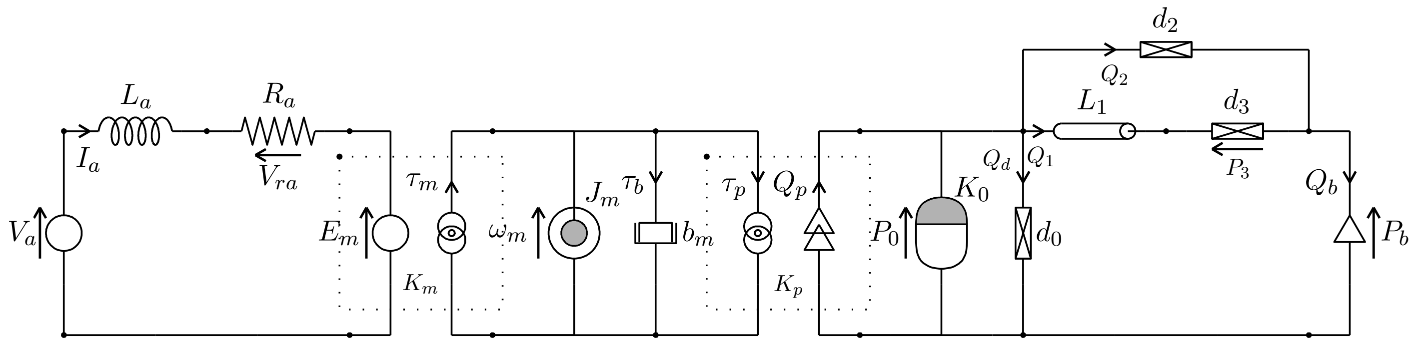
- Consider the following electric circuit:



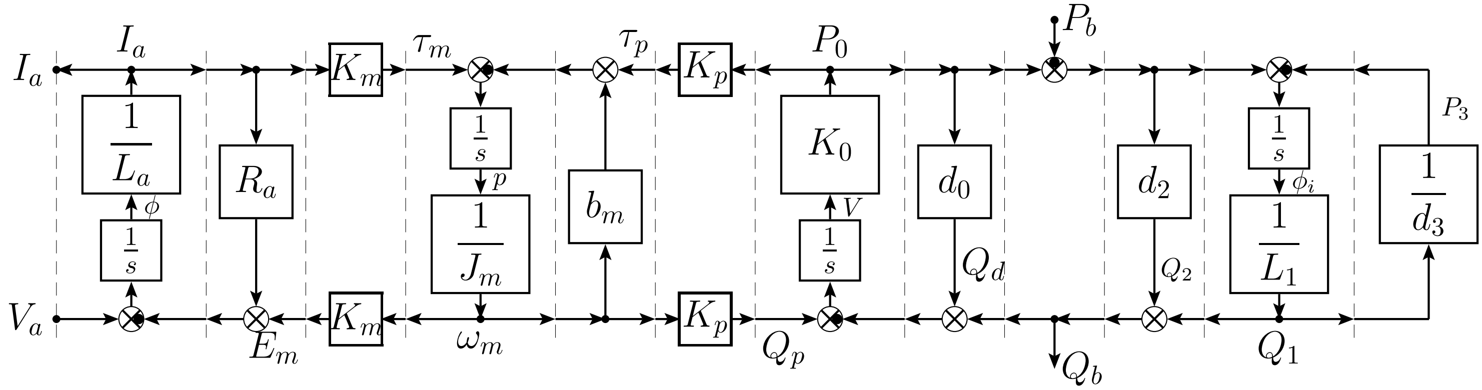
The corresponding POG block scheme is:



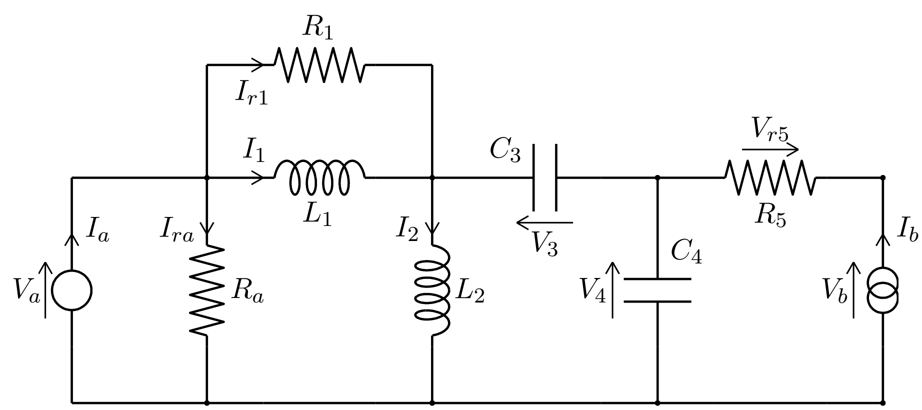
- Consider the following dynamic system (DC motor with an hydraulic pump):



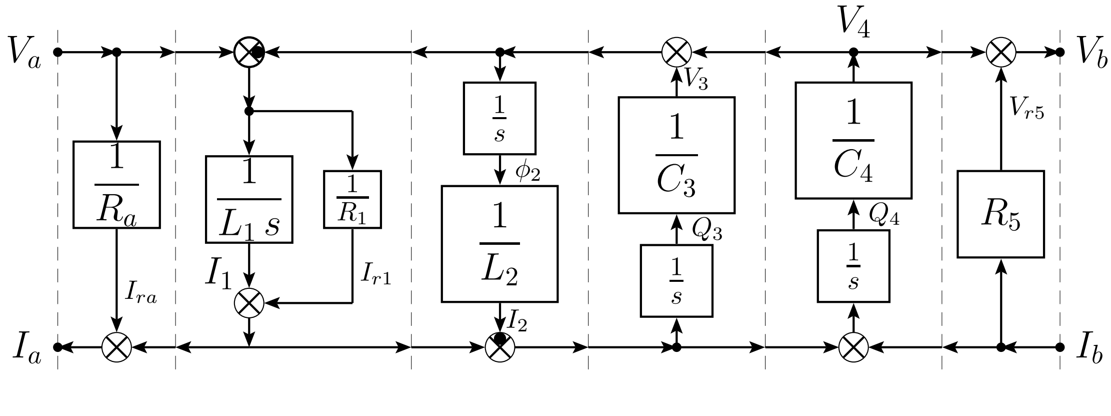
The corresponding POG block scheme is:



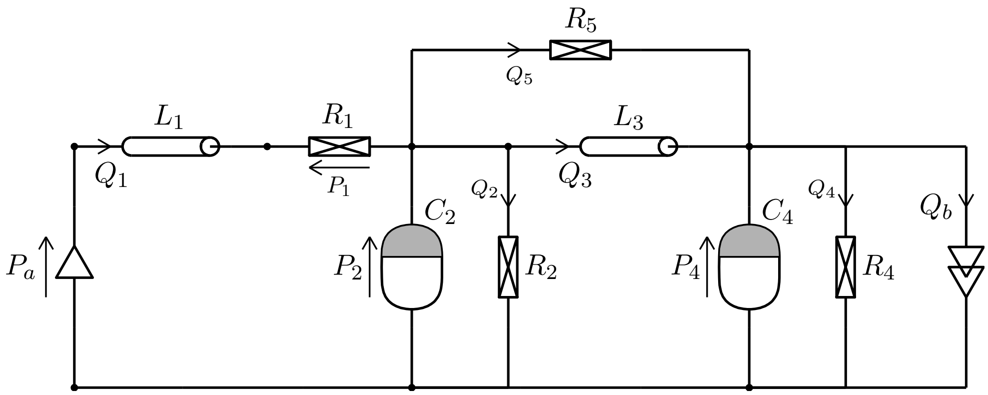
- Consider the following electric circuit:



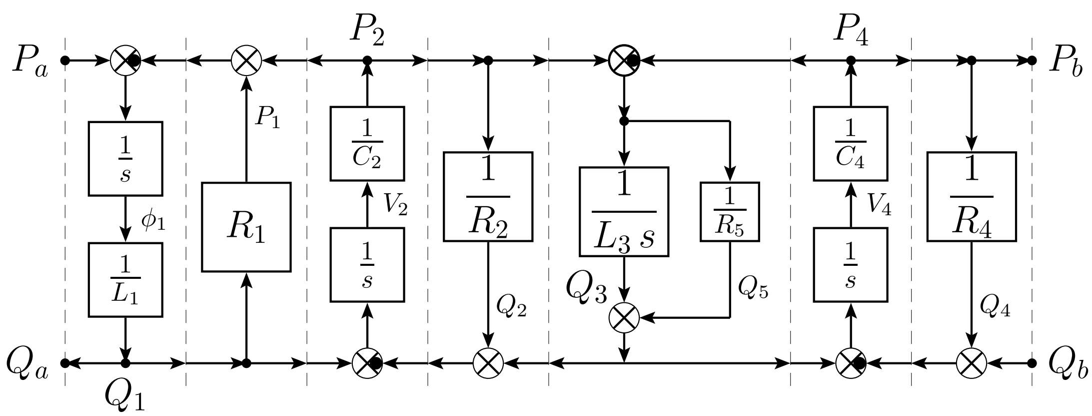
The corresponding POG block scheme is:



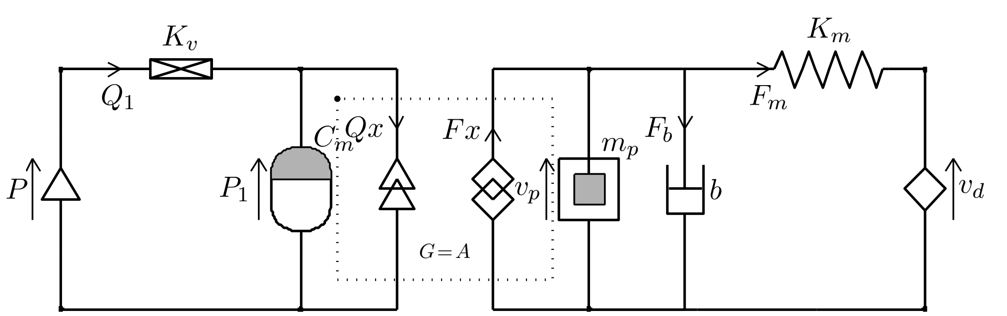
- Consider the following hydraulic circuit:



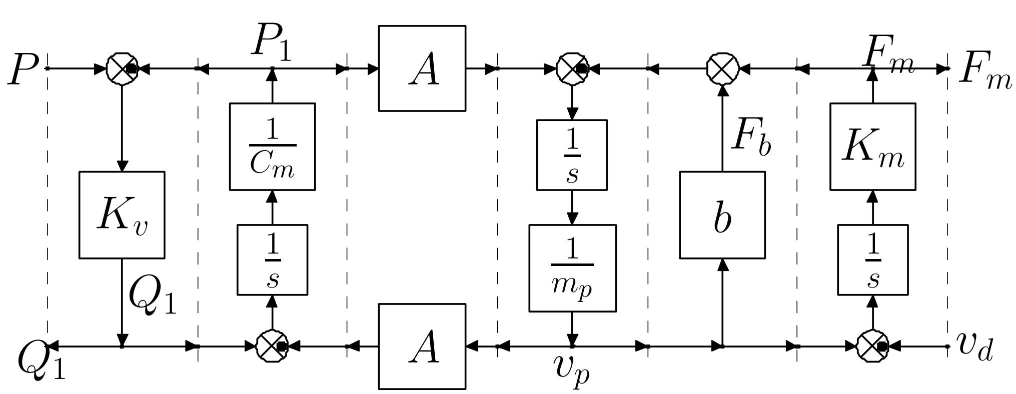
The corresponding POG block scheme is:



- Consider the following friction system:

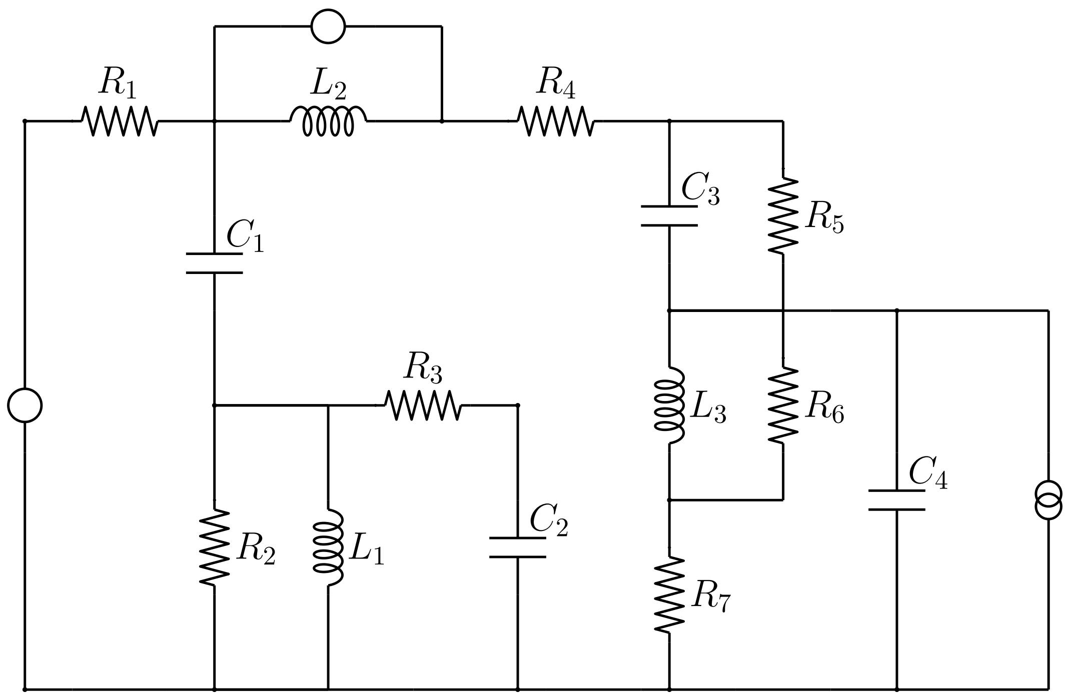


The corresponding POG block scheme is:



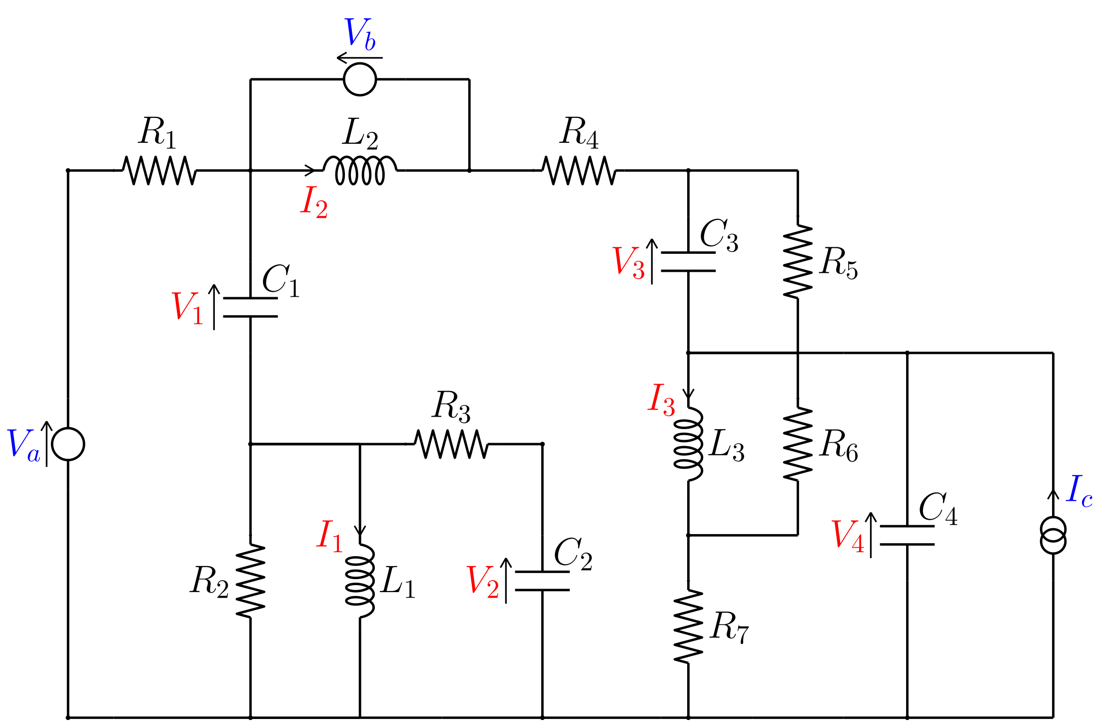
POG model of a large dynamic system

- Example. Consider the following physical scheme of a large electric circuit:

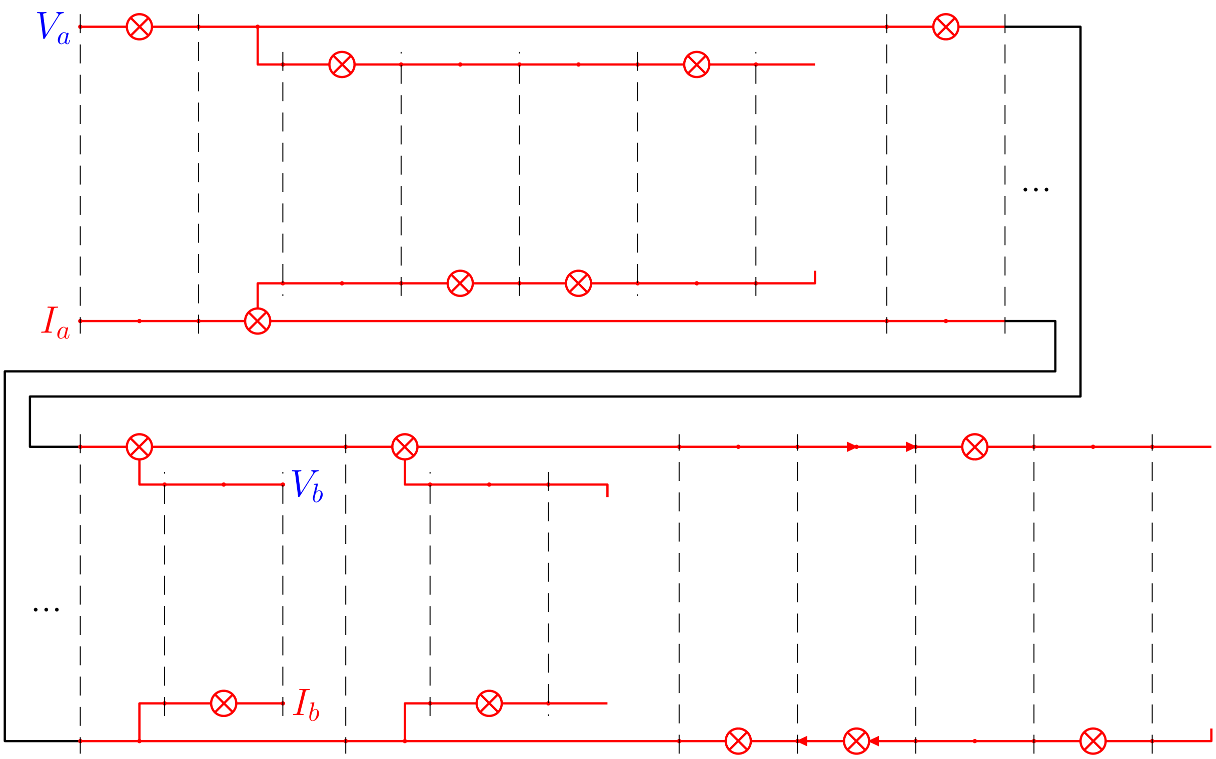


The POG scheme can be obtained by following the steps below.

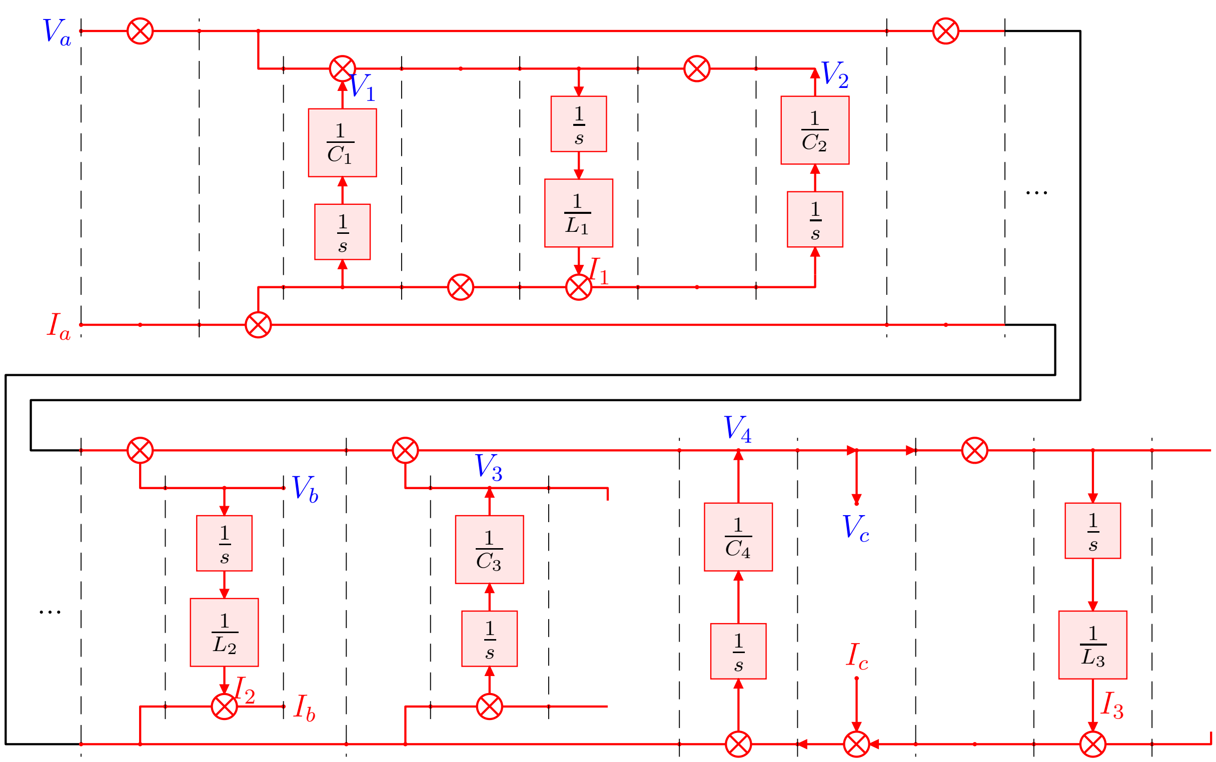
1) Choose the positive directions of all the state variables $(I_1, I_2, I_3, V_1, V_2, V_3, V_4)$ and all the input variables (V_a, V_b, I_c):



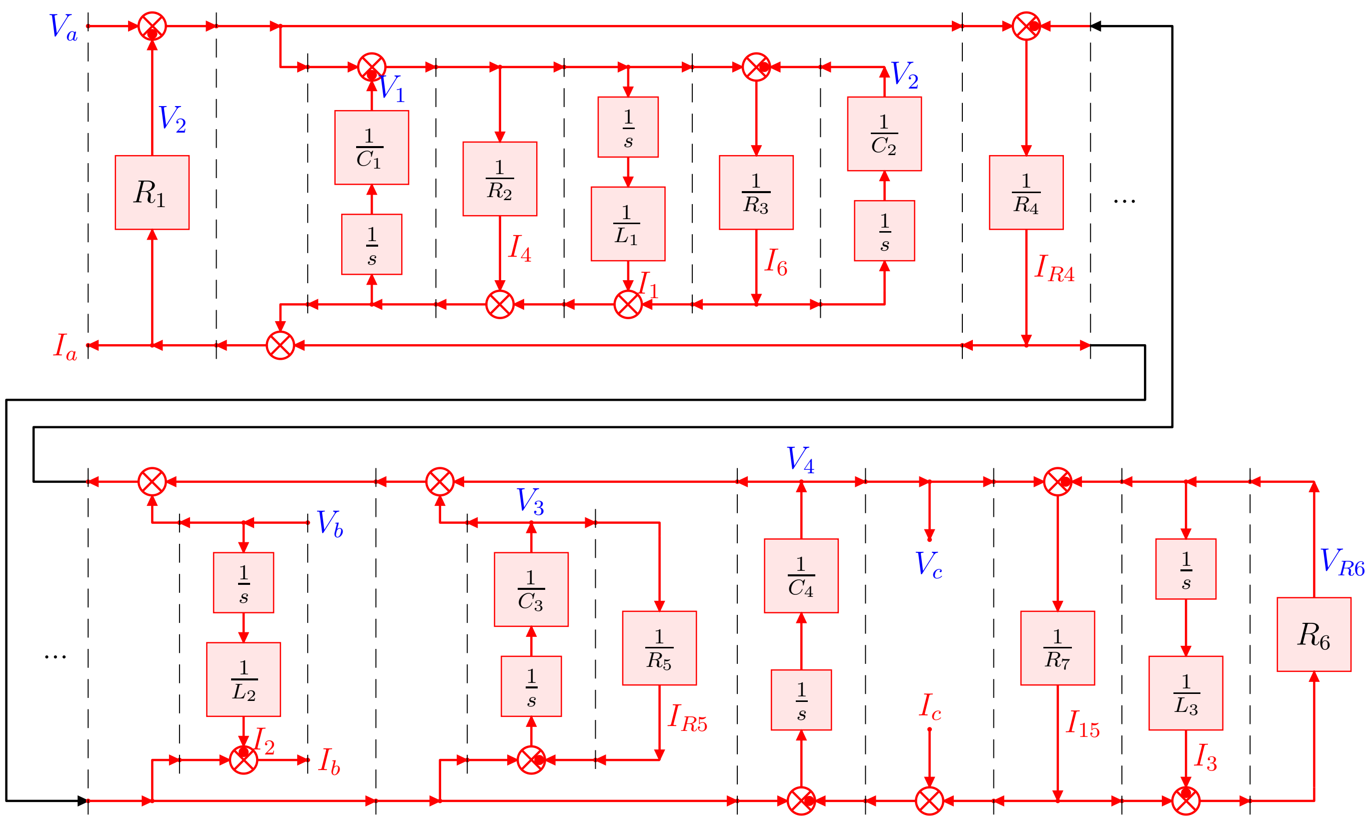
2) Choose an input port (in this case V_a) and draw the series/parallel structure of the corresponding POG block scheme:



3) Add to the POG block scheme the graphical representations of all the dynamic blocks:



3) Add to the POG block scheme the graphical representations of all the dissipative blocks:

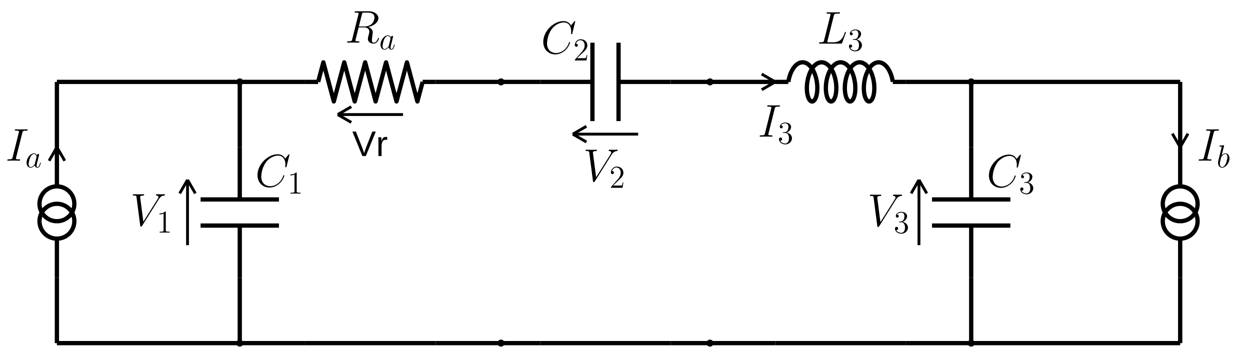


POG modeling of physical systems

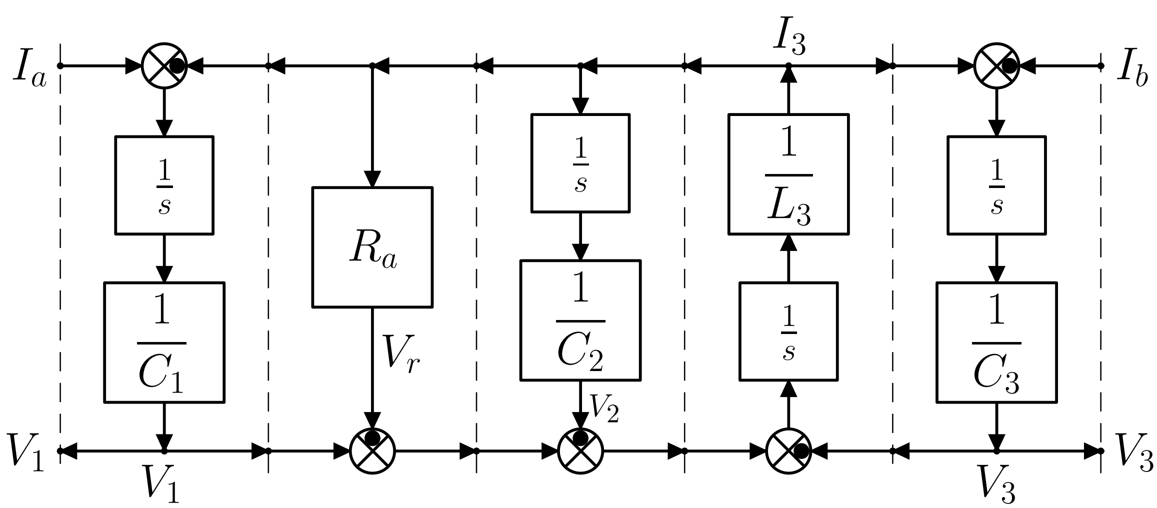
There are two different ways of modeling physical systems using POG.

A) POG modeling from left to right.

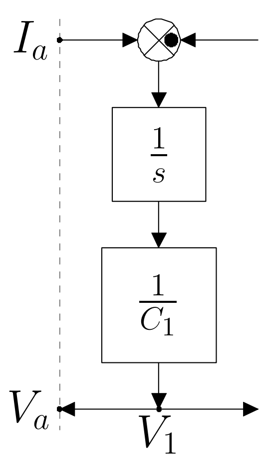
- Let us consider the following electric circuit:



- The corresponding POG block scheme is:



- This POG scheme can be obtained modeling the system from left ro right.

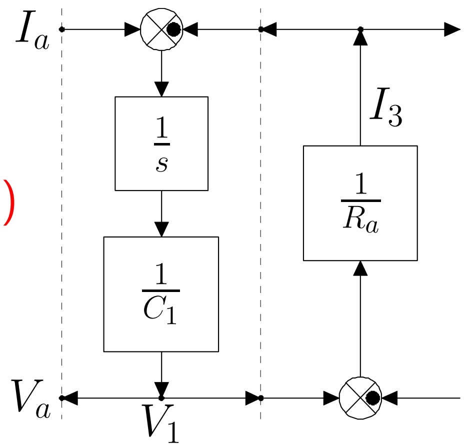


The input current I_a acts on a capacitor (Effort Block) in parallel connection, and therefore there is only one way of modeling the capacitor.

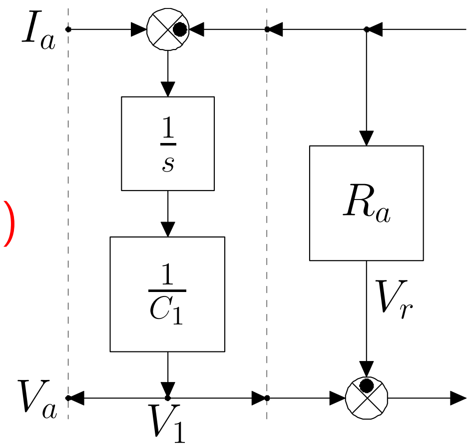
- The next element is the resistor R_a in series with an input voltage V_1 on the left. In this case the resistor can be modeled using two POG schemes.

- The two possible POG schemes are:

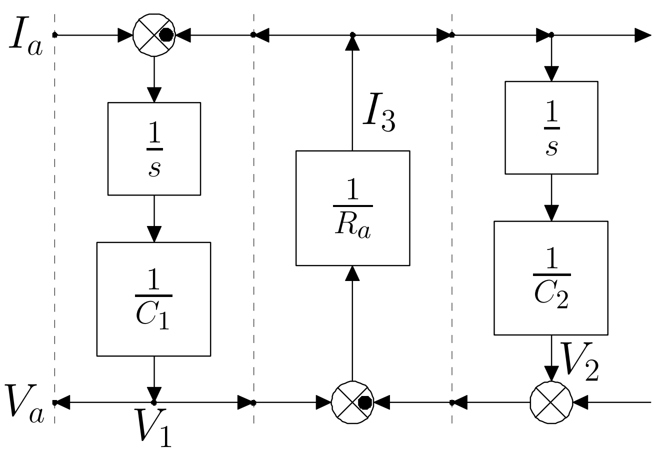
(1)



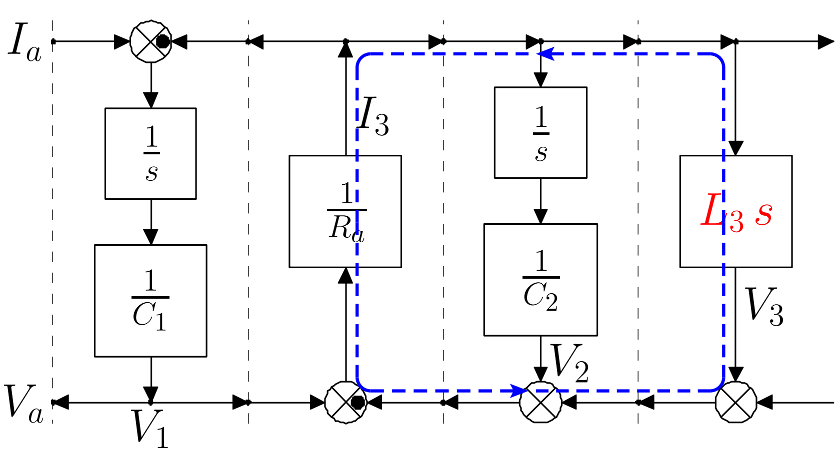
(2)



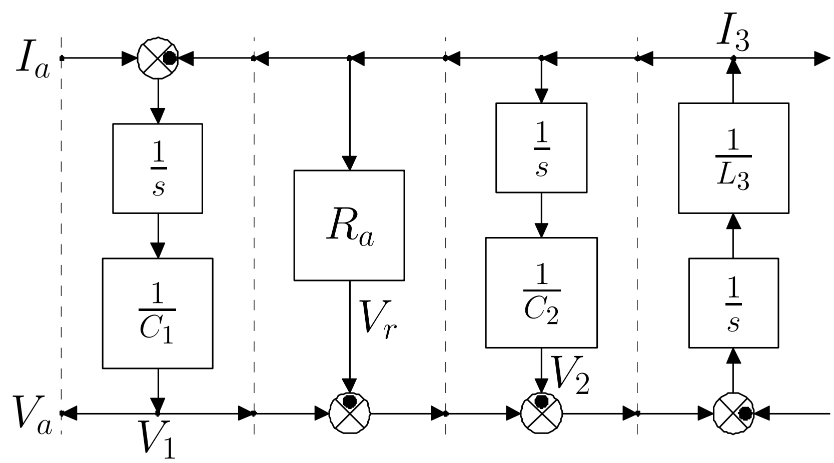
- Let us consider the solution (1). The next element to be modeled is the capacitor C_2 in series with the input current I_3 on the left. In this case the capacitor C_2 can be modeled only as follows:



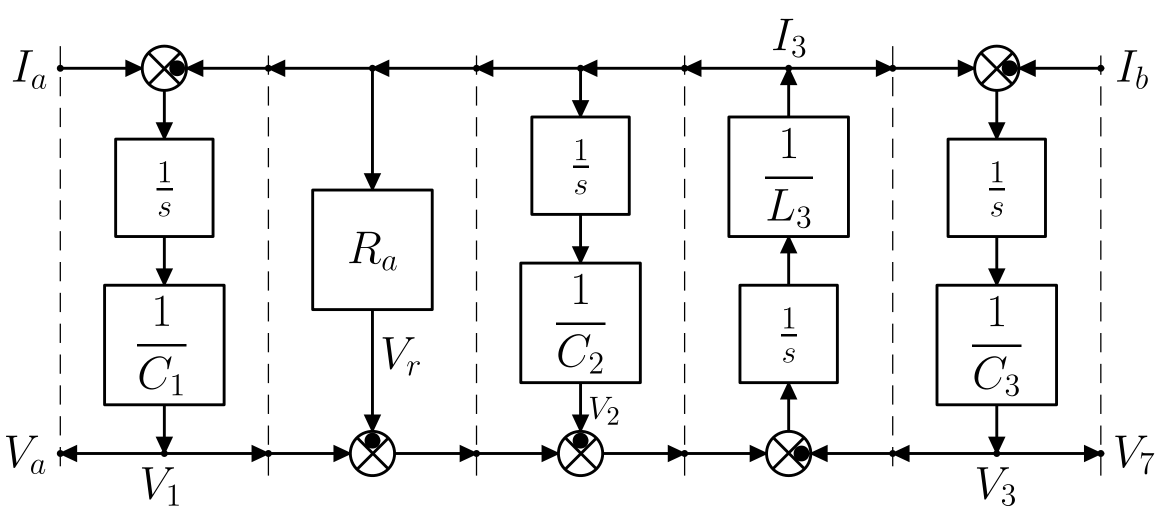
- The next element to be modeled is the inductor L_3 in series. Since the input current I_3 enters on the left, the inductor L_3 can be modeled only using the derivative causality:



- In this case the causality problem can be solved inverting the blue dashed path shown in the picture.
- After inverting the path, the POG scheme has the following structure:



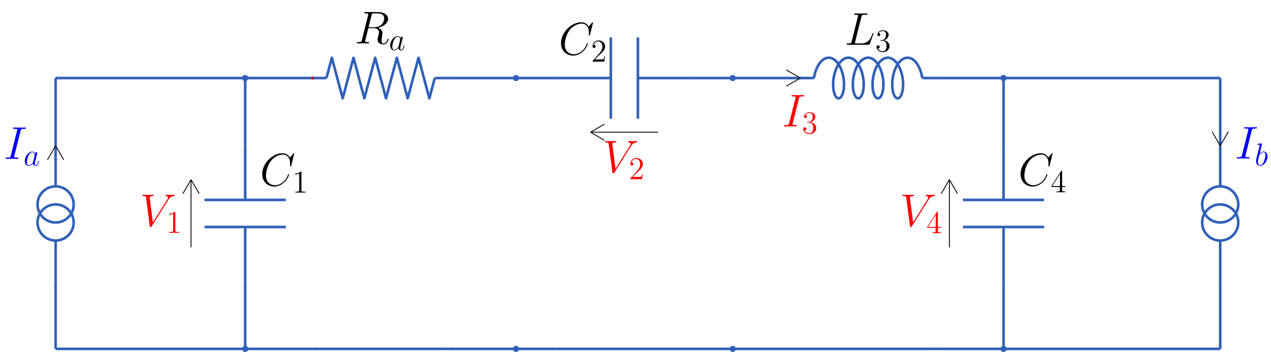
- This is exactly the POG scheme that coluld have been obtained choosing the solution (2) when the resistor R_a was modeled.
- The last two elements to be modeled are capacitor C_3 and the current generator I_b. The only possible POG solution is the following:



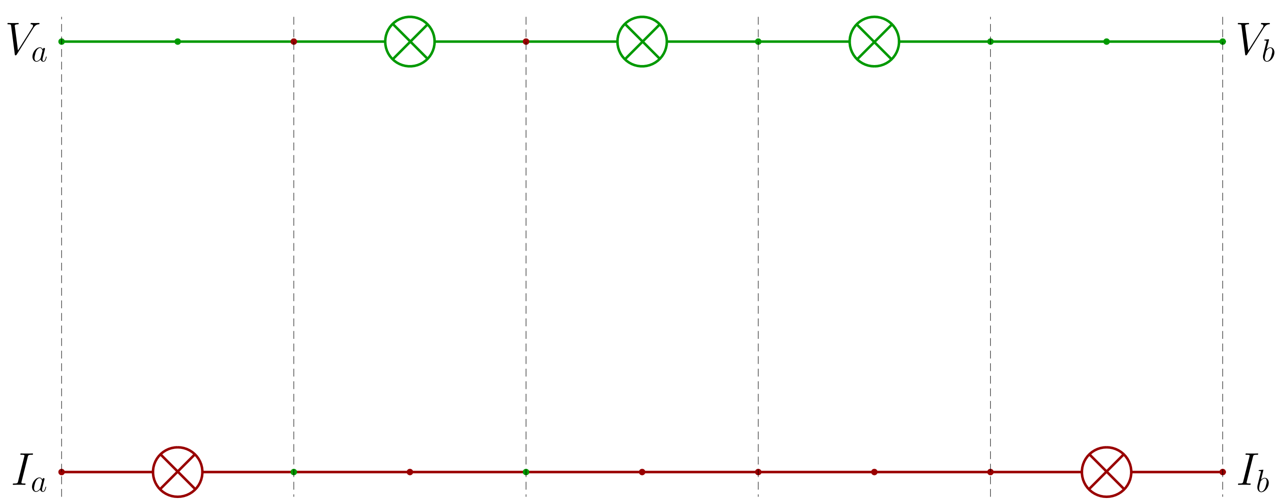
- In fact, the capacitor C_3 is connected in parallel and can be modeled only using the POG structure shown in the picture.

B) POG Direct modeling.

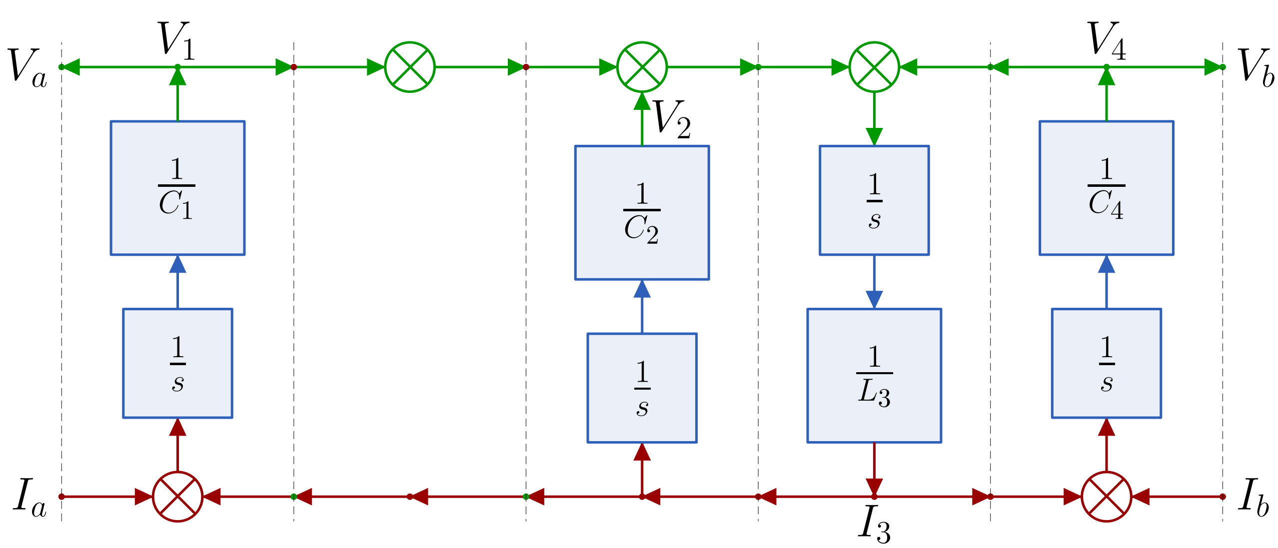
a) Choose the positive directions of the state and input variables of the considered physical system:



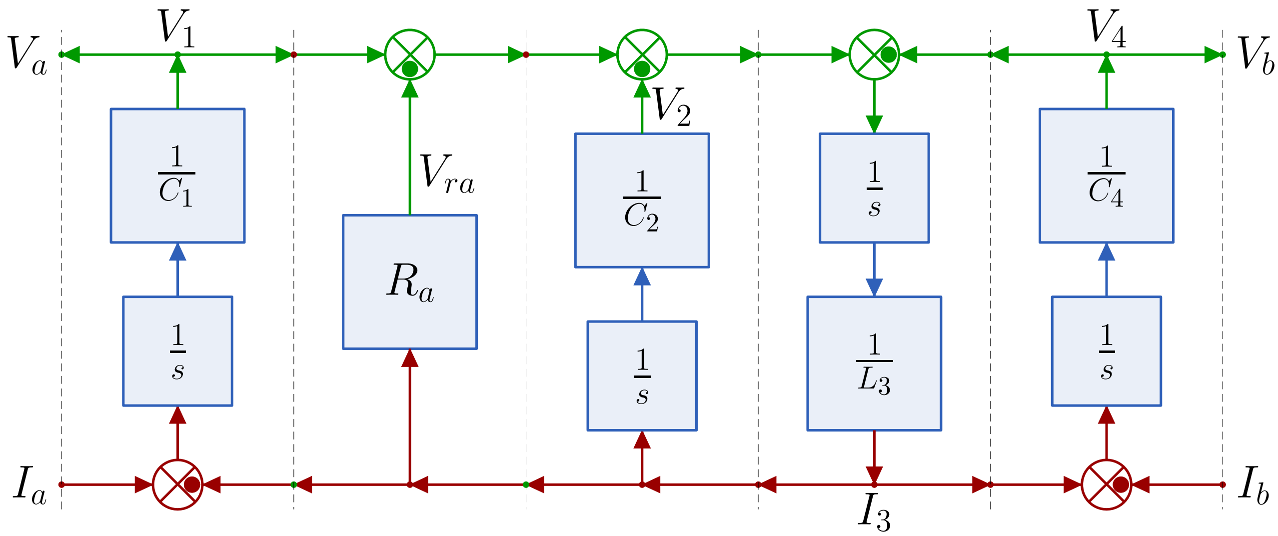
b) Draw the series/parallel structure of the POG block scheme:



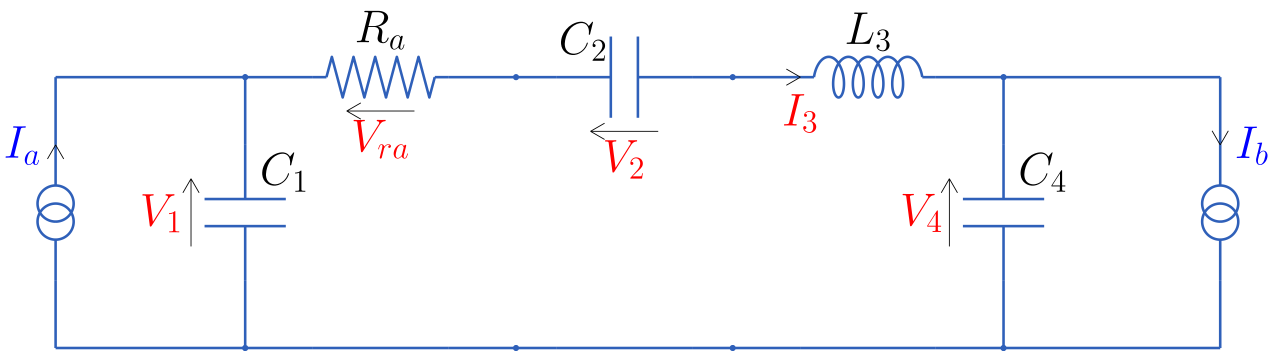
c) Add the dynamic blocks to the POG scheme:



d) Add the dissipative blocks and the summation signs to the POG scheme:

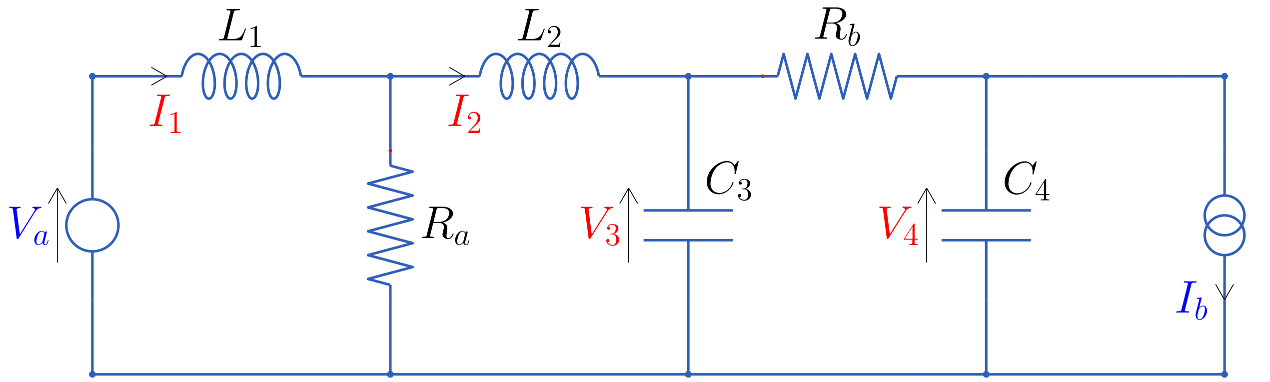


e) Check the signs by comparing with the considered physical system:

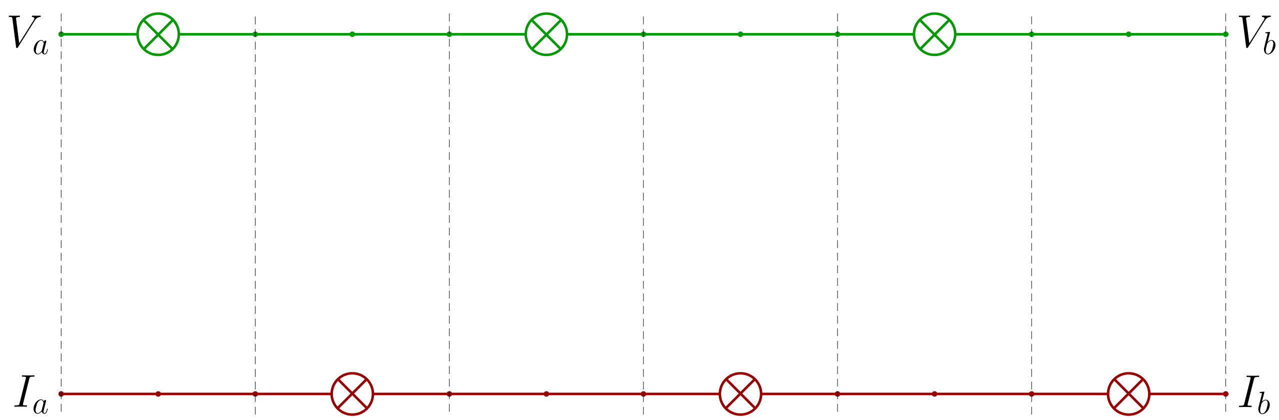


POG direct modeling: simple examples

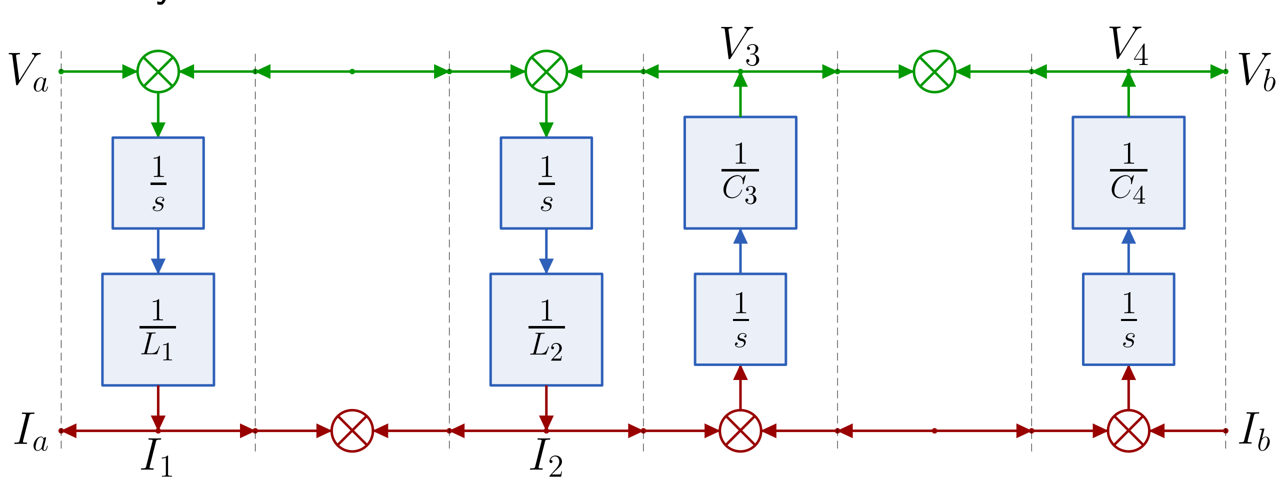
1.a) Choose the positive directions of the state and input variables of the considered physical system:



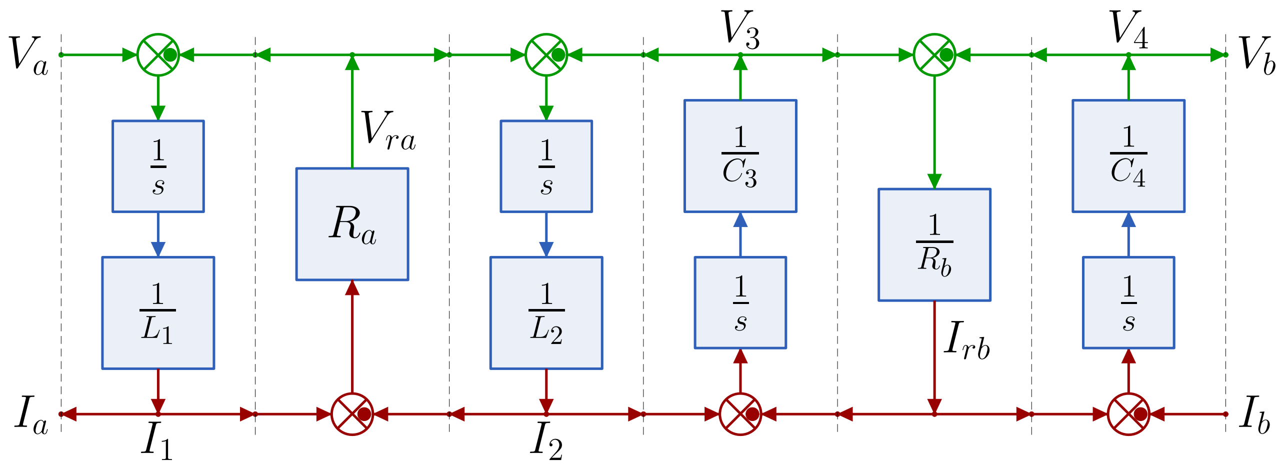
1.b) Draw the series/parallel structure of the POG block scheme:



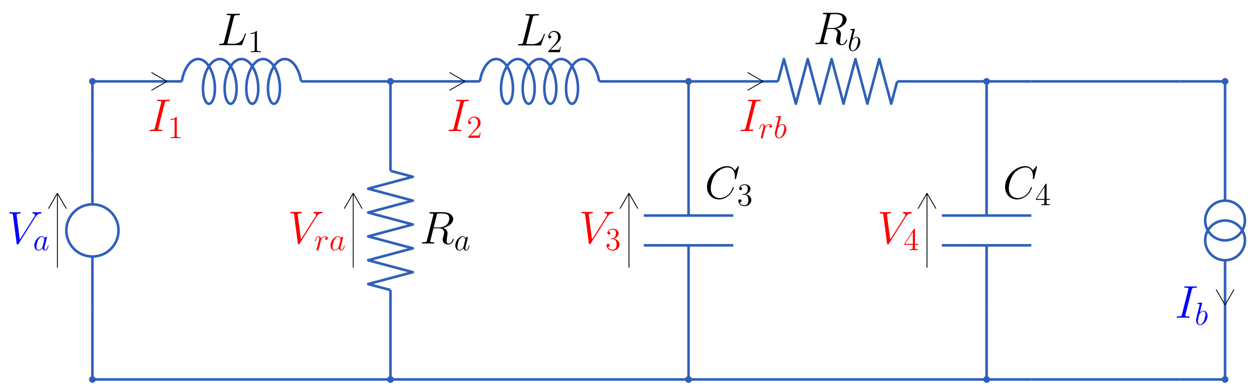
1.c) Add the dynamic blocks to the POG scheme:



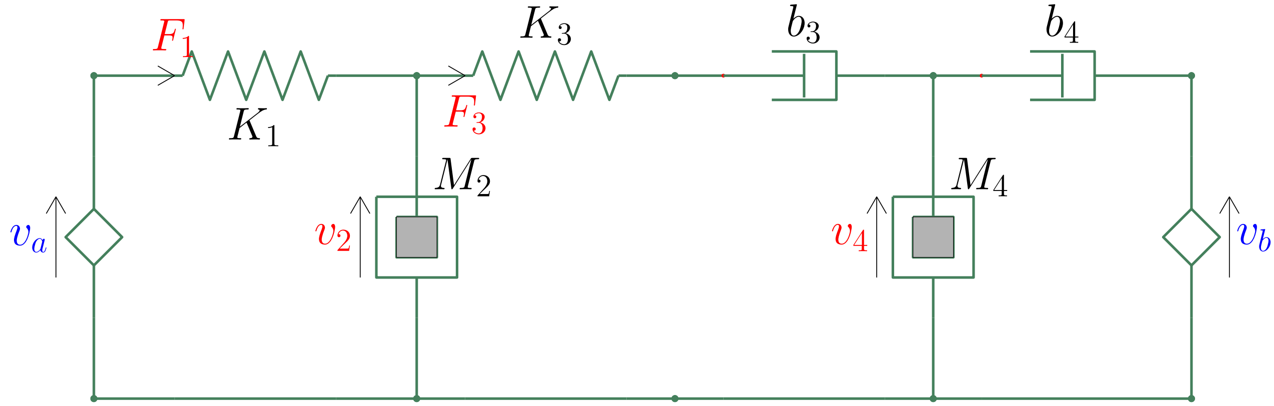
1.d) Add the dissipative blocks and the summation signs to the POG scheme:



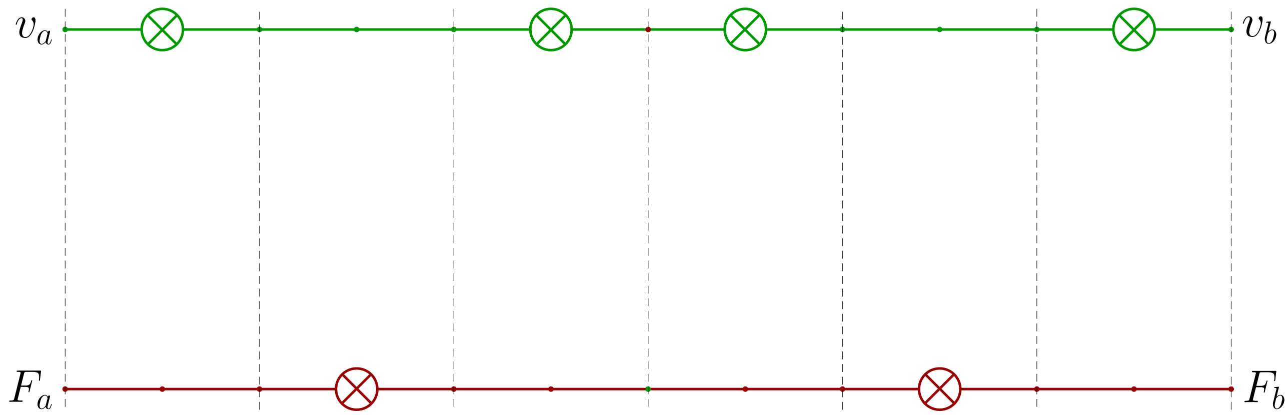
1.e) Check the signs by comparing with the considered physical system:



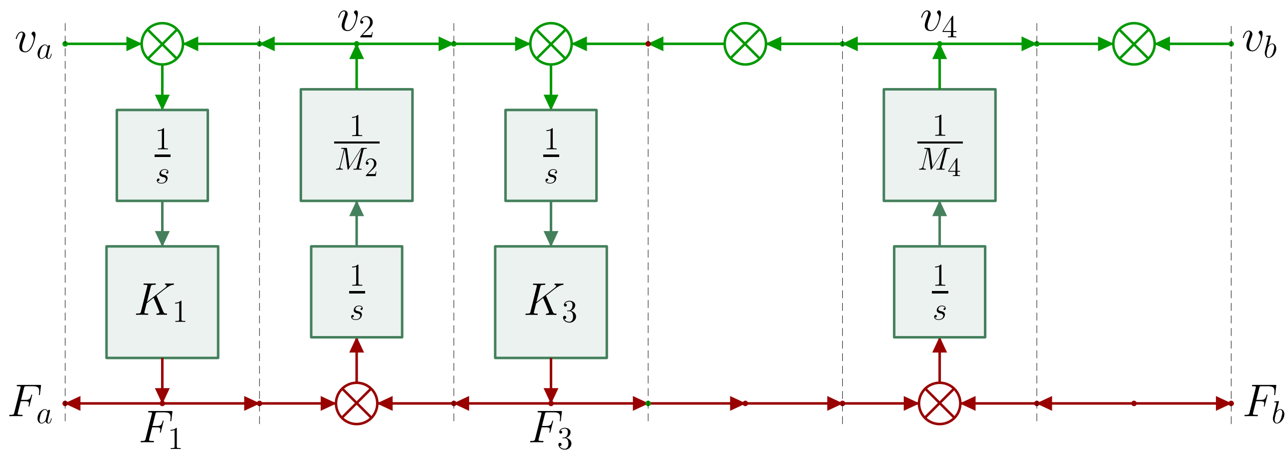
2.a) Choose the positive directions of the state and input variables of the considered physical system:



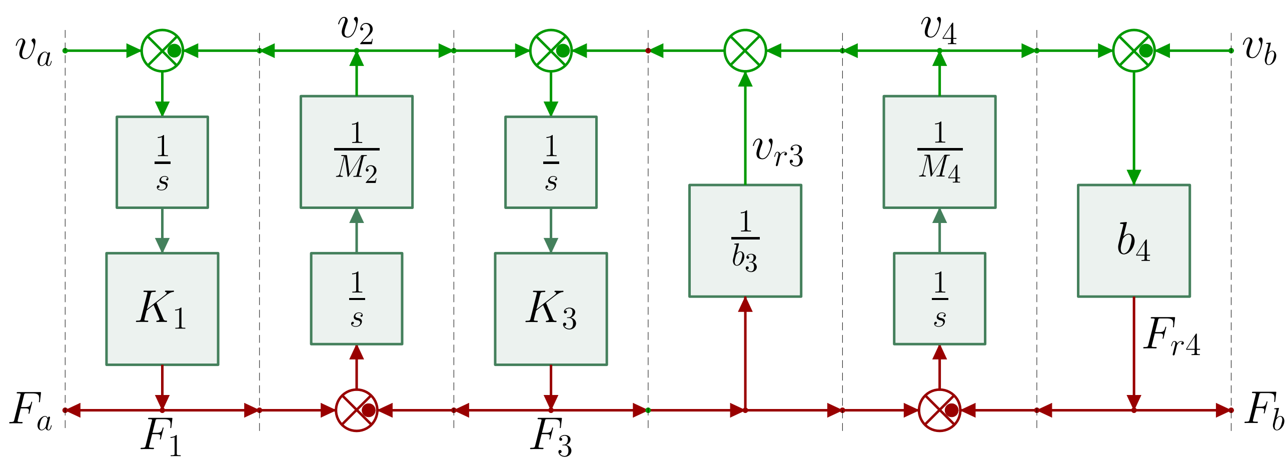
2.b) Draw the series/parallel structure of the POG block scheme:



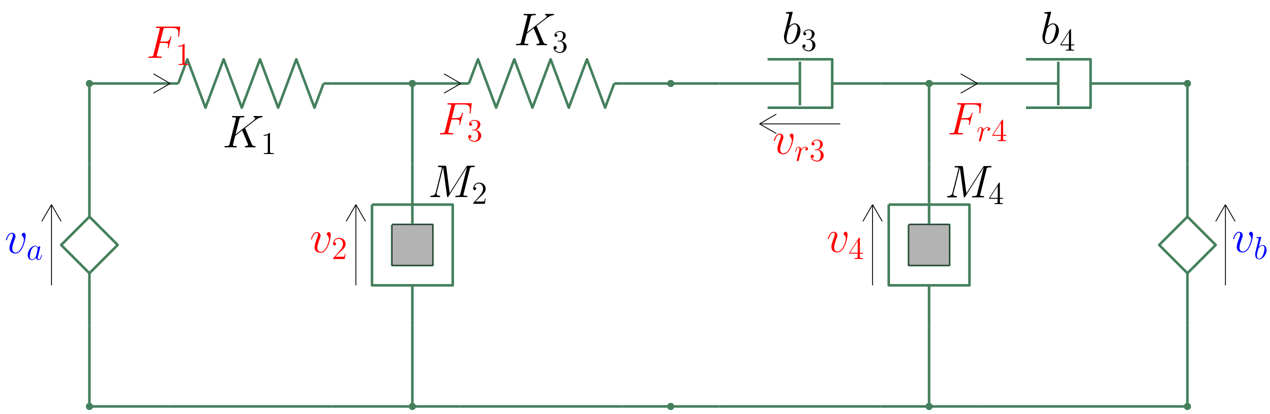
2.c) Add the dynamic blocks to the POG scheme:



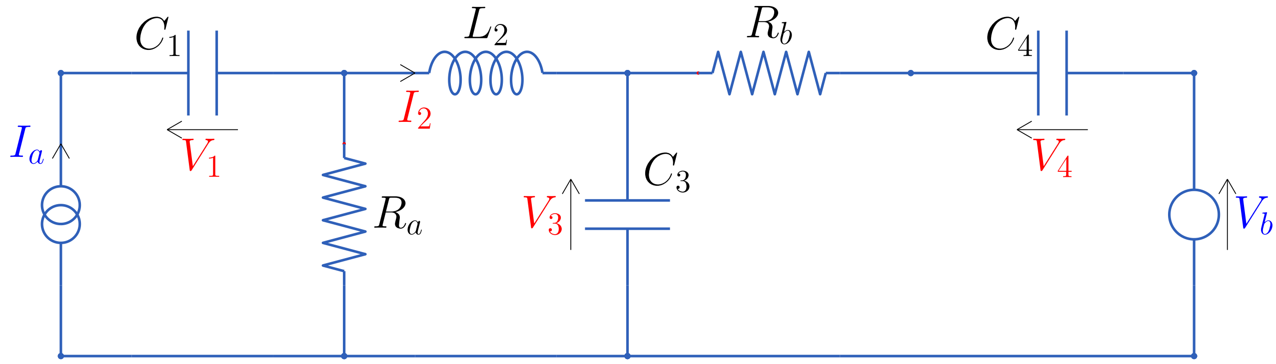
2.d) Add the dissipative blocks and the summation signs to the POG scheme:



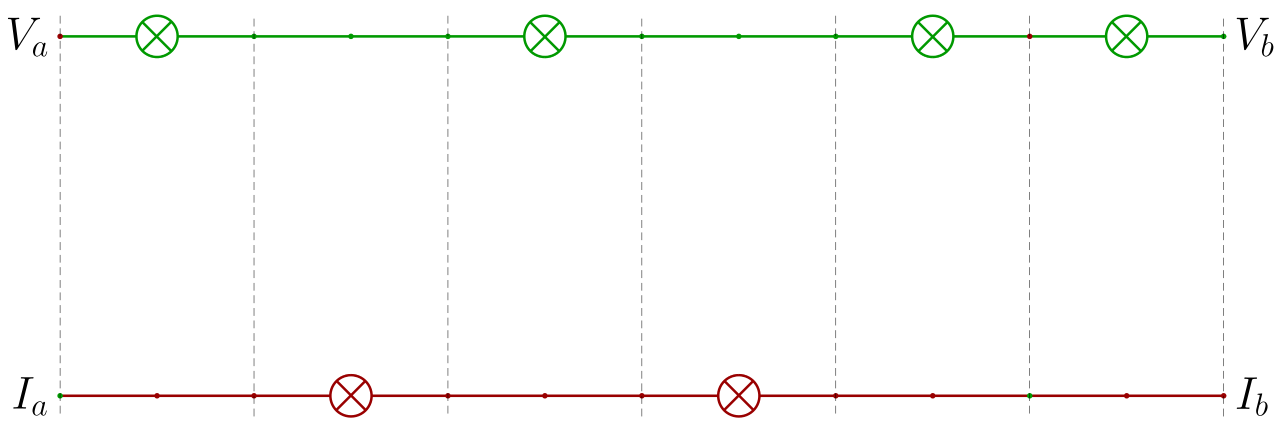
2.e) Check the signs by comparing with the considered physical system:



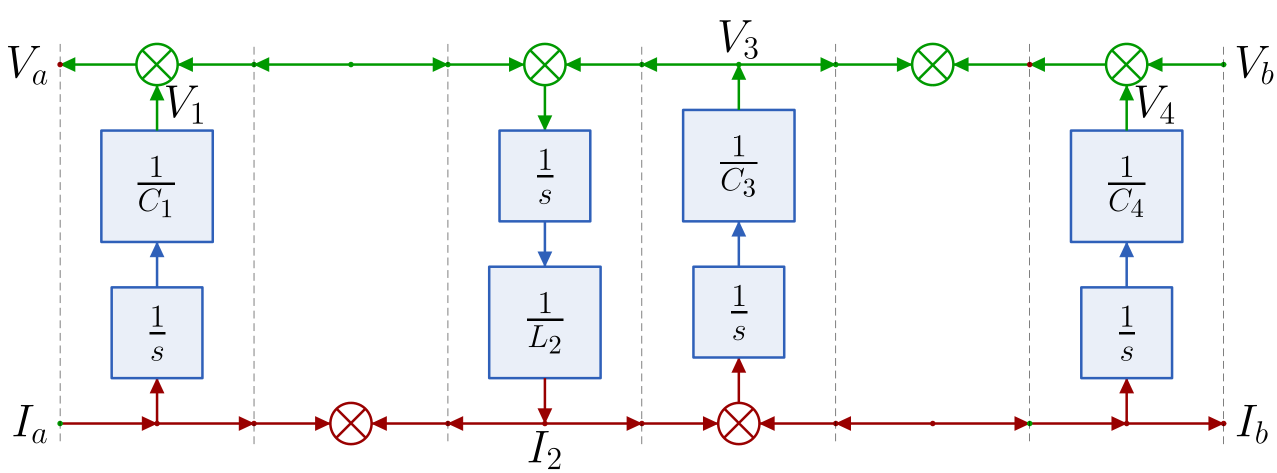
3.a) Choose the positive directions of the state and input variables of the considered physical system:



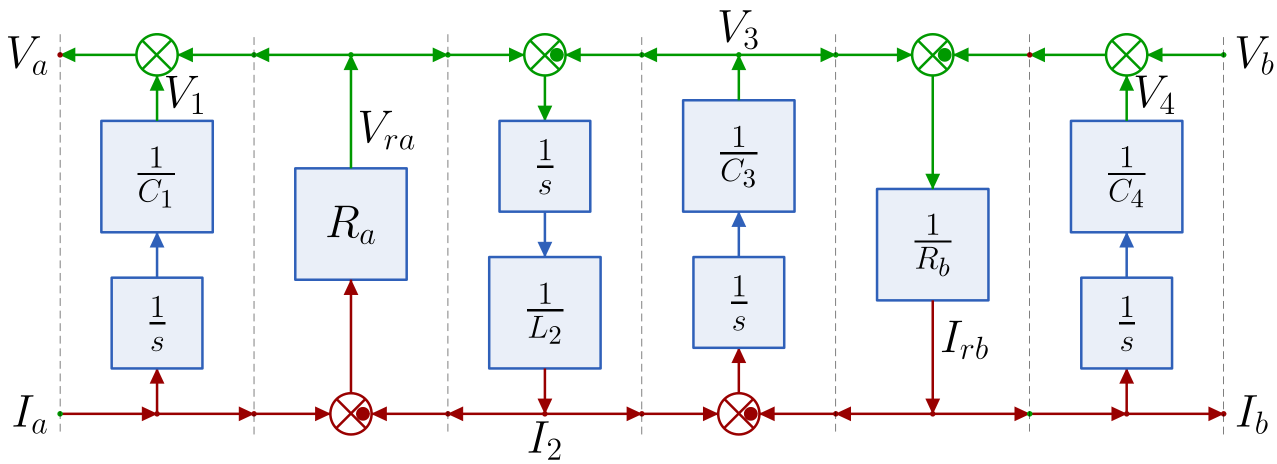
3.b) Draw the series/parallel structure of the POG block scheme:



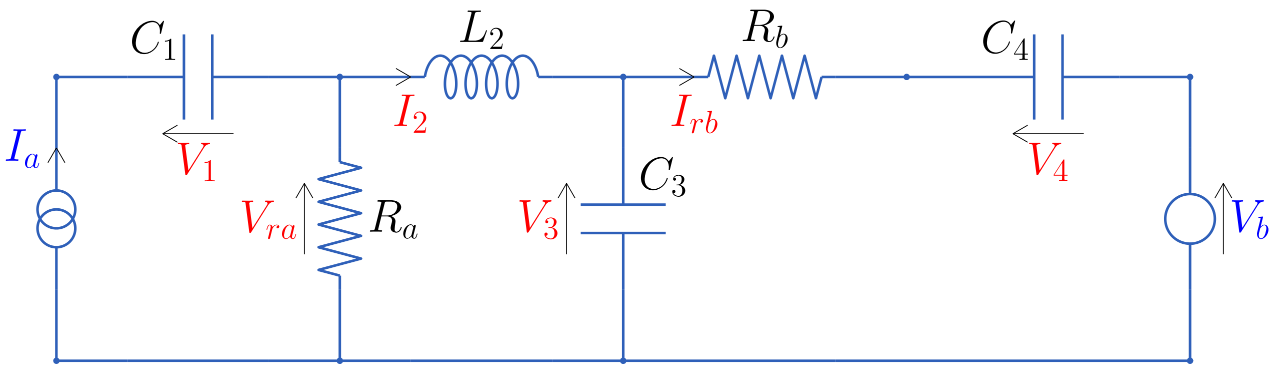
3.c) Add the dynamic blocks to the POG scheme:



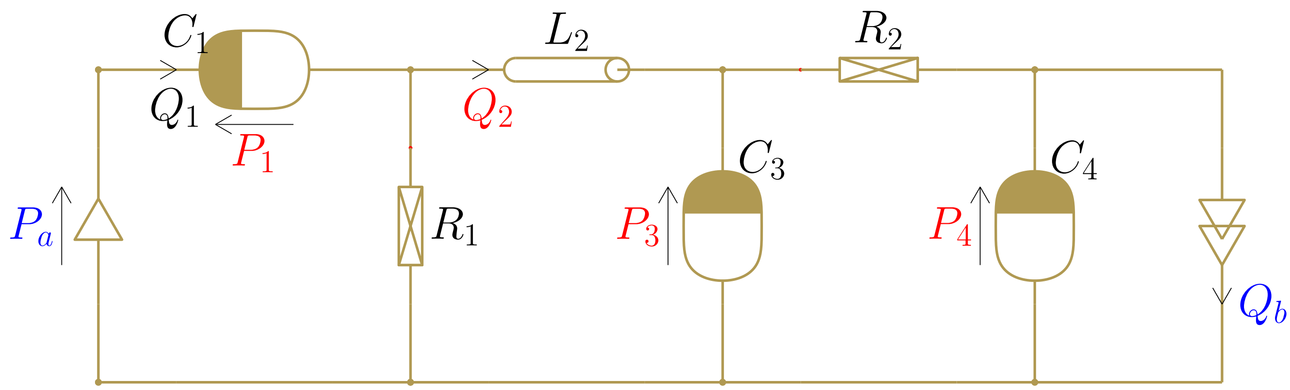
3.d) Add the dissipative blocks and the summation signs to the POG scheme:



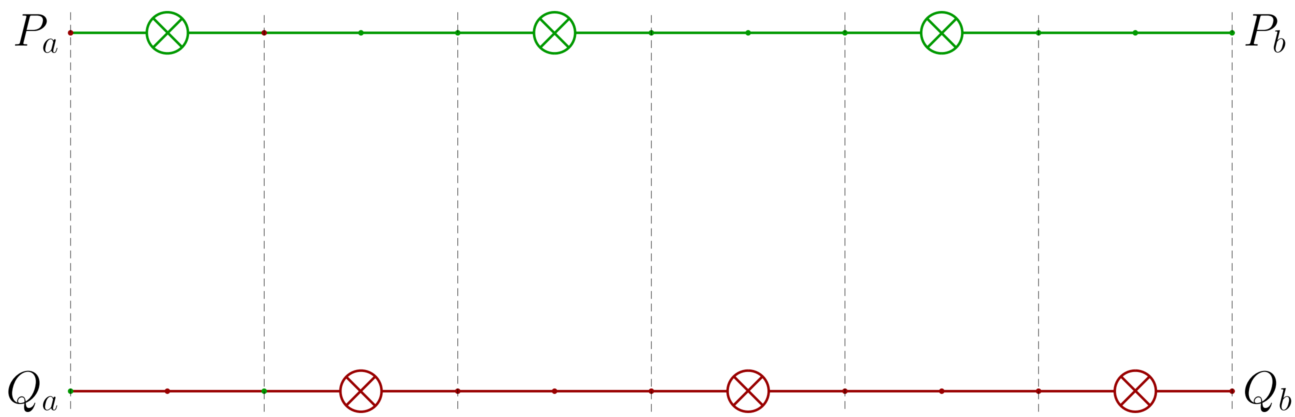
3.e) Check the signs by comparing with the considered physical system:



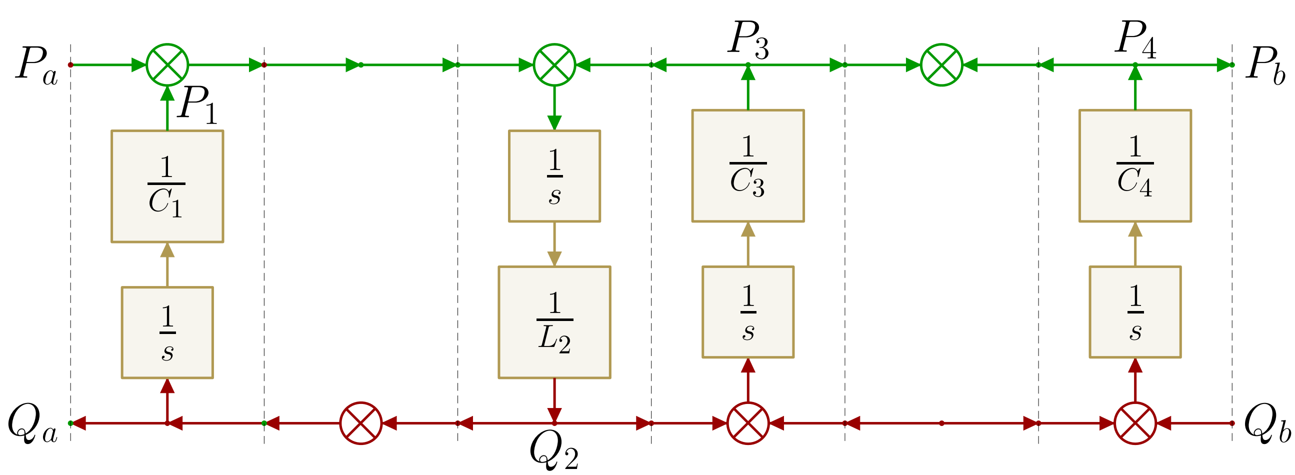
4.a) Choose the positive directions of the state and input variables of the considered physical system:



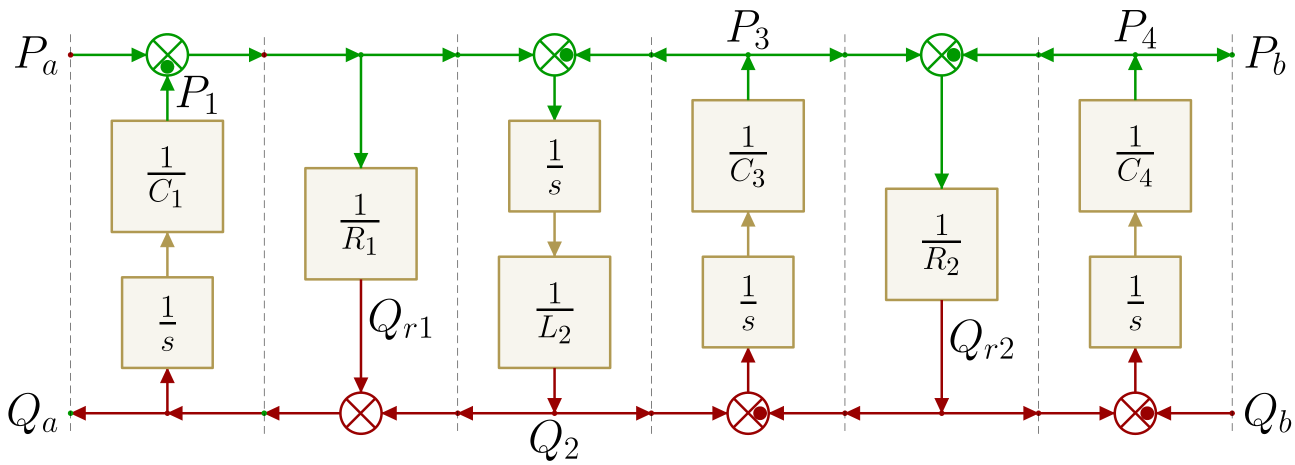
4.b) Draw the series/parallel structure of the POG block scheme:



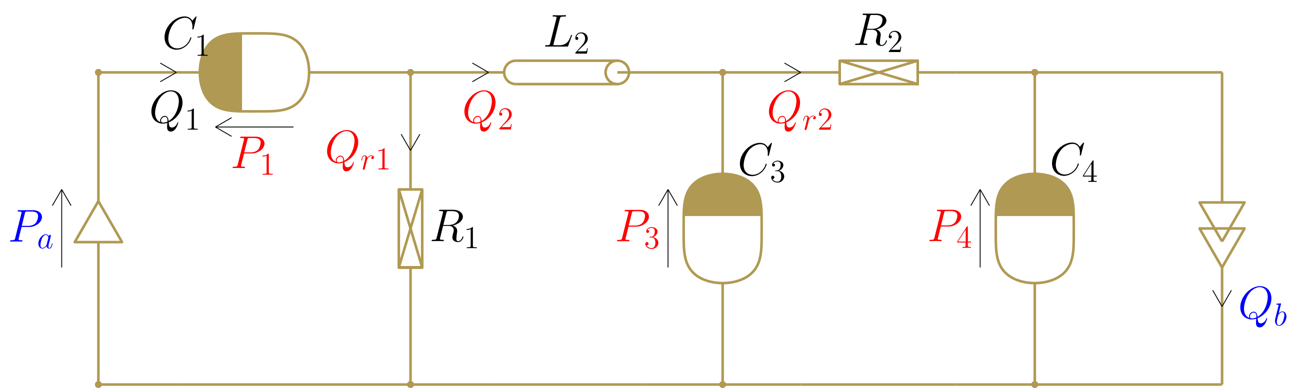
4.c) Add the dynamic blocks to the POG scheme:



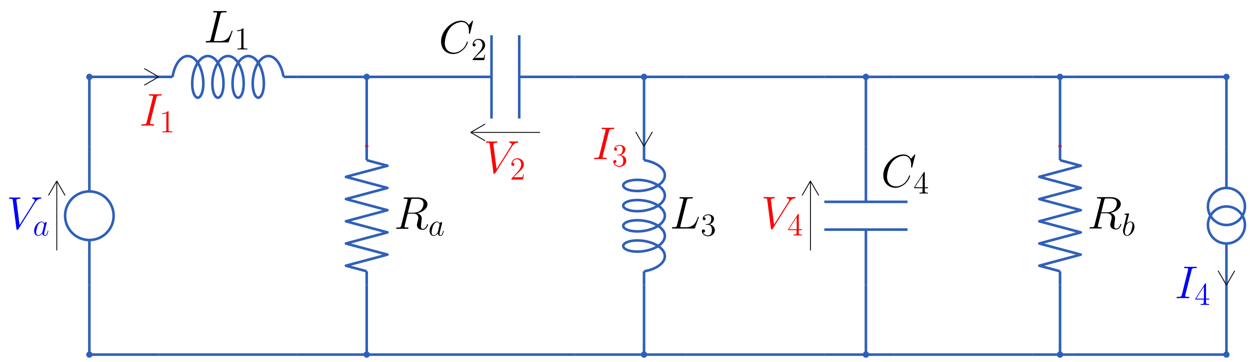
4.d) Add the dissipative blocks and the summation signs to the POG scheme:



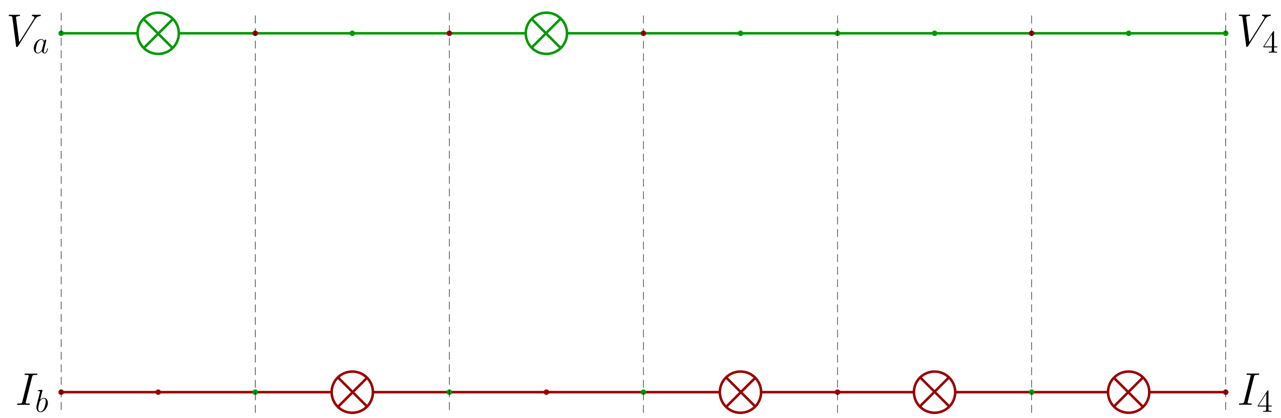
4.e) Check the signs by comparing with the considered physical system:



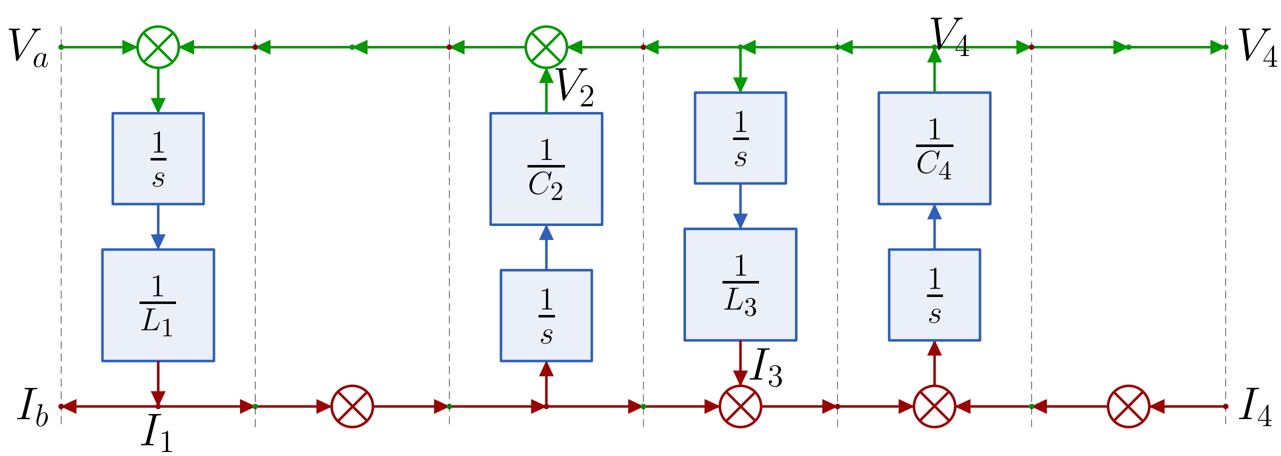
5.a) Choose the positive directions of the state and input variables of the considered physical system:



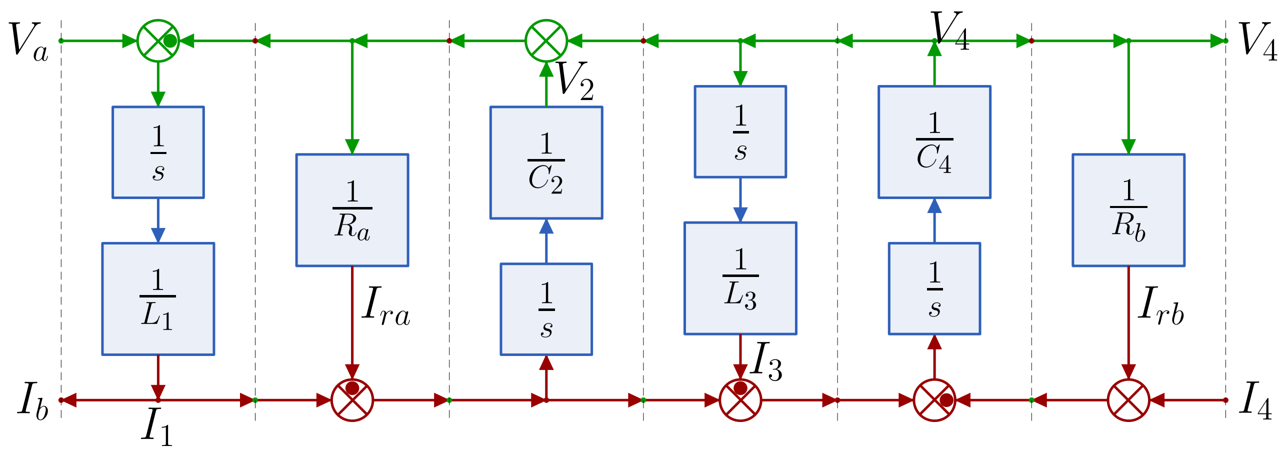
5.b) Draw the series/parallel structure of the POG block scheme:



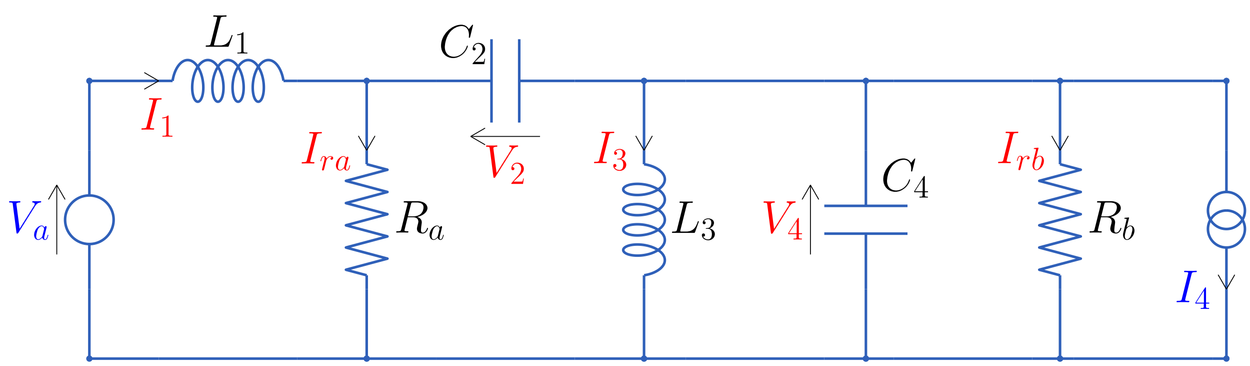
5.c) Add the dynamic blocks to the POG scheme:



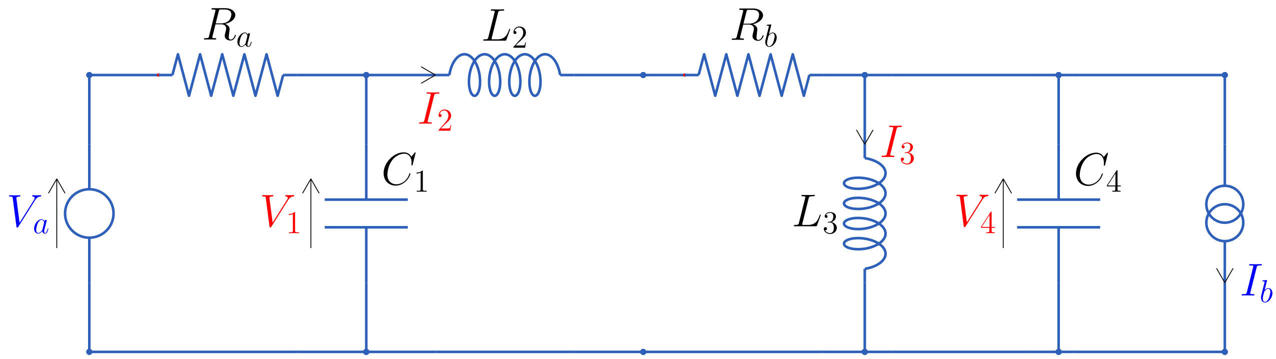
5.d) Add the dissipative blocks and the summation signs to the POG scheme:



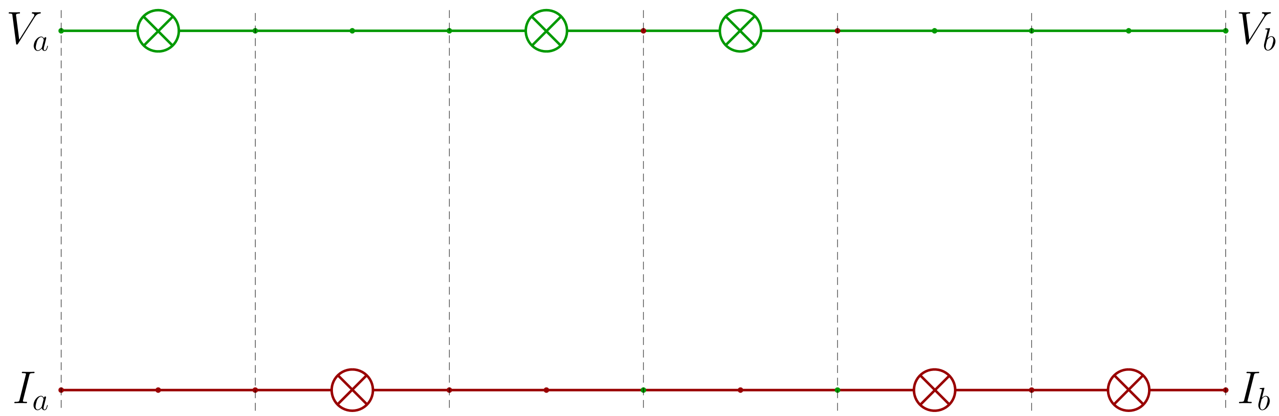
5.e) Check the signs by comparing with the considered physical system:



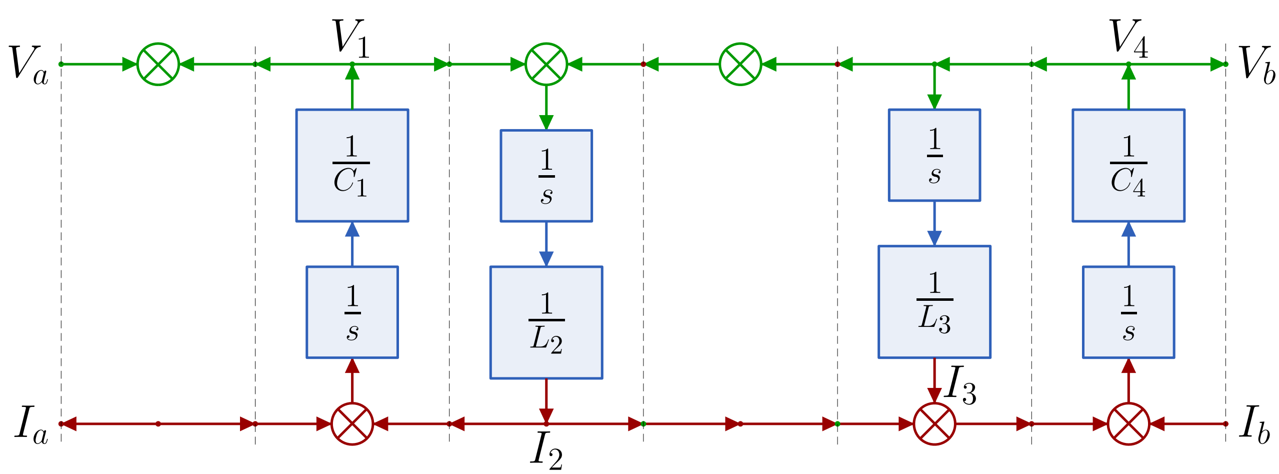
6.a) Choose the positive directions of the state and input variables of the considered physical system:



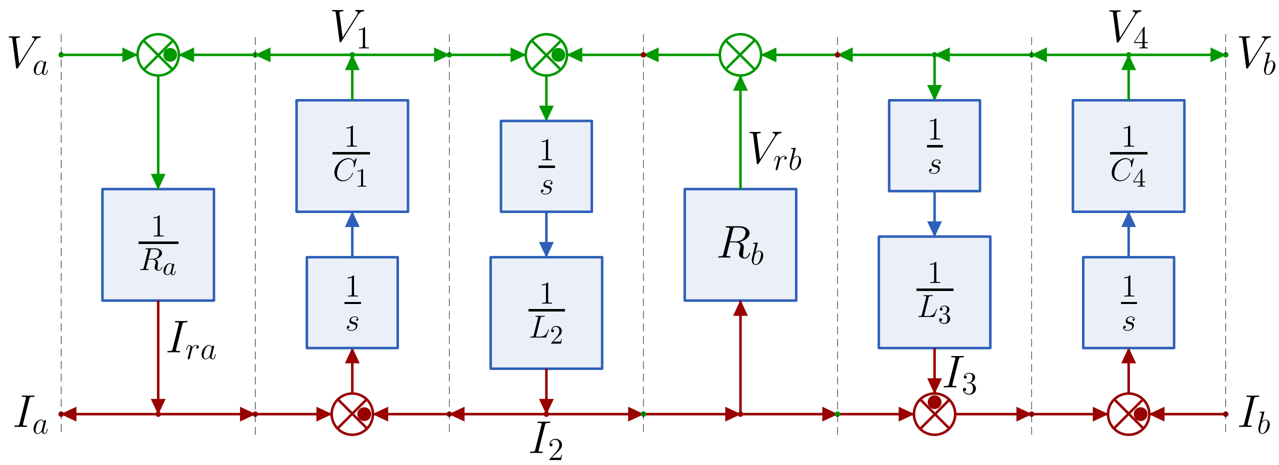
6.b) Draw the series/parallel structure of the POG block scheme:



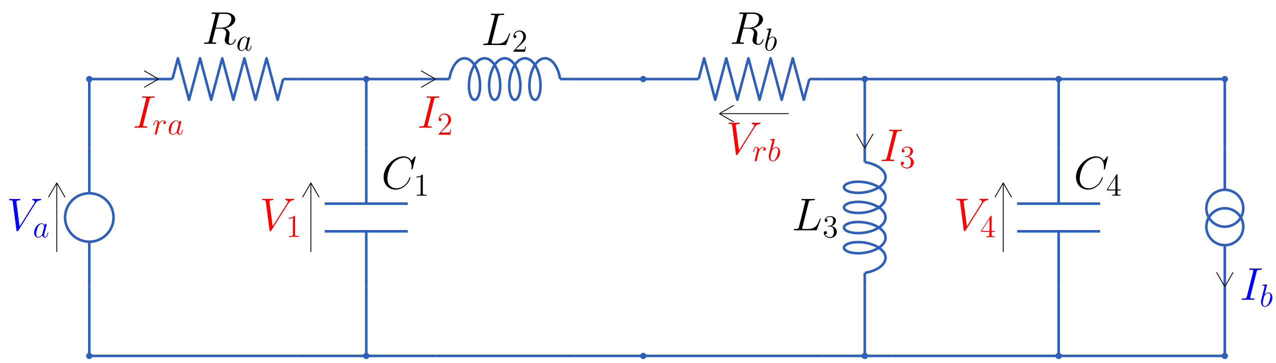
6.c) Add the dynamic blocks to the POG scheme:



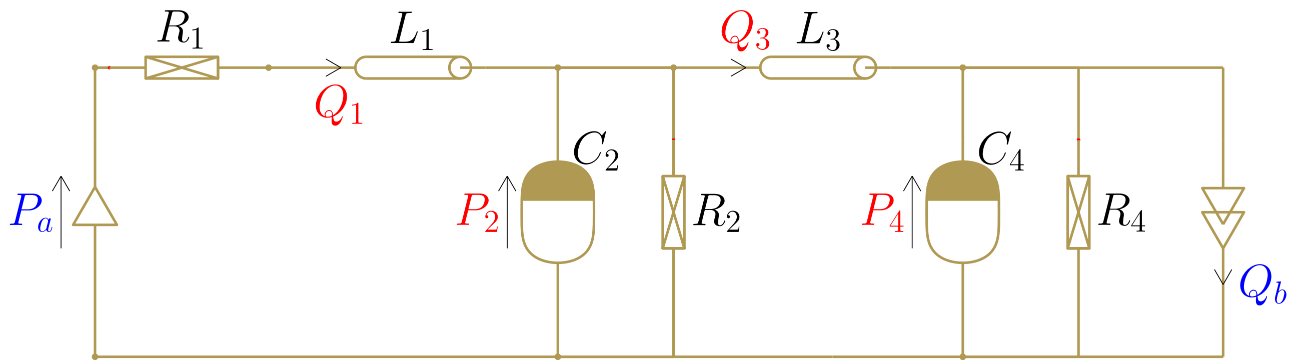
6.d) Add the dissipative blocks and the summation signs to the POG scheme:



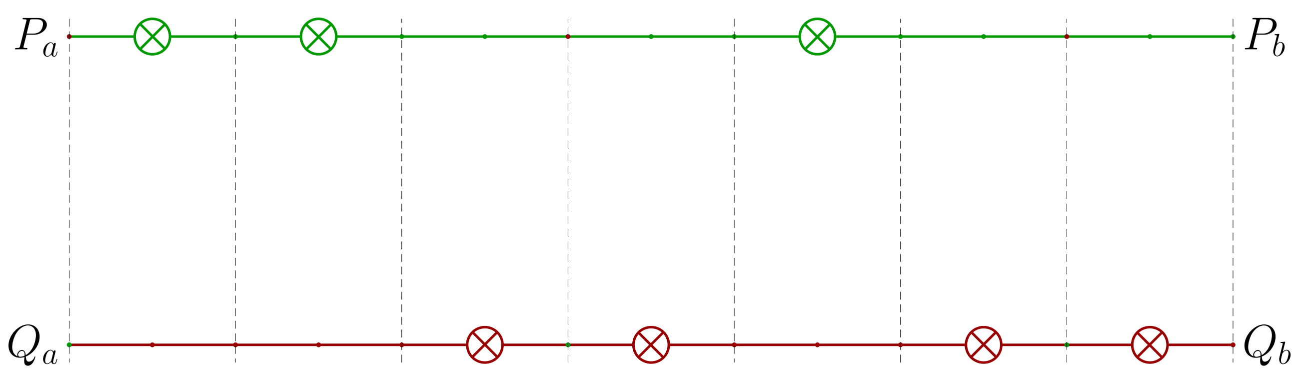
6.e) Check the signs by comparing with the considered physical system:



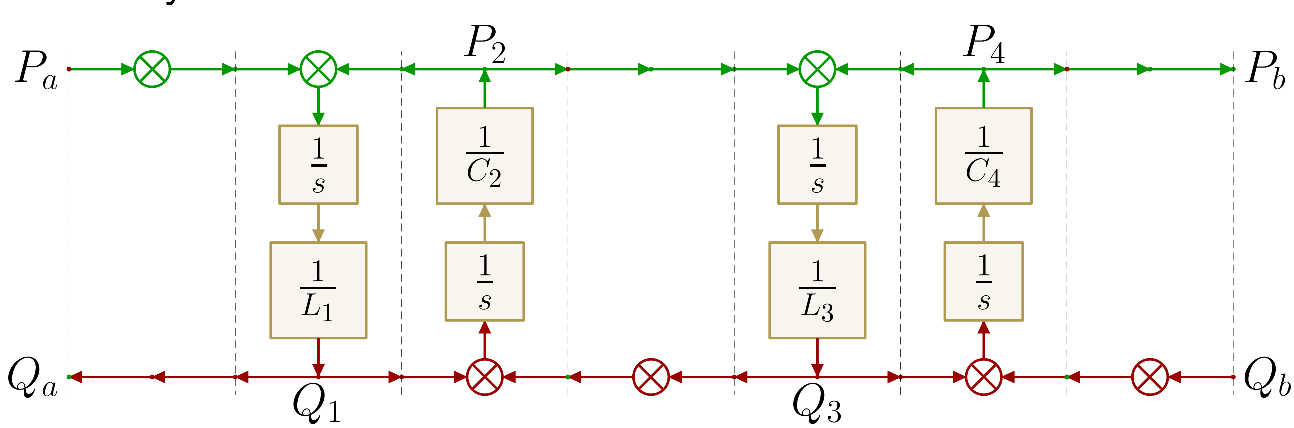
7.a) Choose the positive directions of the state and input variables of the considered physical system:



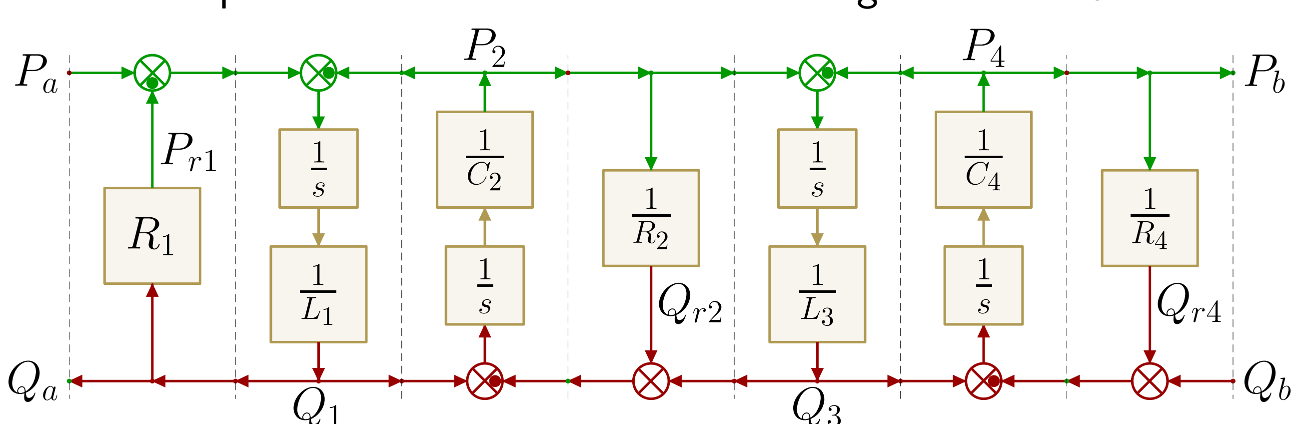
7.b) Draw the series/parallel structure of the POG block scheme:



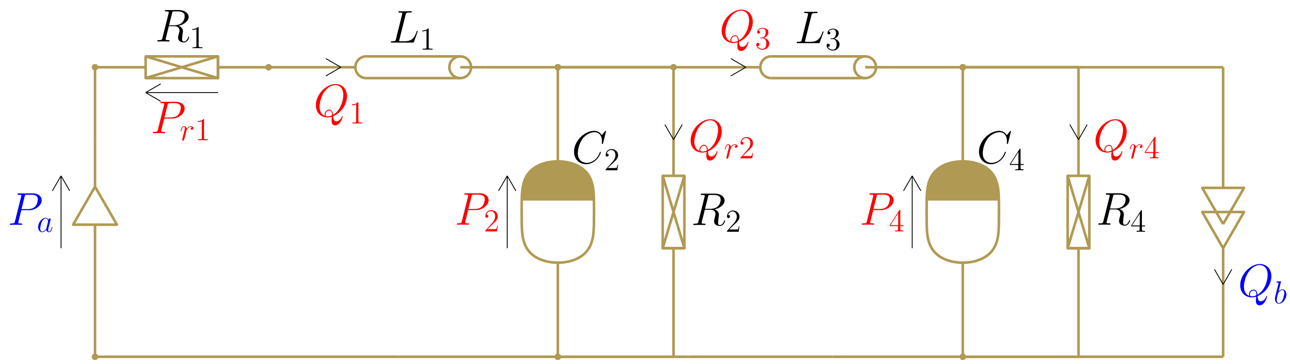
7.c) Add the dynamic blocks to the POG scheme:



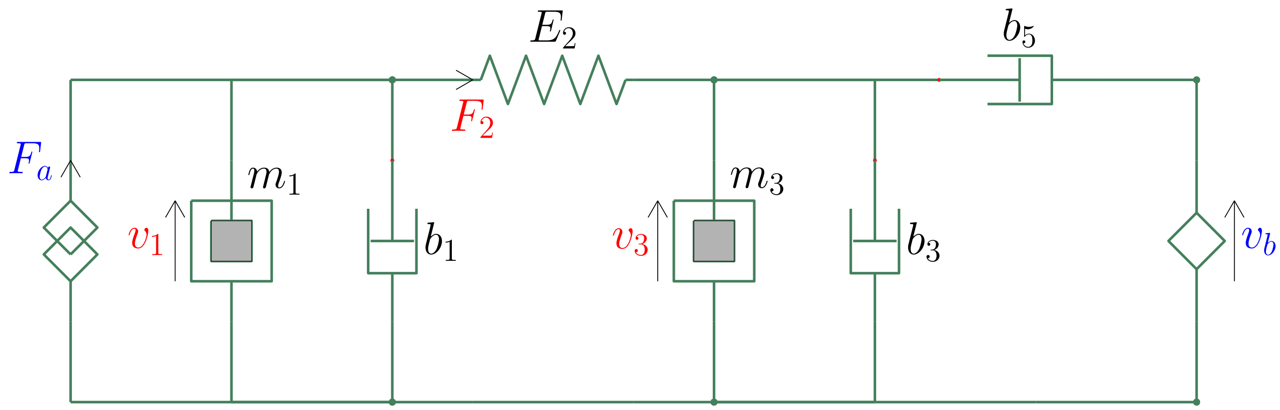
7.d) Add the dissipative blocks and the summation signs to the POG scheme:



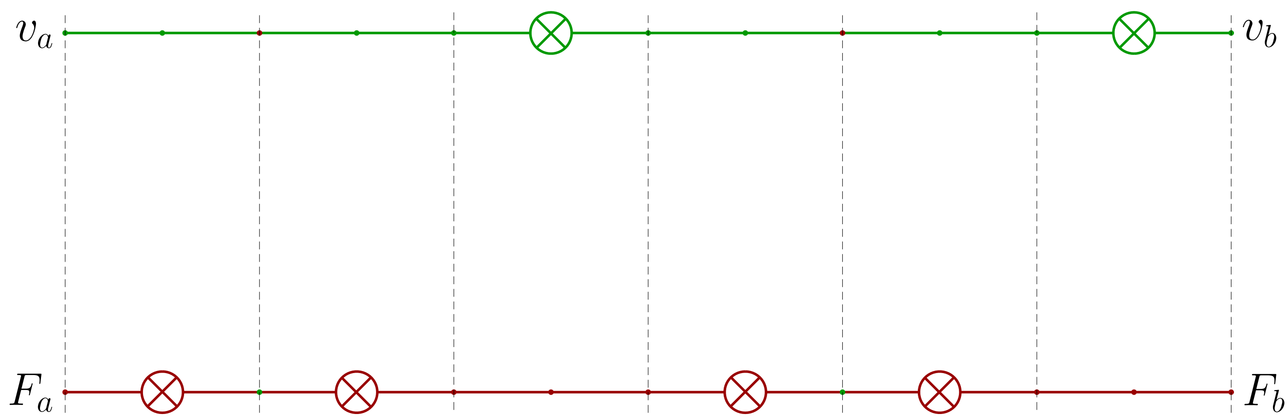
7.e) Check the signs by comparing with the considered physical system:



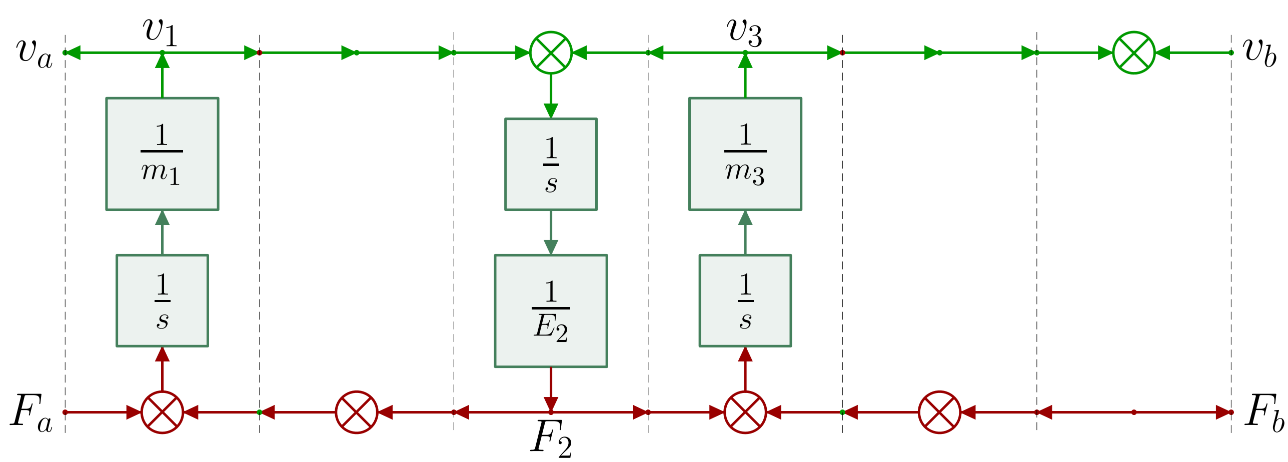
8.a) Choose the positive directions of the state and input variables of the considered physical system:



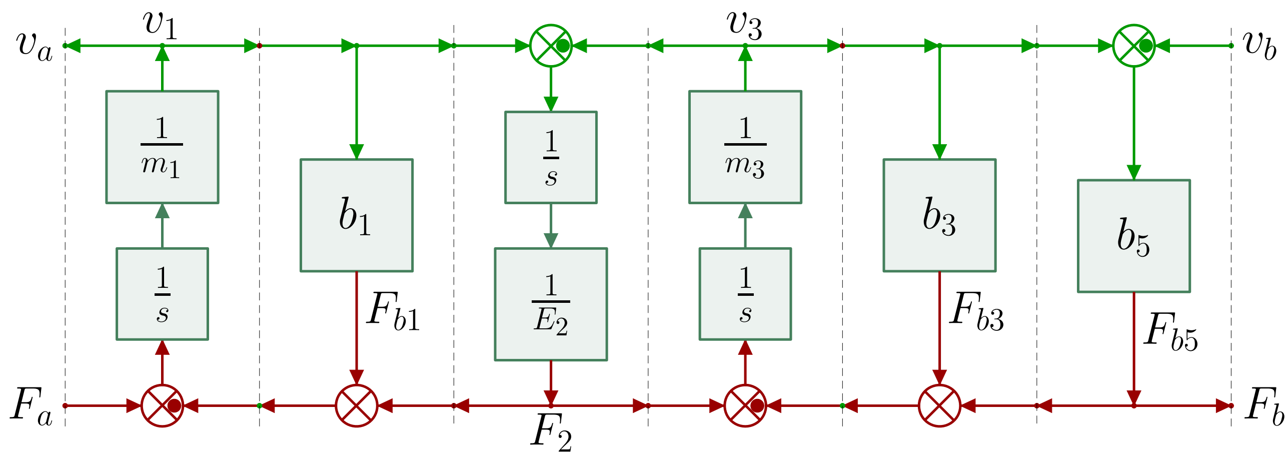
8.b) Draw the series/parallel structure of the POG block scheme:



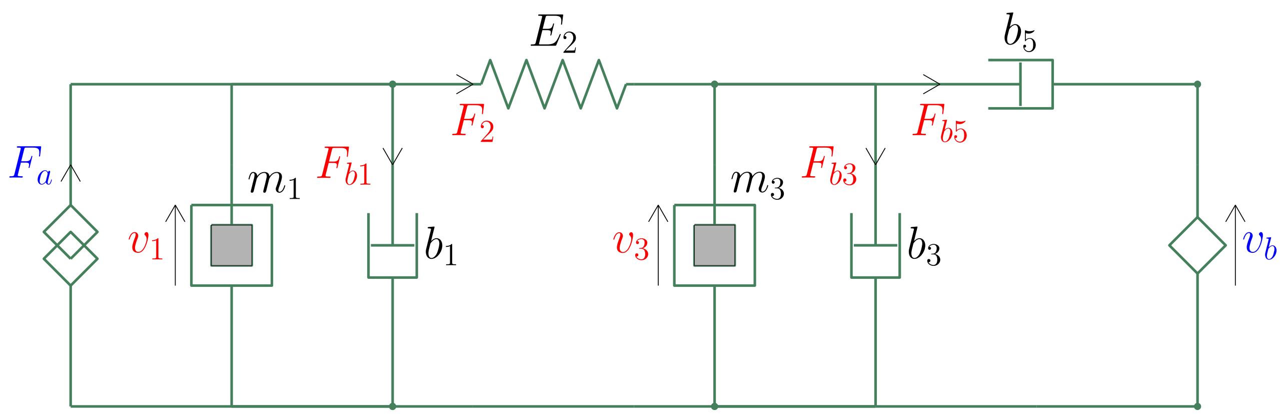
8.c) Add the dynamic blocks to the POG scheme:



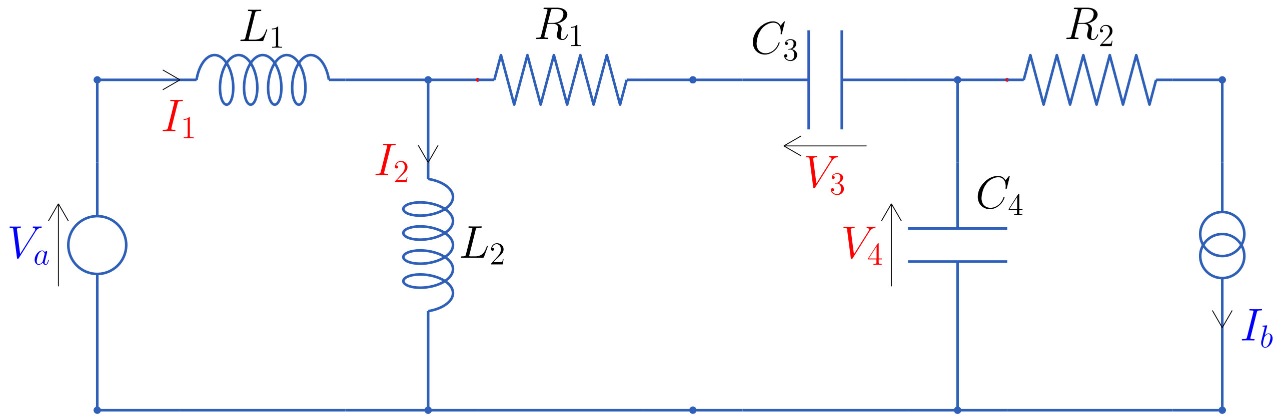
8.d) Add the dissipative blocks and the summation signs to the POG scheme:



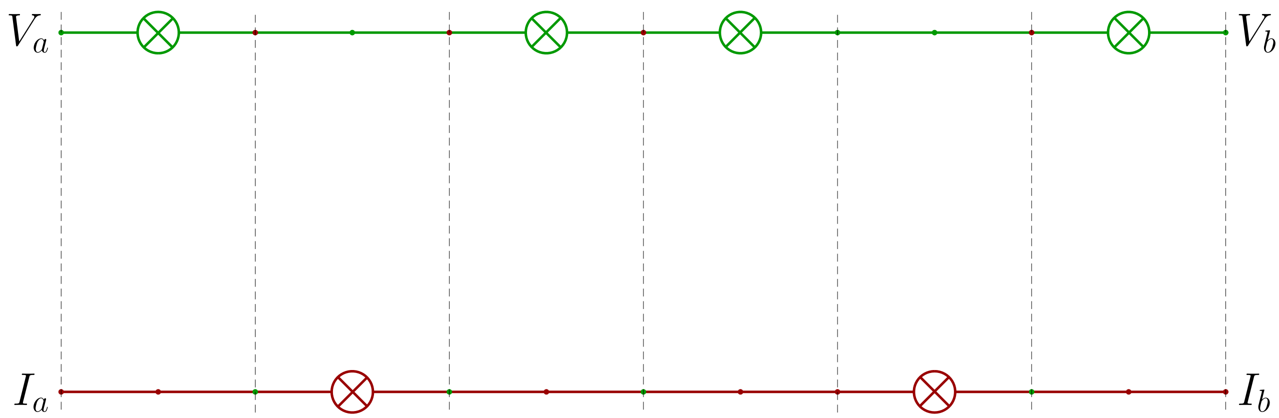
8.e) Check the signs by comparing with the considered physical system:



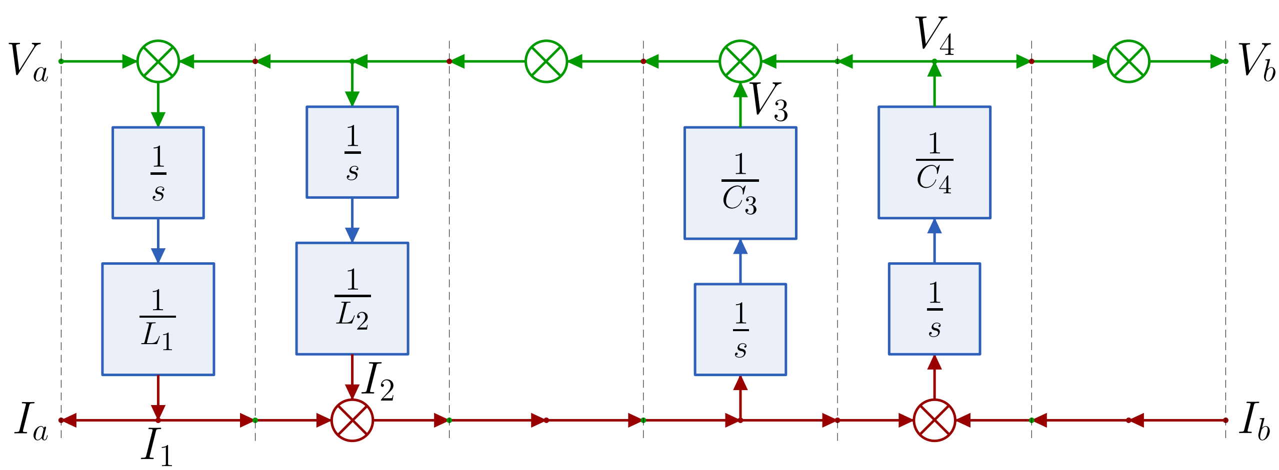
9.a) Choose the positive directions of the state and input variables of the considered physical system:



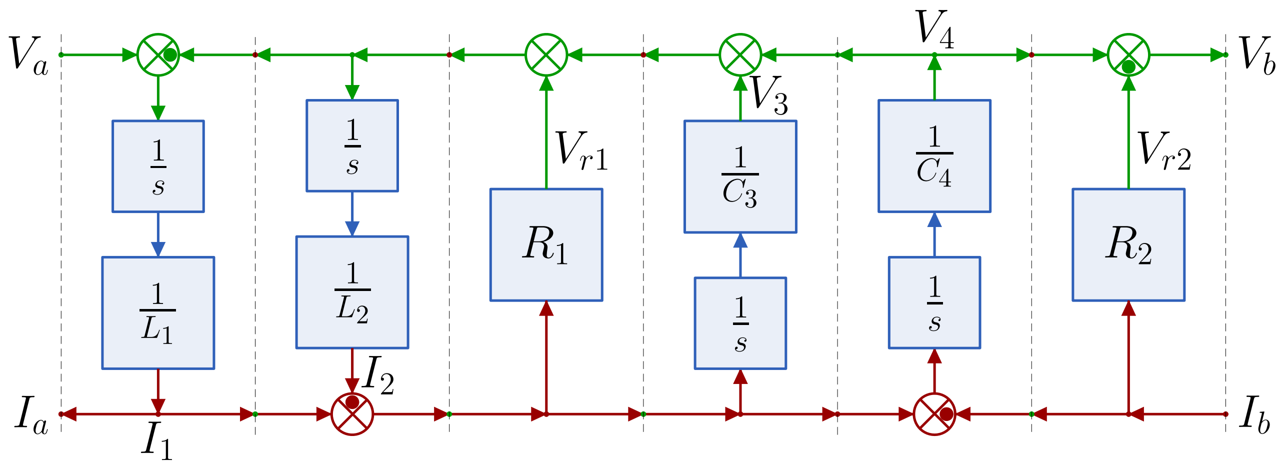
9.b) Draw the series/parallel structure of the POG block scheme:



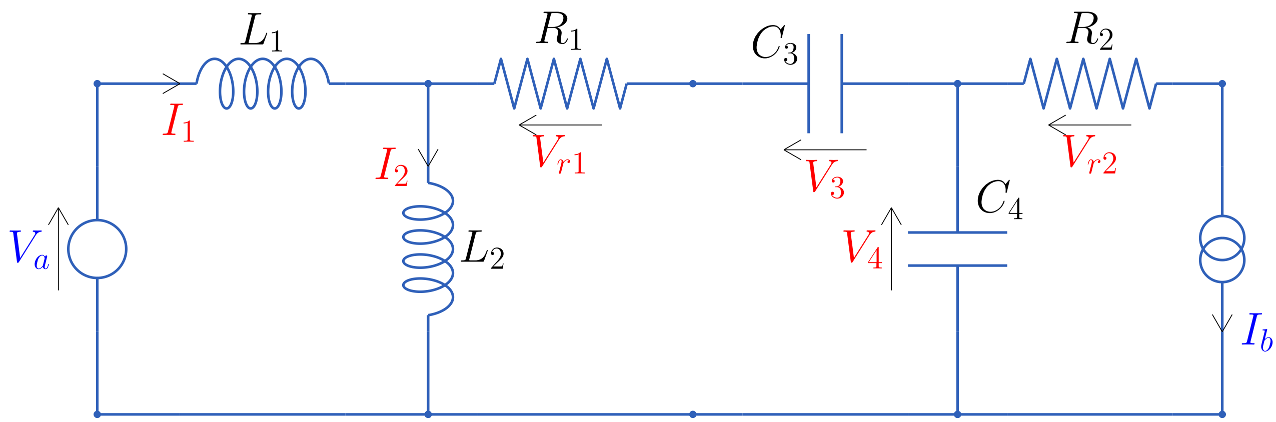
9.c) Add the dynamic blocks to the POG scheme:



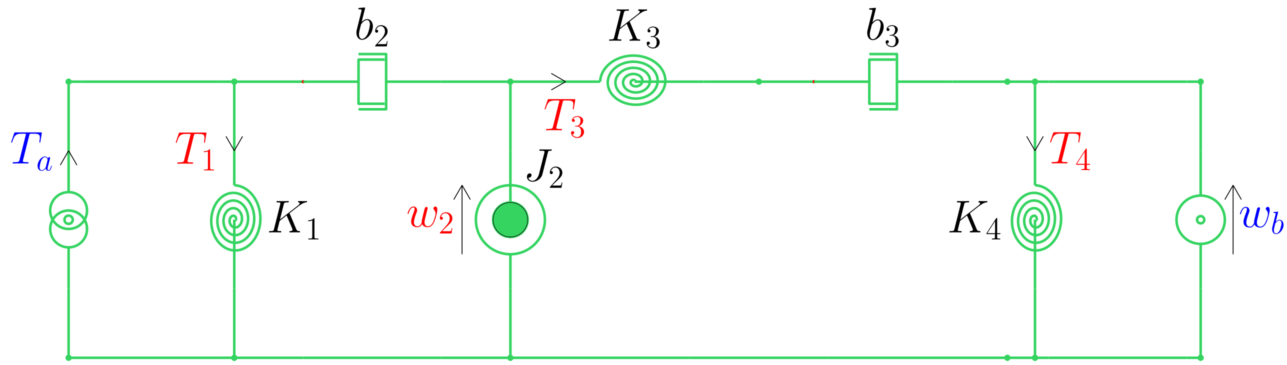
9.d) Add the dissipative blocks and the summation signs to the POG scheme:



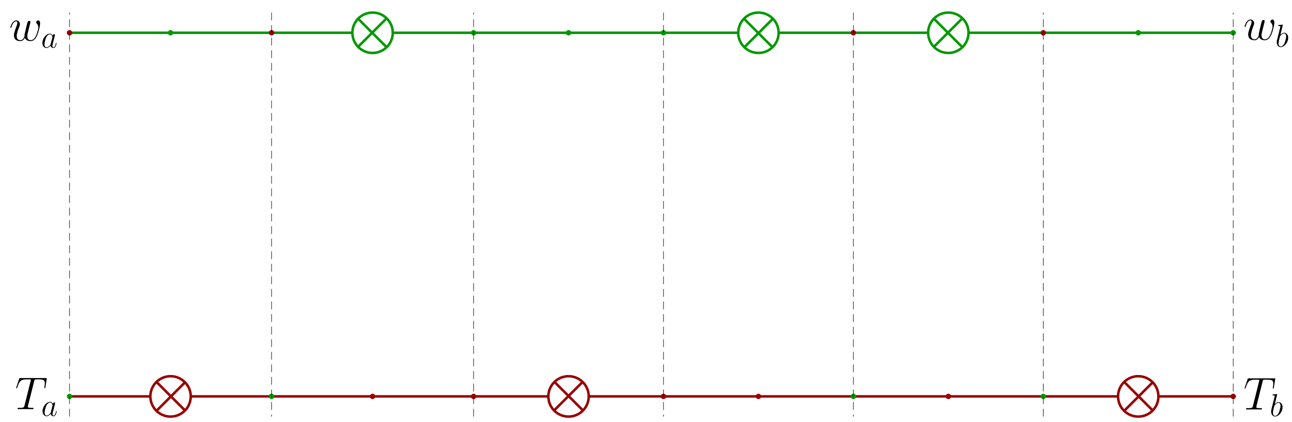
9.e) Check the signs by comparing with the considered physical system:



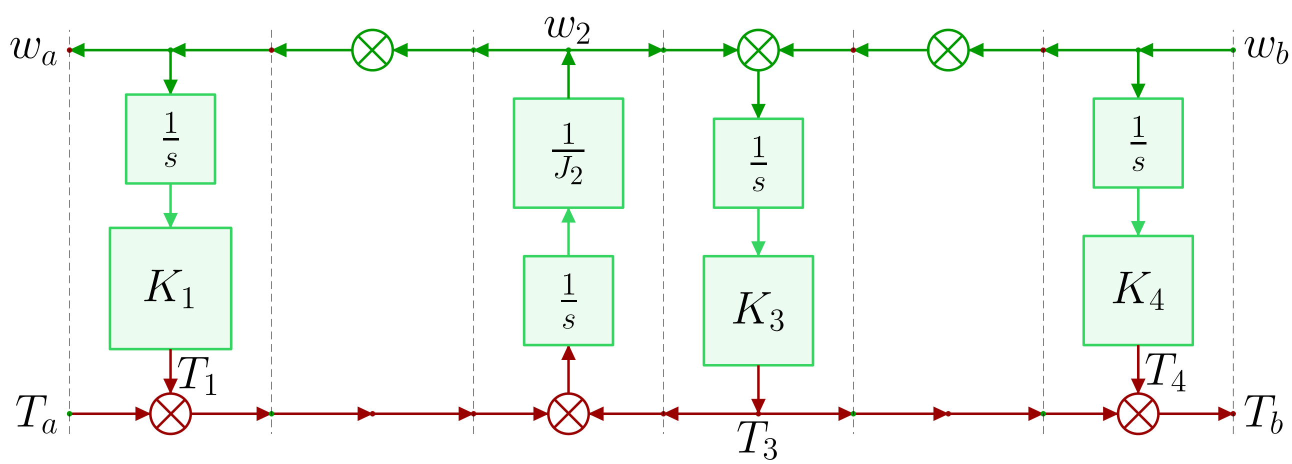
10.a) Choose the positive directions of the state and input variables of the considered physical system:



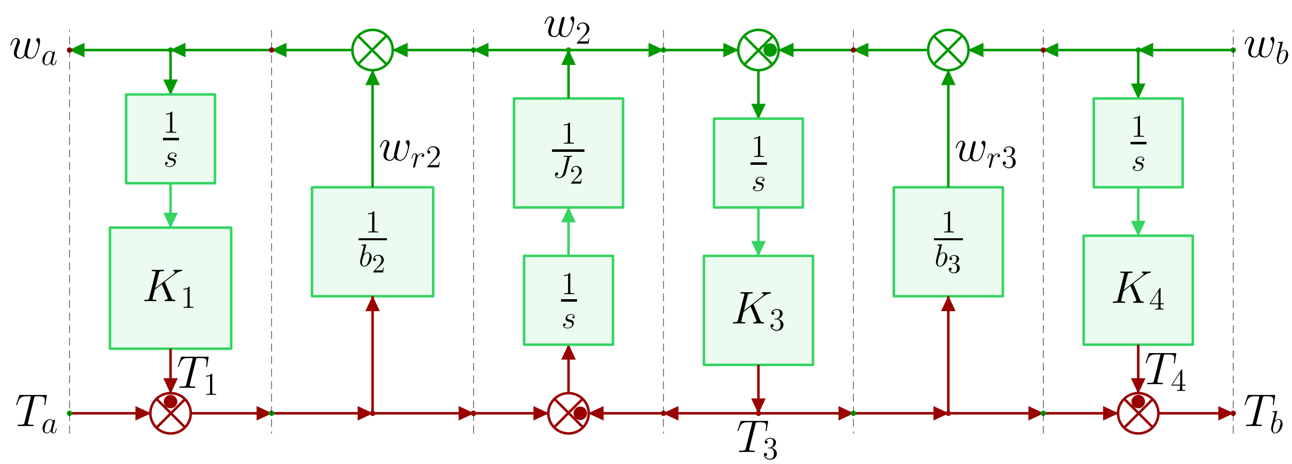
10.b) Draw the series/parallel structure of the POG block scheme:



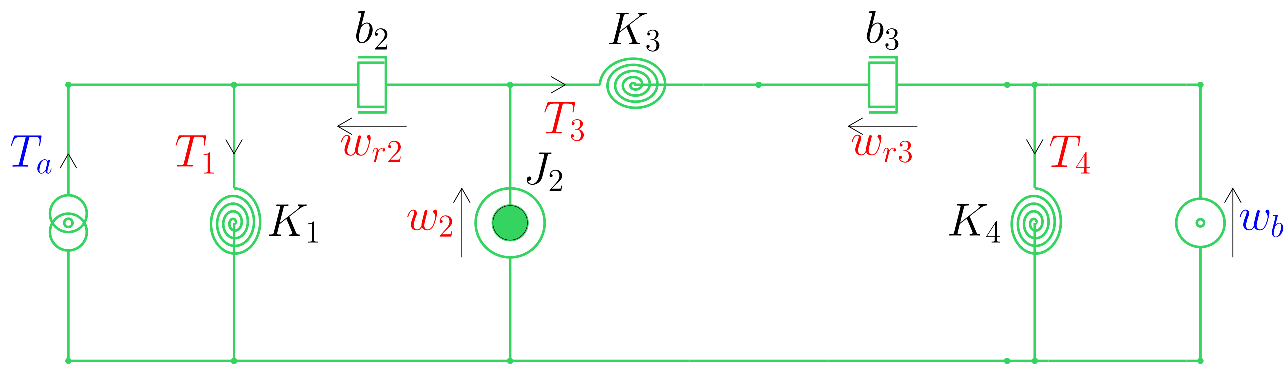
10.c) Add the dynamic blocks to the POG scheme:



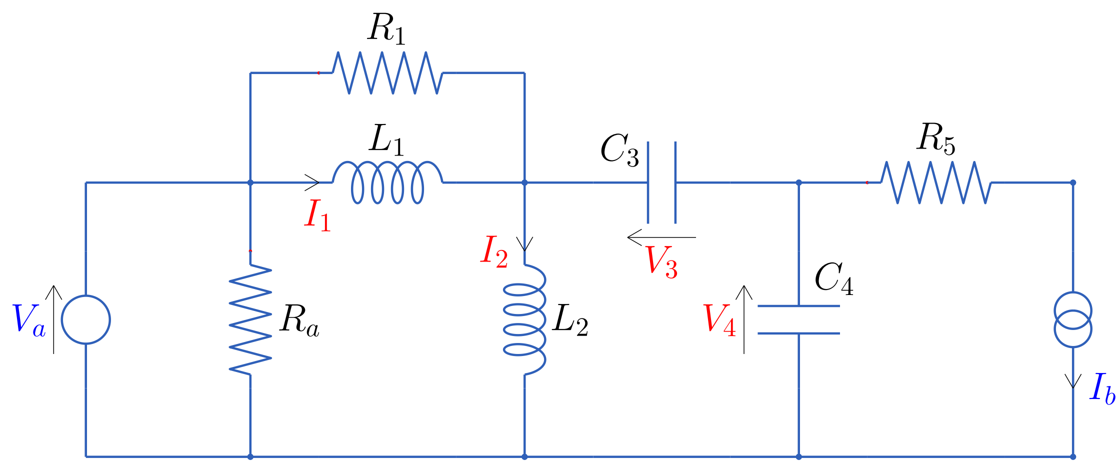
10.d) Add the dissipative blocks and the summation signs to the POG scheme:



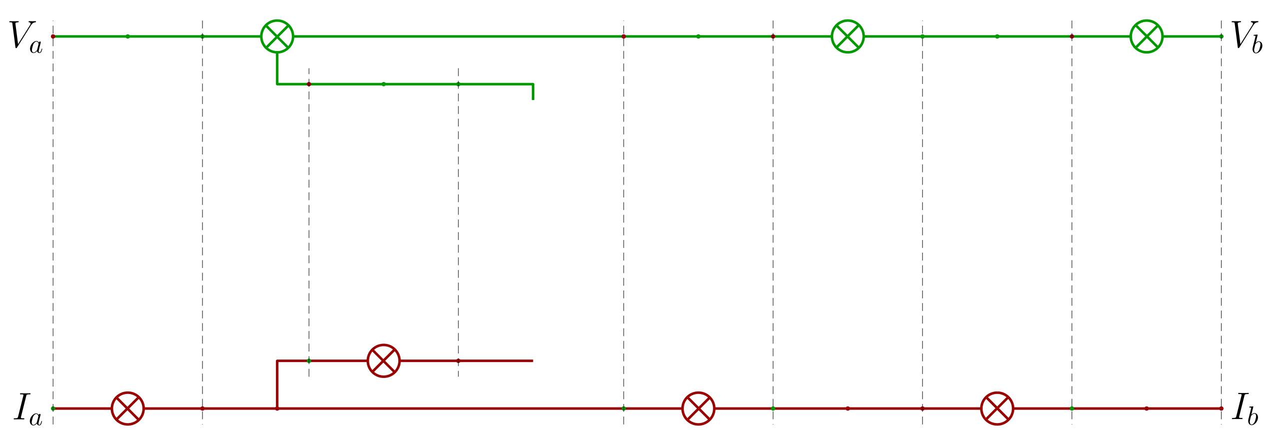
10.e) Check the signs by comparing with the considered physical system:



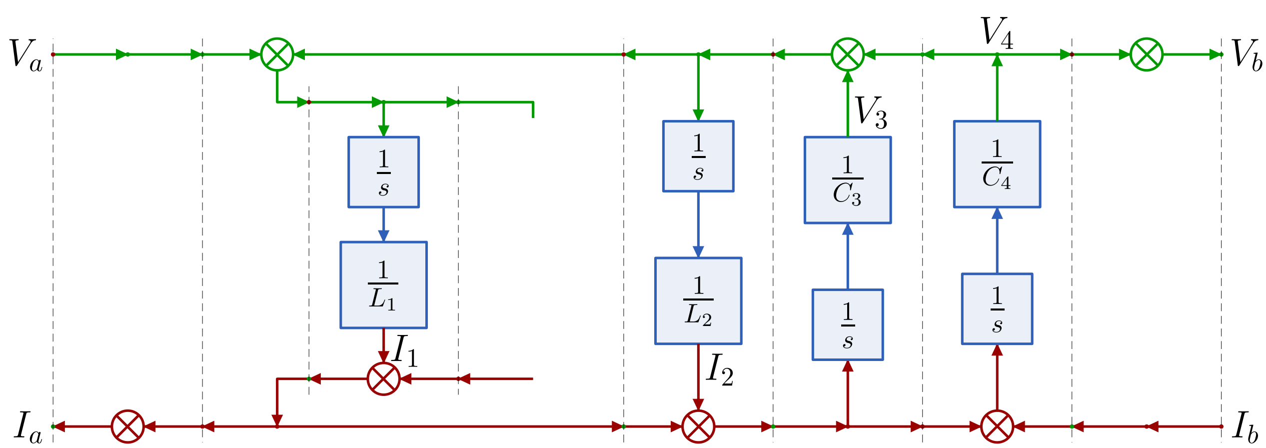
11.a) Choose the positive directions of the state and input variables of the considered physical system:



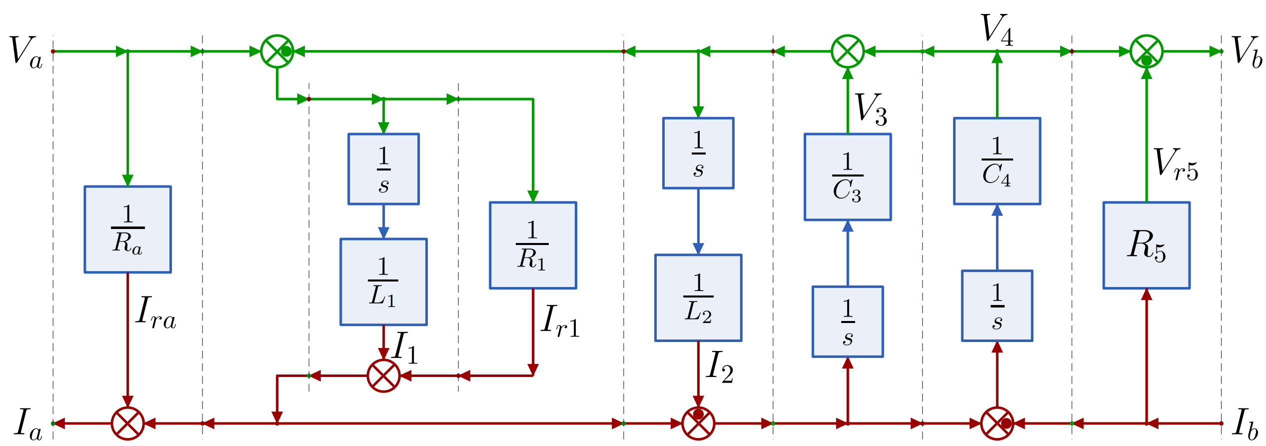
11.b) Draw the series/parallel structure of the POG block scheme:



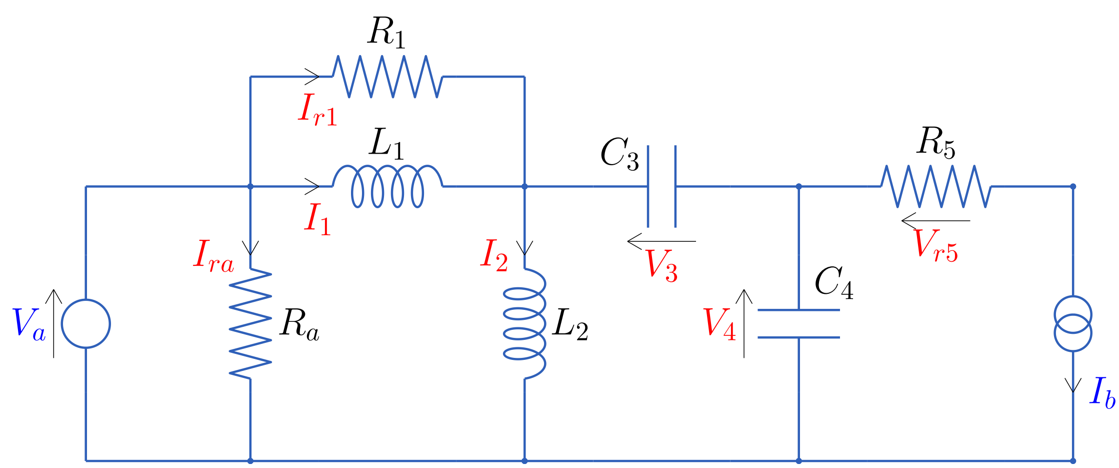
11.c) Add the dynamic blocks to the POG scheme:



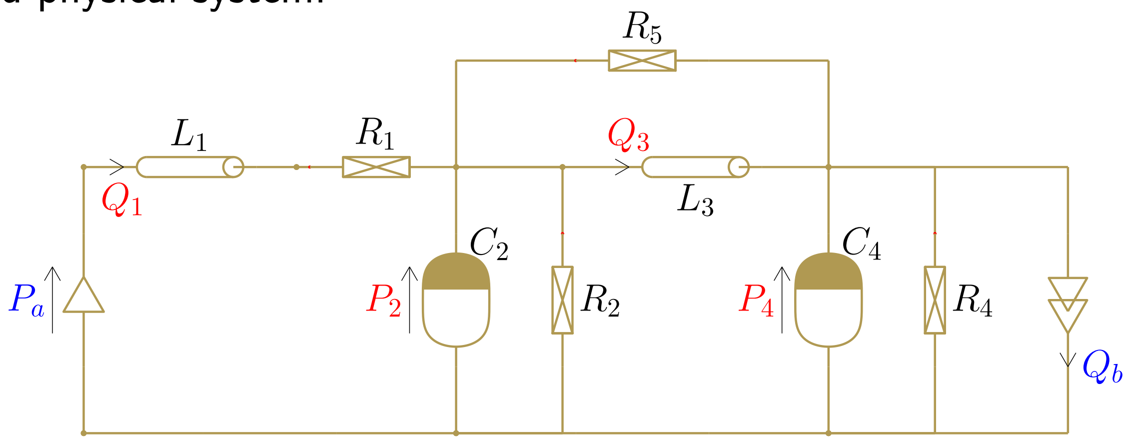
11.d) Add the dissipative blocks and the summation signs to the POG scheme:



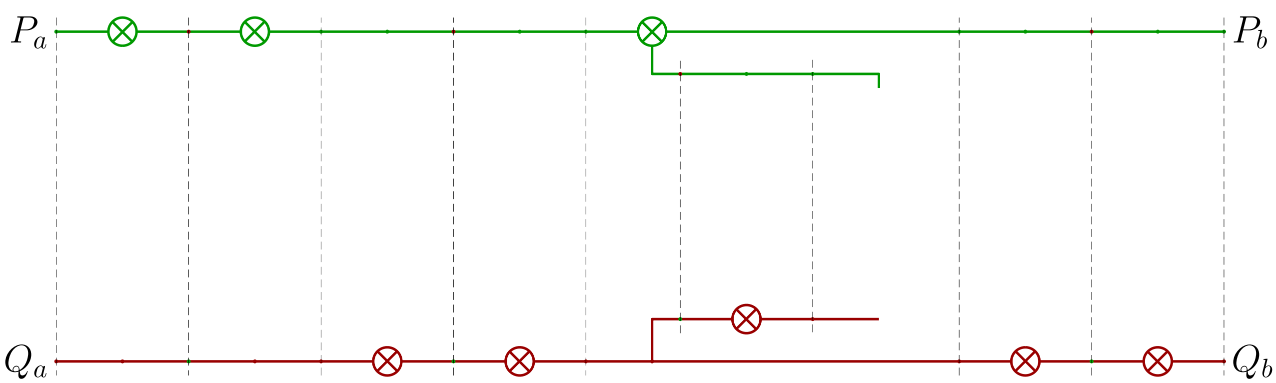
11.e) Check the signs by comparing with the considered physical system:



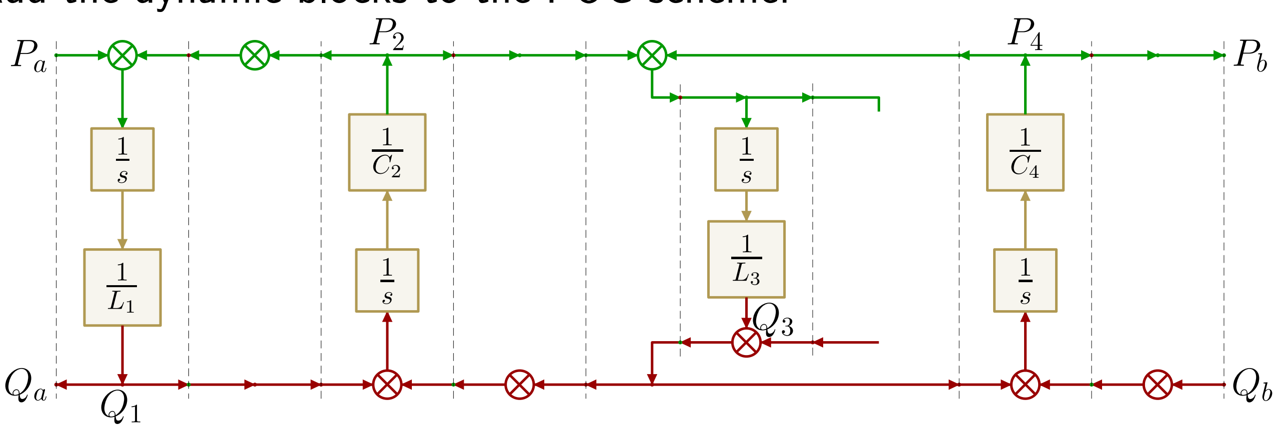
12.a) Choose the positive directions of the state and input variables of the considered physical system:



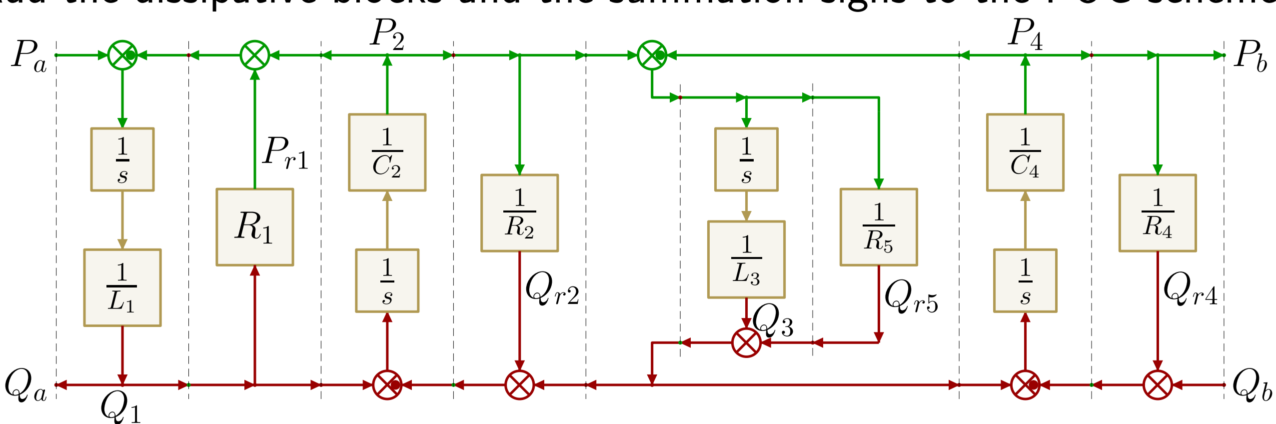
12.b) Draw the series/parallel structure of the POG block scheme:



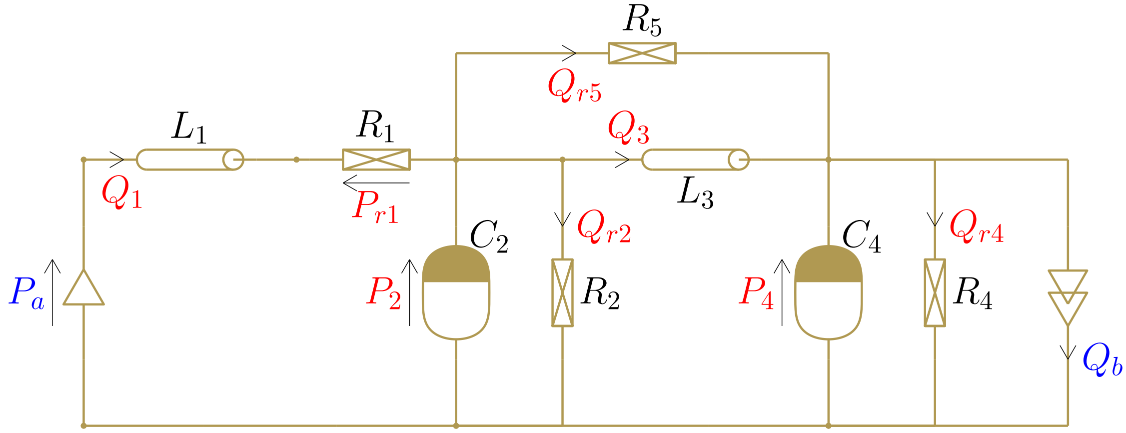
12.c) Add the dynamic blocks to the POG scheme:



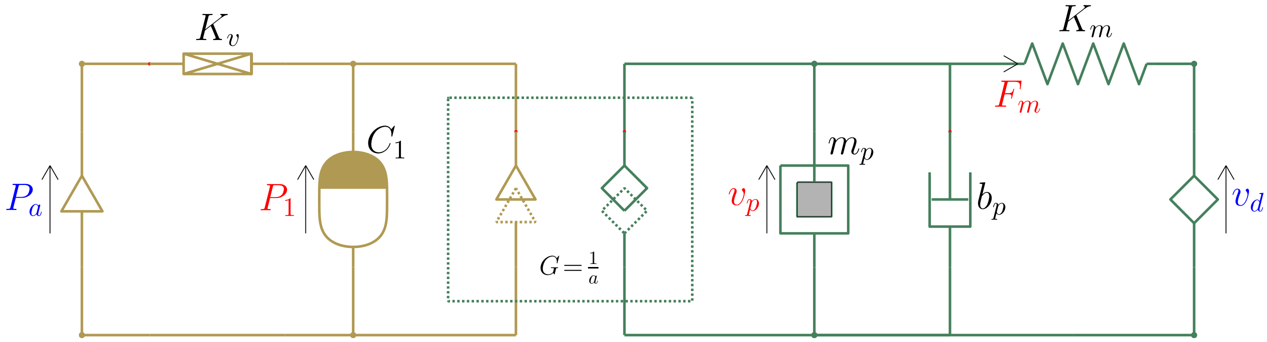
12.d) Add the dissipative blocks and the summation signs to the POG scheme:



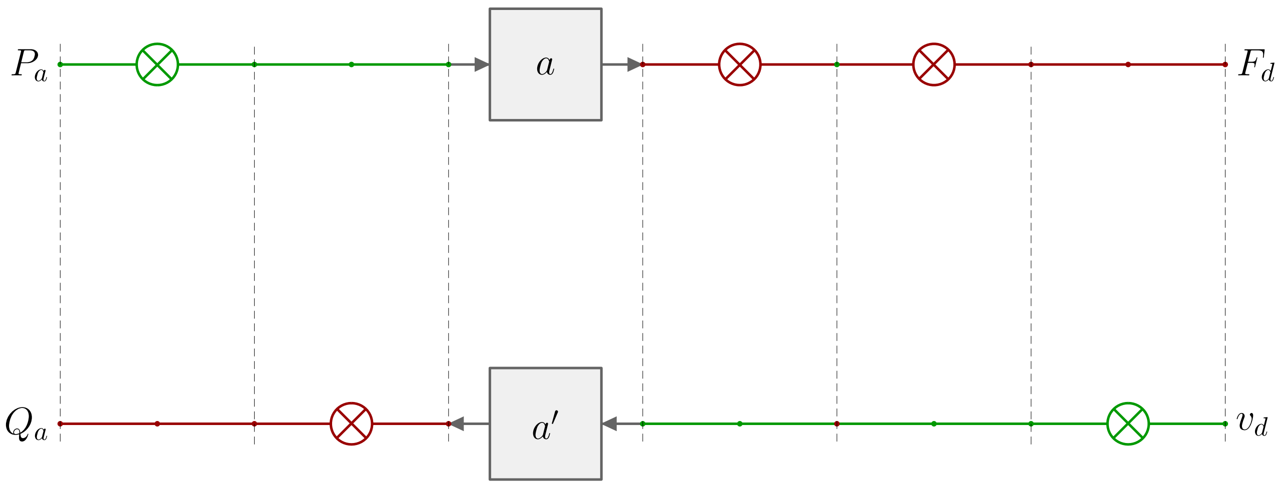
12.e) Check the signs by comparing with the considered physical system:



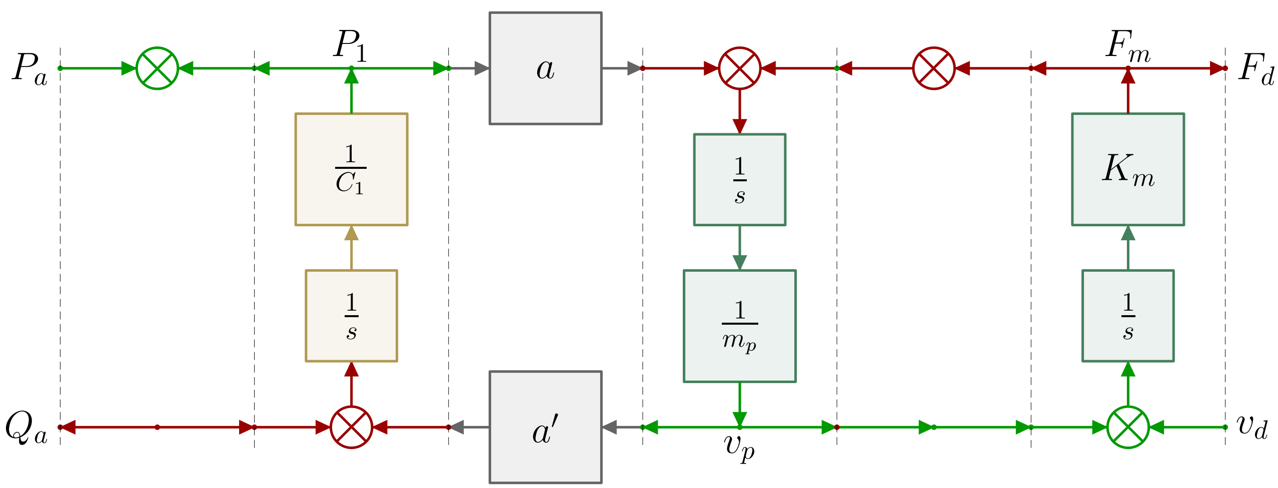
13.a) Choose the positive directions of the state and input variables of the considered physical system:



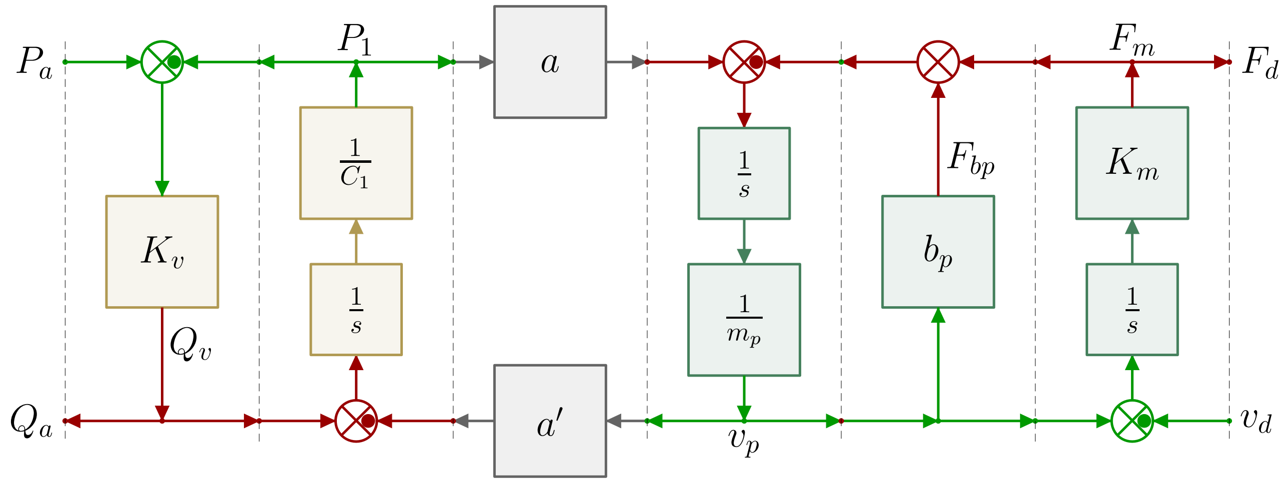
13.b) Draw the series/parallel structure of the POG block scheme:



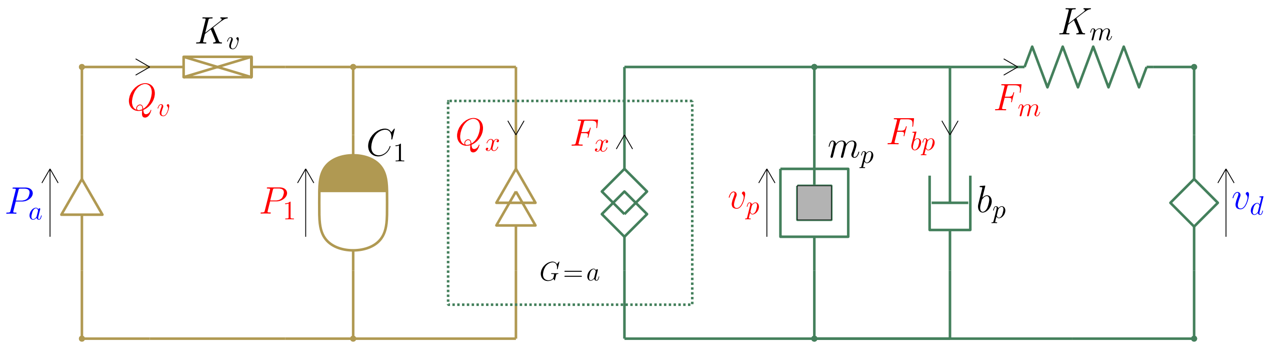
13.c) Add the dynamic blocks to the POG scheme:



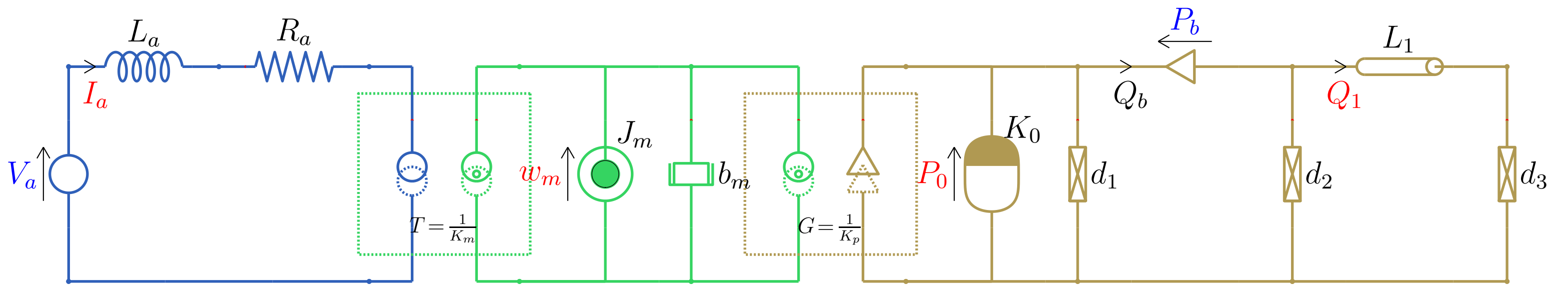
13.d) Add the dissipative blocks and the summation signs to the POG scheme:



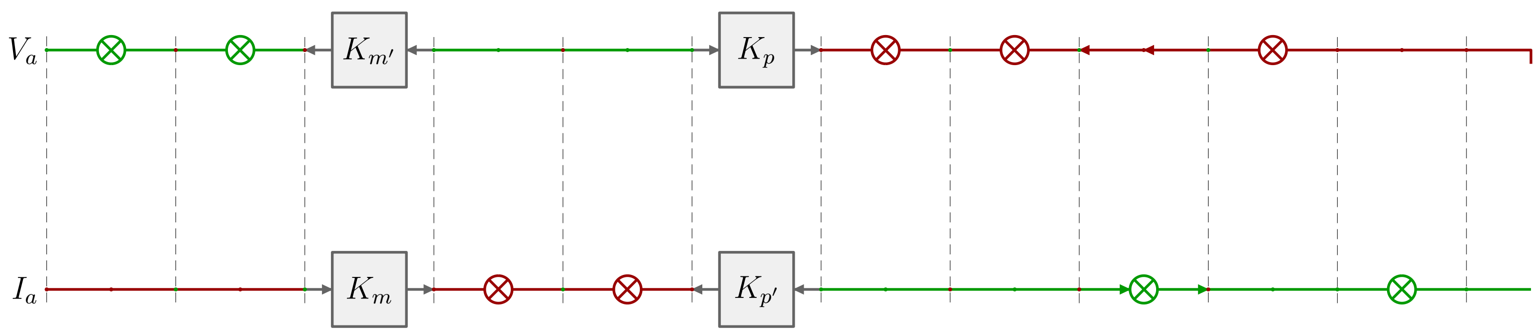
13.e) Check the signs by comparing with the considered physical system:



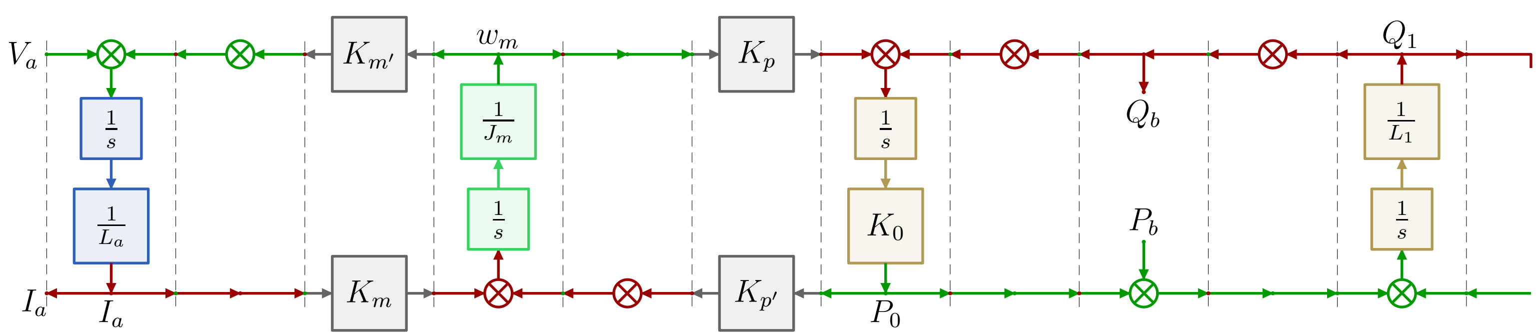
14.a) Choose the positive directions of the state and input variables of the considered physical system:



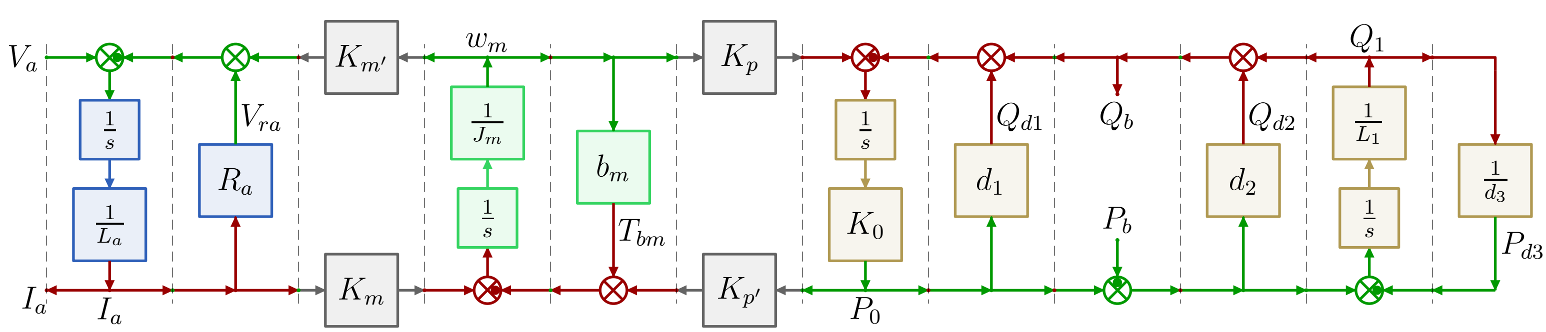
14.b) Draw the series/parallel structure of the POG block scheme:



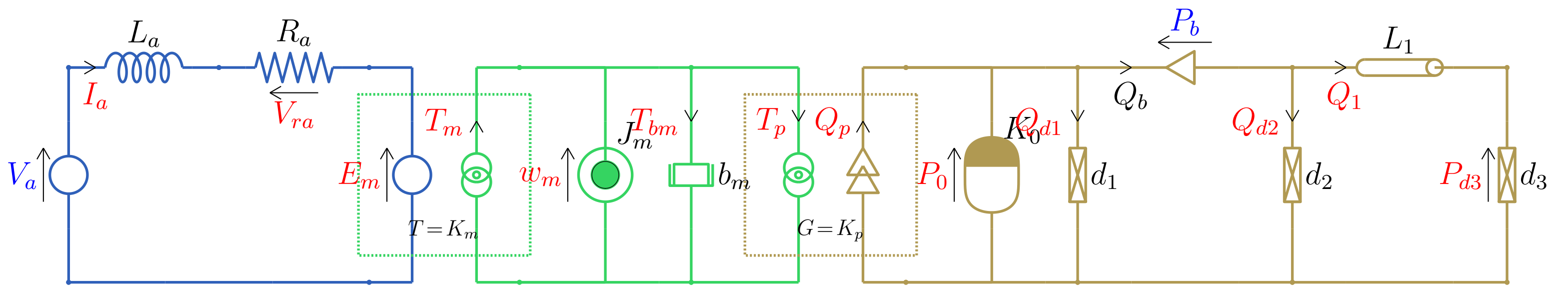
14.c) Add the dynamic blocks to the POG scheme:



14.d) Add the dissipative blocks and the summation signs to the POG scheme:

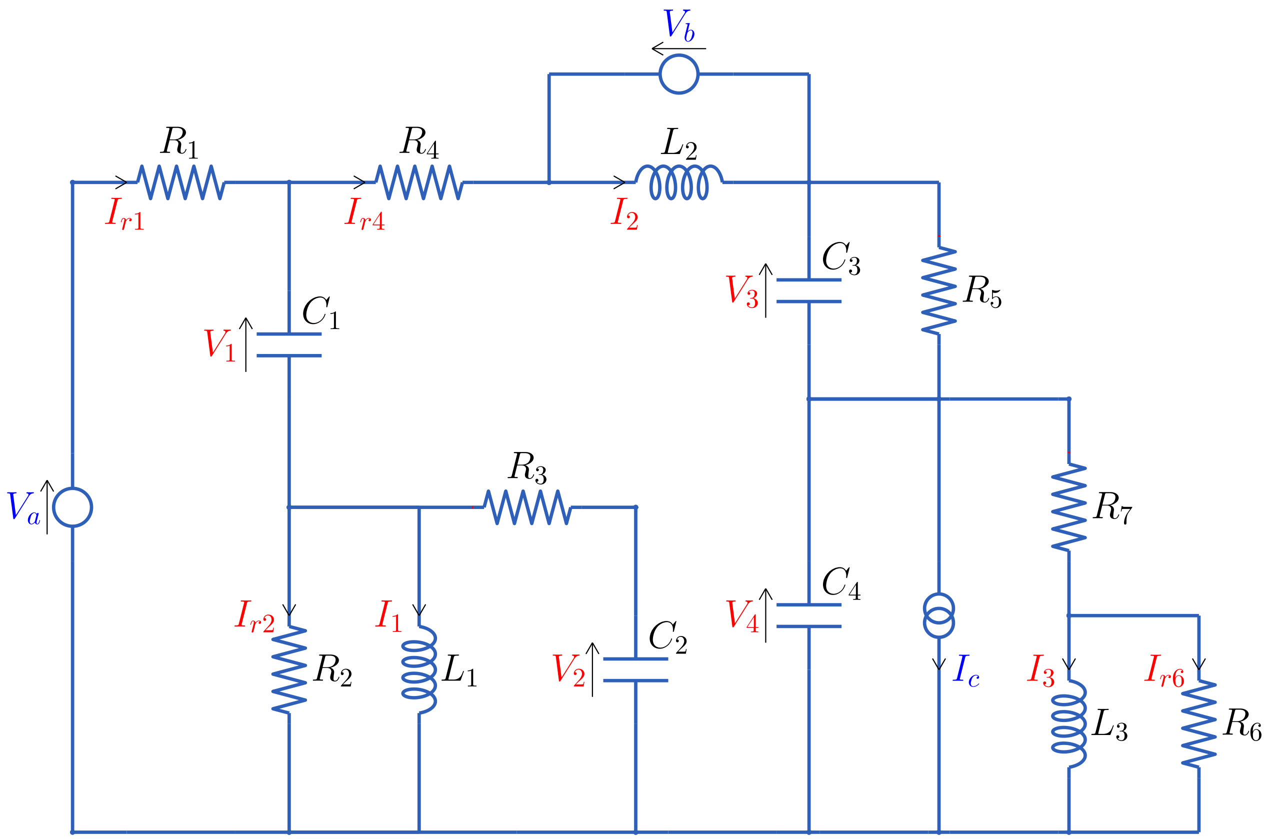


14.e) Check the signs by comparing with the considered physical system:

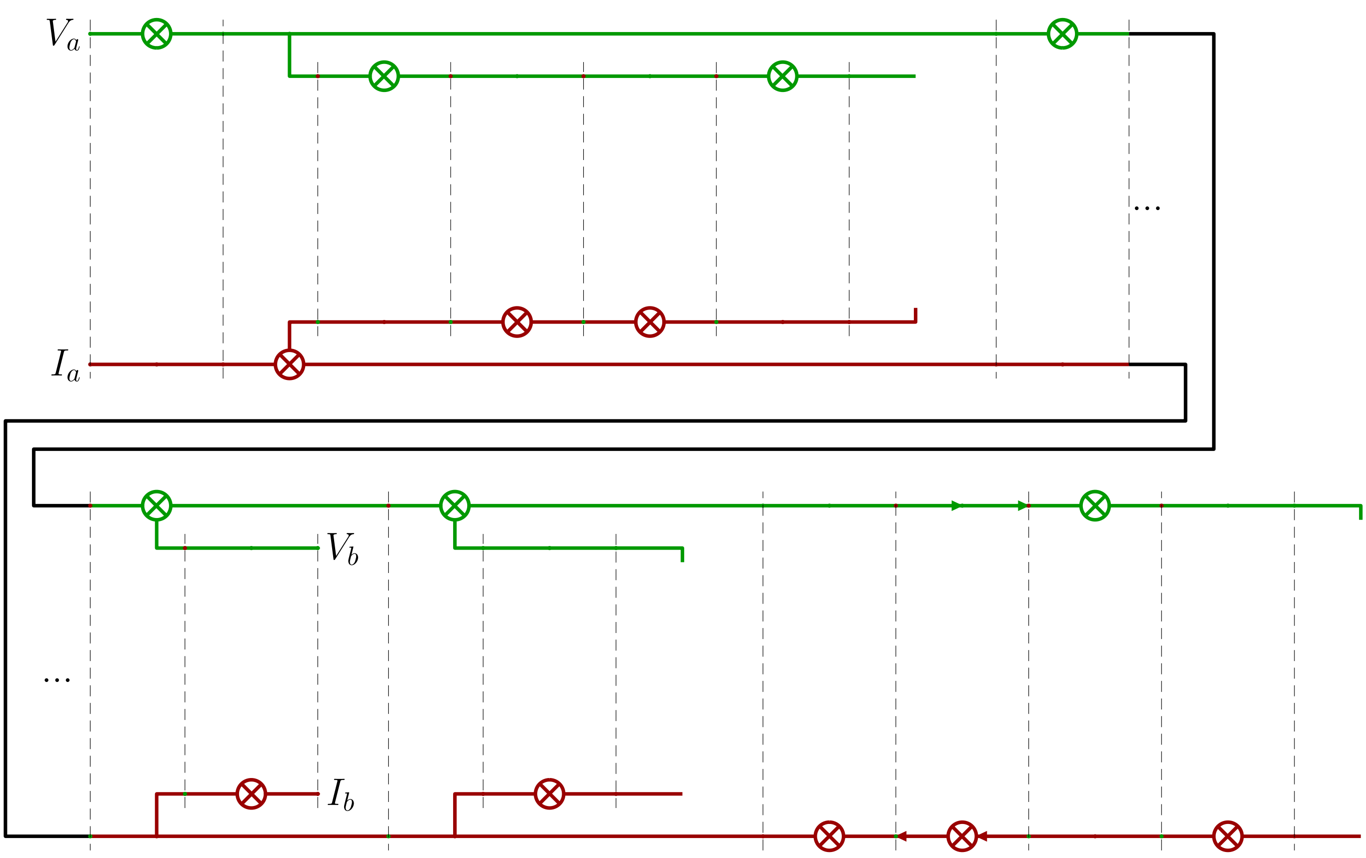


L_a
R_a
I_a
V_{ra}
V_a
E_m
T_m
$T = K_m$
ω_m
J_m
T_{bm}
b_m
T_p
Q_p
$G = K_p$
P_0
K_0
Q_{d1}
d_1
P_b
Q_b
L_1
Q_1
Q_{d2}
d_2
P_{d3}
d_3

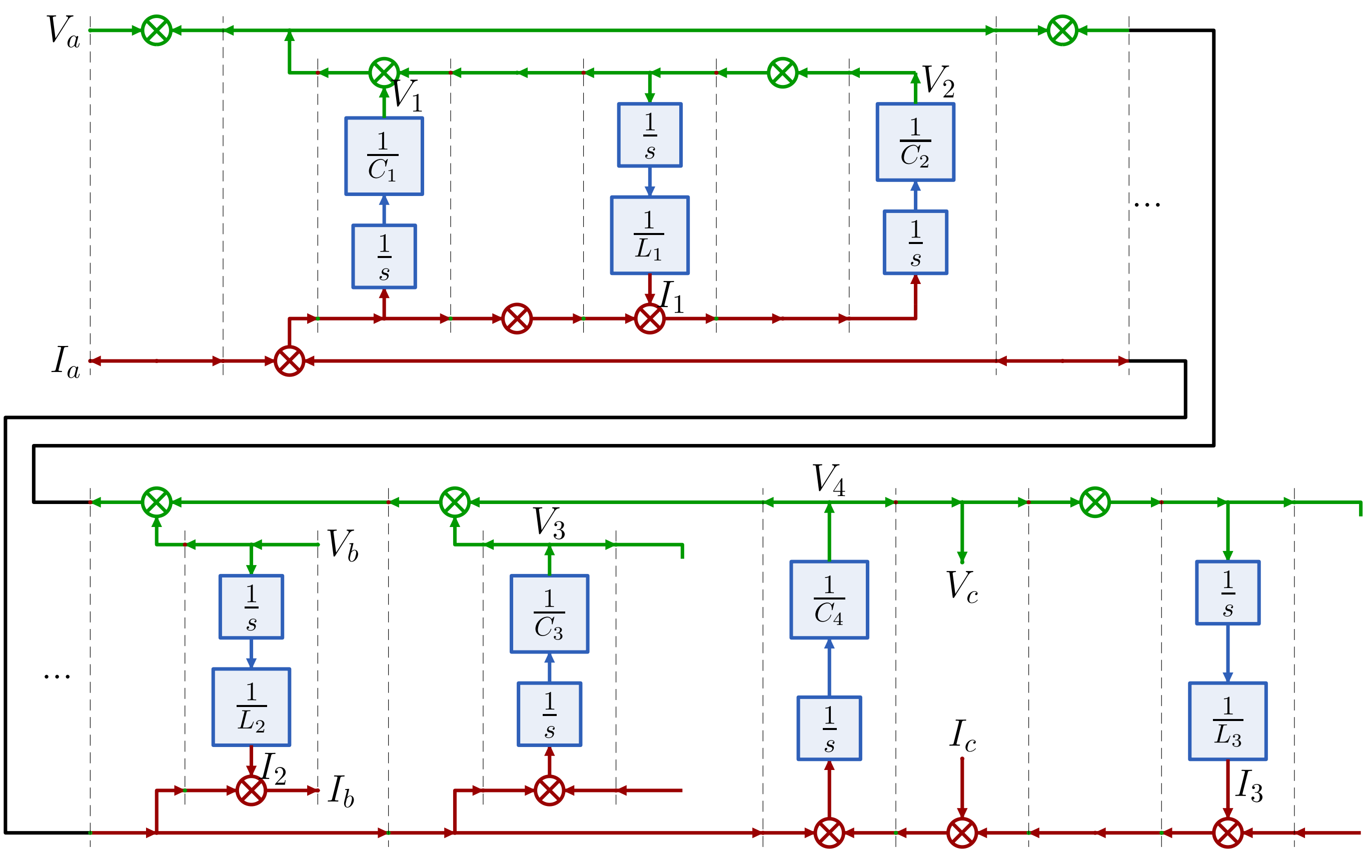
15.a) Choose the positive directions of the state and input variables of the considered physical system:



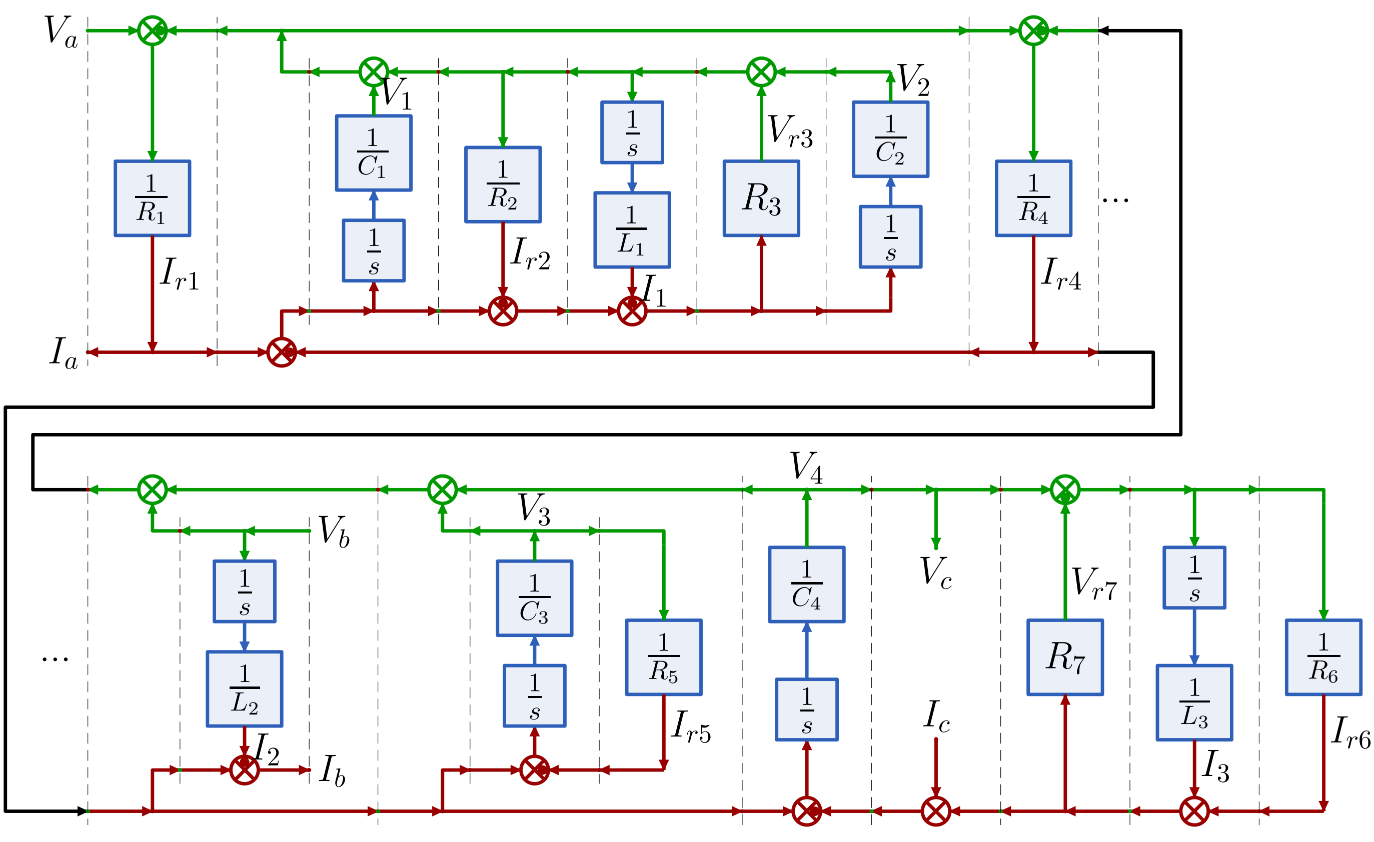
15.b) Draw the series/parallel structure of the POG block scheme:



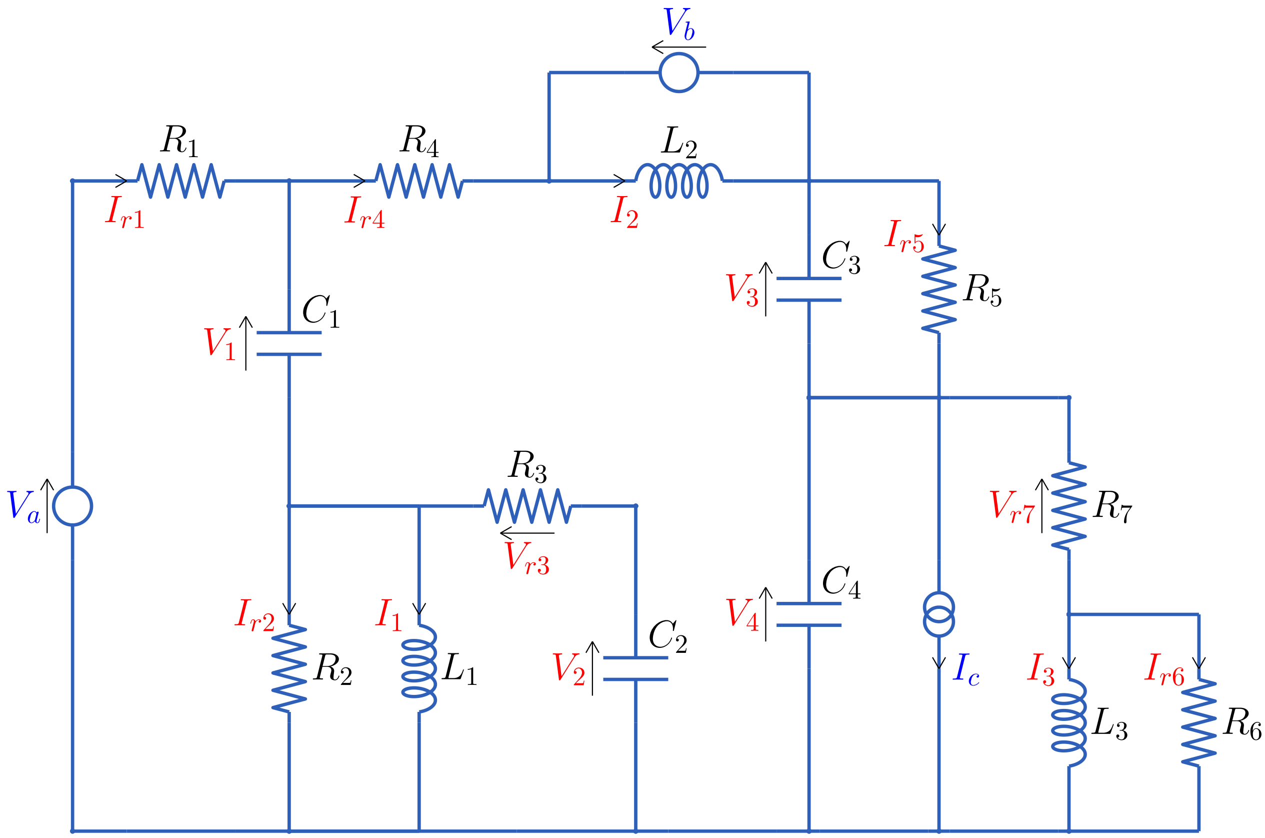
15.c) Add the dynamic blocks to the POG scheme:



15.d) Add the dissipative blocks and the summation signs to the POG scheme:



15.e) Check the signs by comparing with the considered physical system:

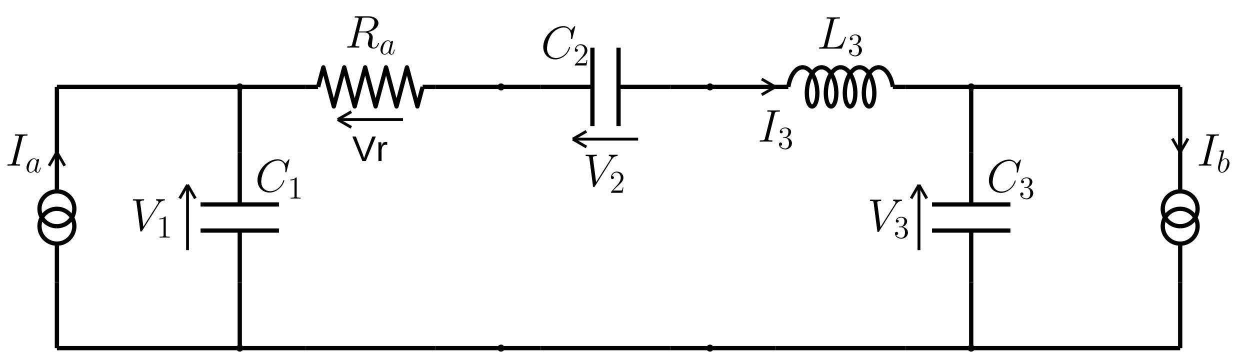


POG modeling of physical systems

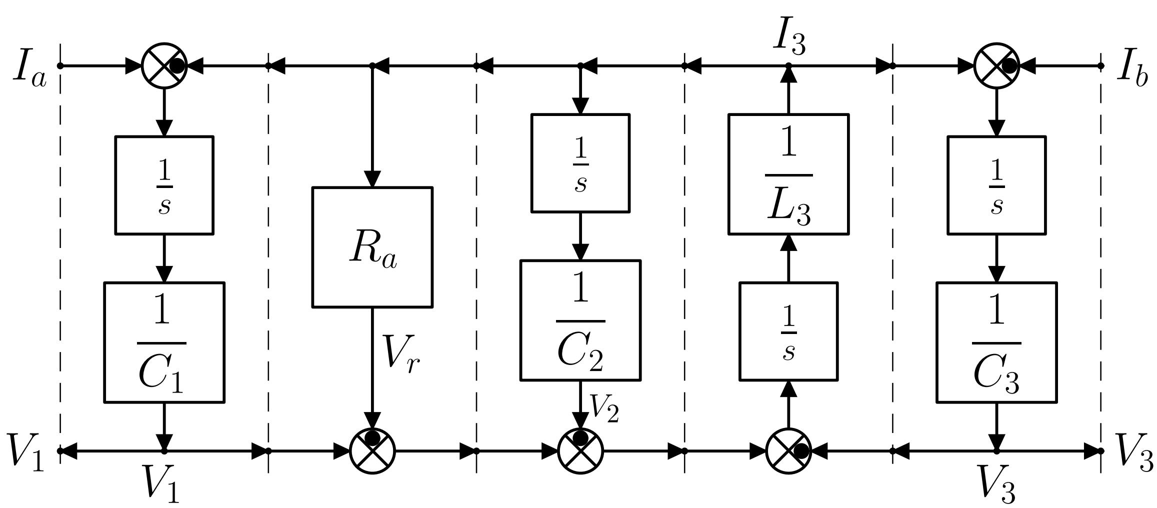
There are two different ways of modeling physical systems using POG.

A) POG modeling from left to right.

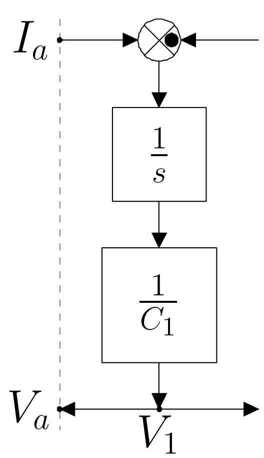
- Let us consider the following electric circuit:



- The corresponding POG block scheme is:

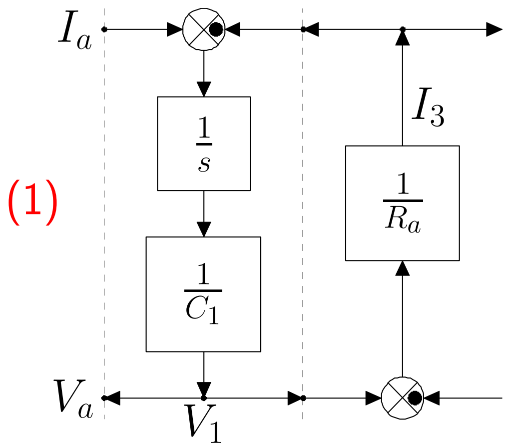


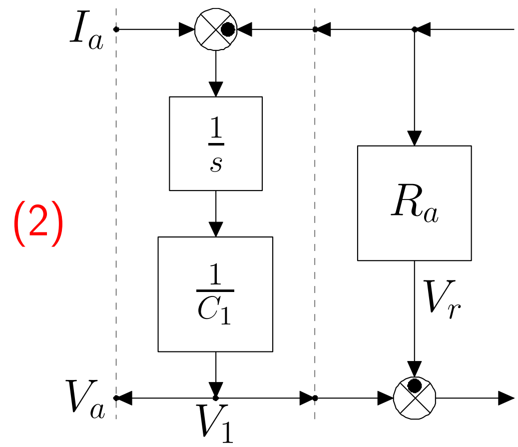
- This POG scheme can be obtained modeling the system from left ro right.



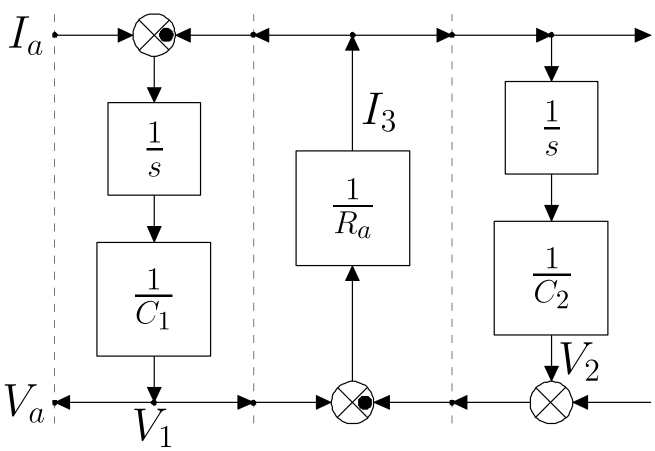
The input current I_a acts on a capacitor (Effort Block) in parallel connection, and therefore there is only one way of modeling the capacitor.

- The next element is the resistor R_a in series with an input voltage V_1 on the left. In this case the resistor can be modeled using two POG schemes.
- The two possible POG schemes are:

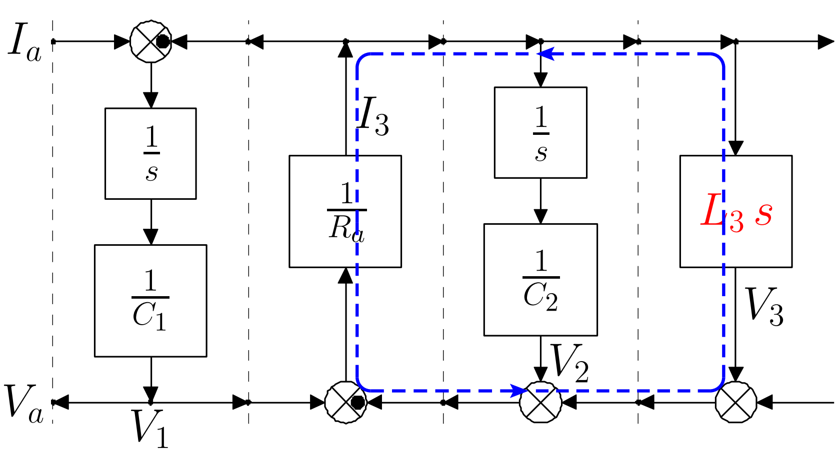




- Let us consider the solution (1). The next element to be modeled is the capacitor C_2 in series with the input current I_3 on the left. In this case the capacitor C_2 can be modeled only as follows:

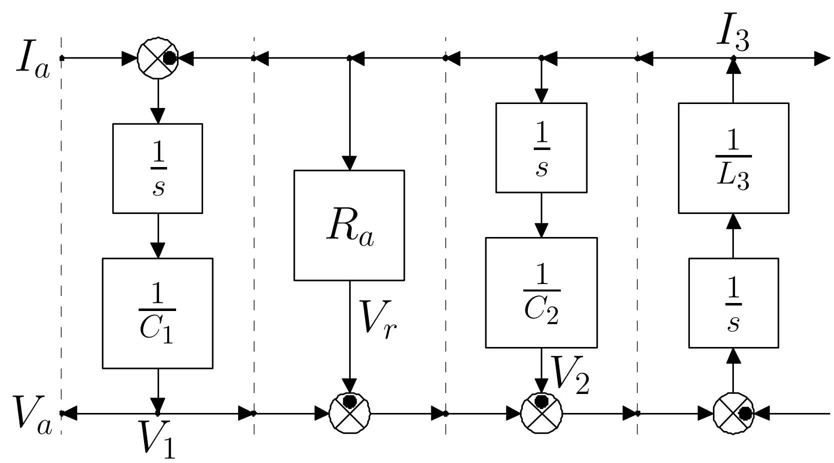


- The next element to be modeled is the inductor L_3 in series. Since the input current I_3 enters on the left, the inductor L_3 can be modeled only using the derivative causality:



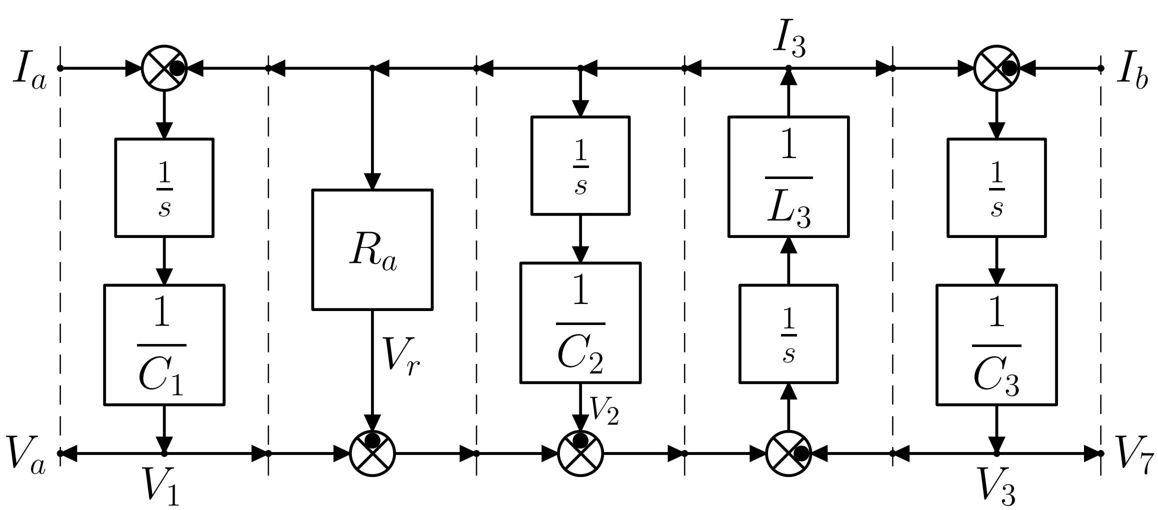
- In this case the causality problem can be solved inverting the blue dashed path shown in the picture.

- After inverting the path, the POG scheme has the following structure:



- This is exactly the POG scheme that coluld have been obtained choosing the solution (2) when the resistor R_a was modeled.

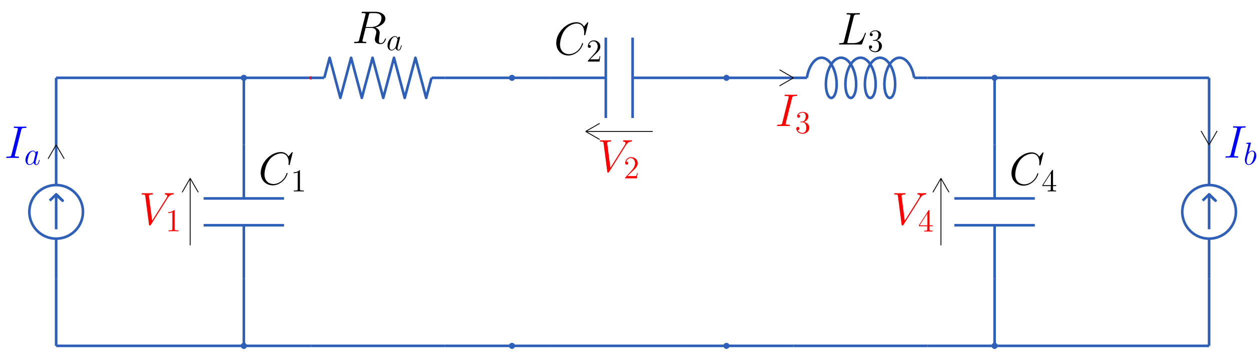
- The last two elements to be modeled are capacitor C_3 and the current generator I_b. The only possible POG solution is the following:



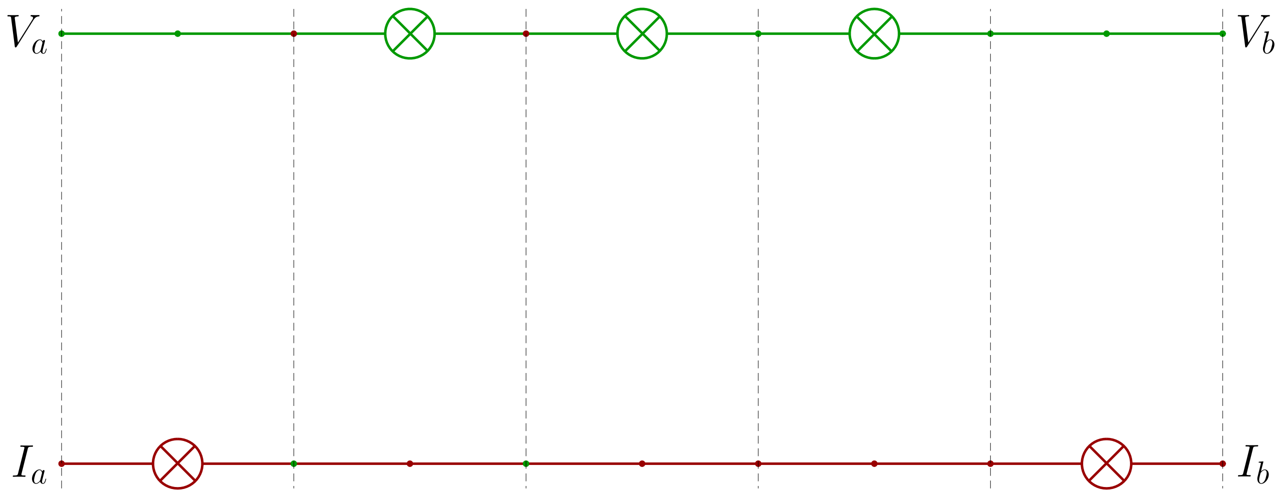
- In fact, the capacitor C_3 is connected in parallel and can be modeled only using the POG structure shown in the picture.

B) POG Direct modeling.

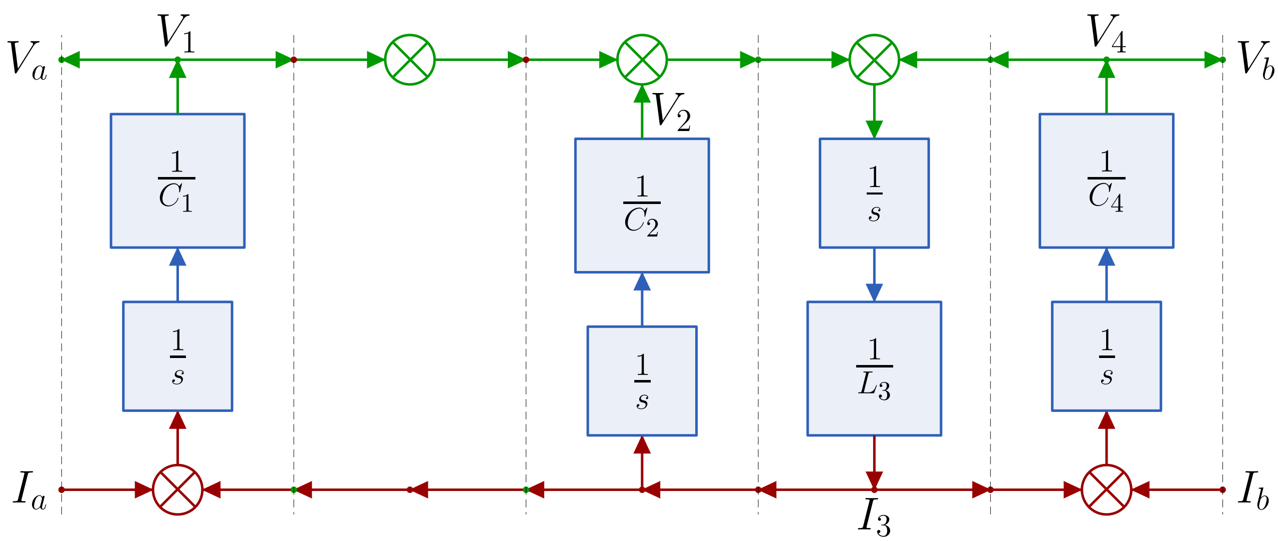
a) Choose the positive directions of the state and input variables of the considered physical system:



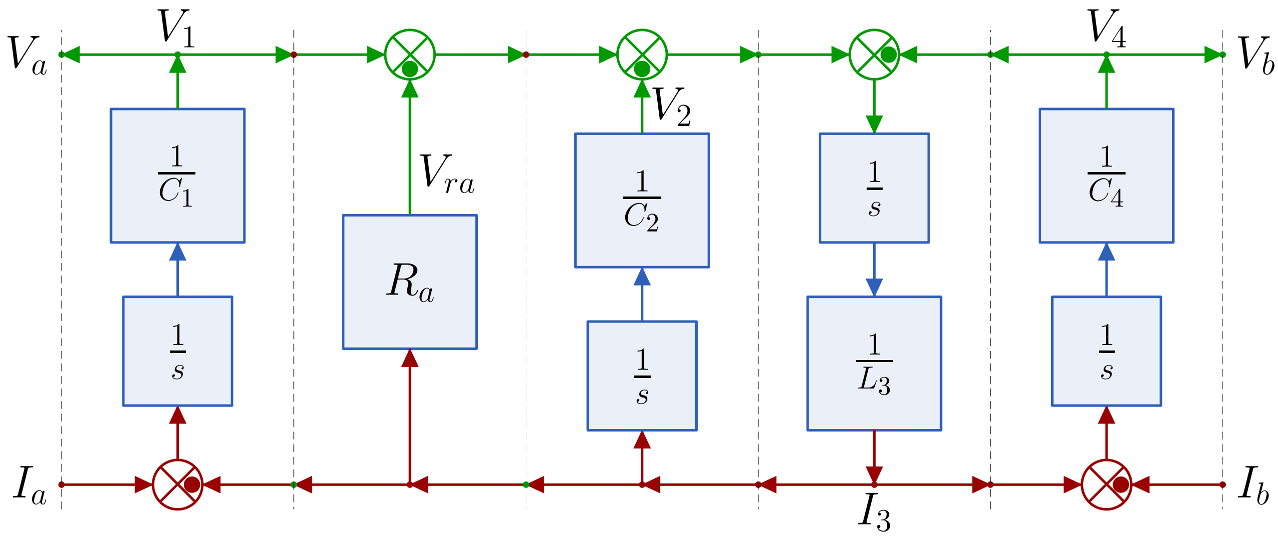
b) Draw the series/parallel structure of the POG block scheme:



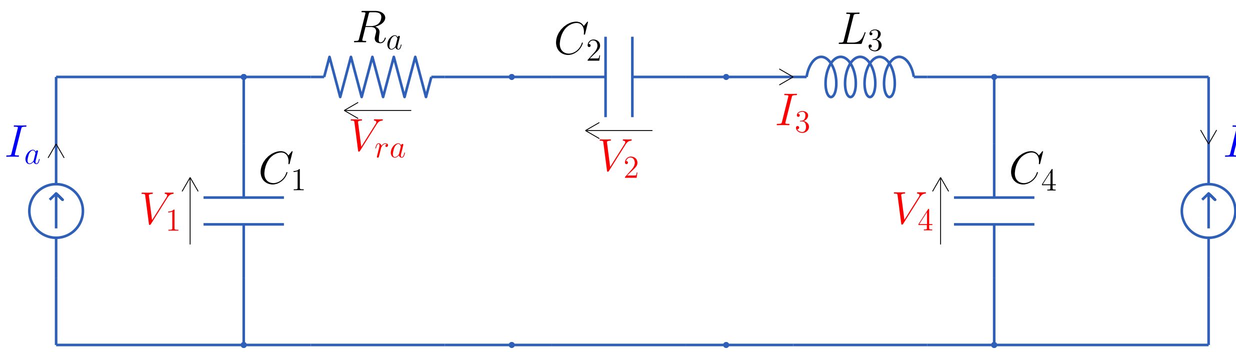
c) Add the dynamic blocks to the POG scheme:



d) Add the dissipative blocks and the summation signs to the POG scheme:

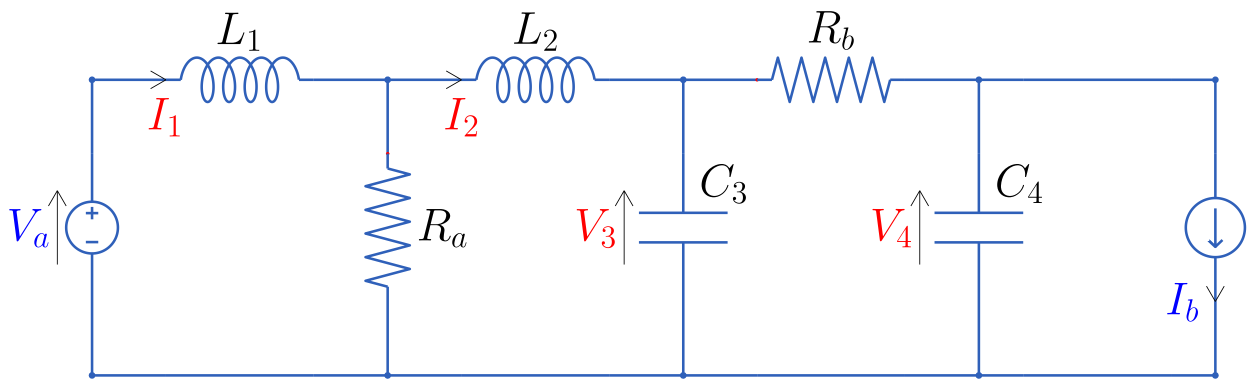


e) Check the signs by comparing with the considered physical system:

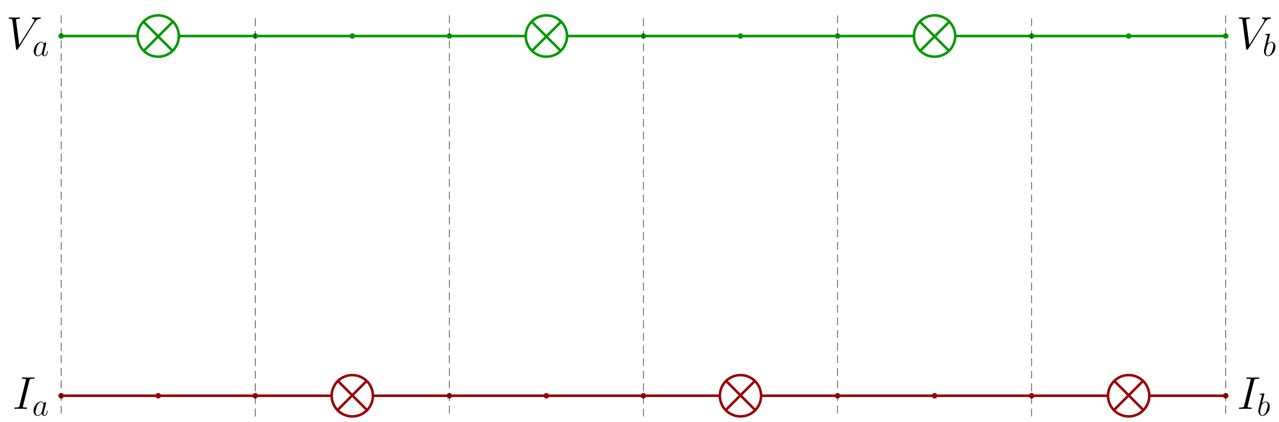


POG direct modeling: simple examples

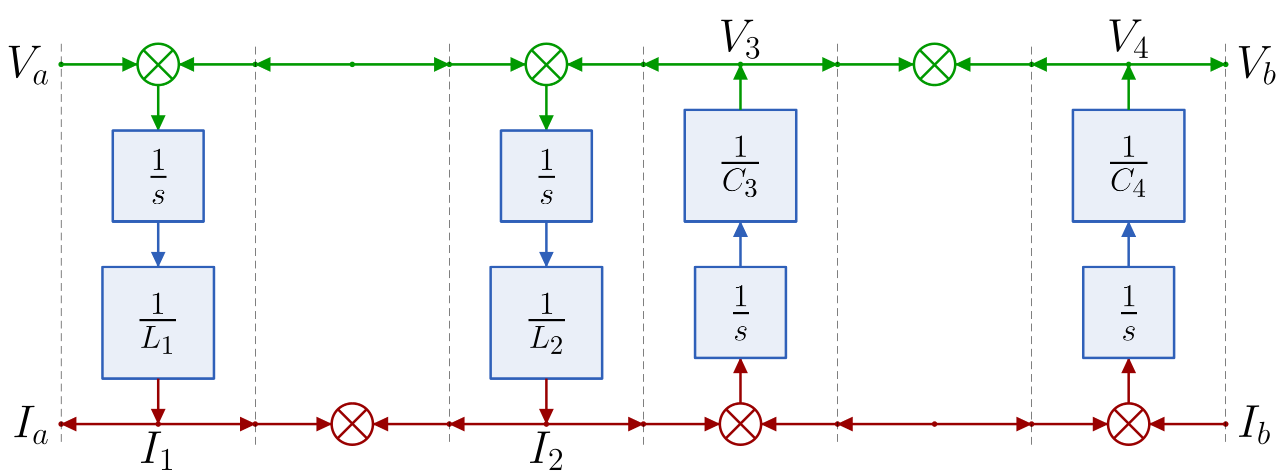
1.a) Choose the positive directions of the state and input variables of the considered physical system:



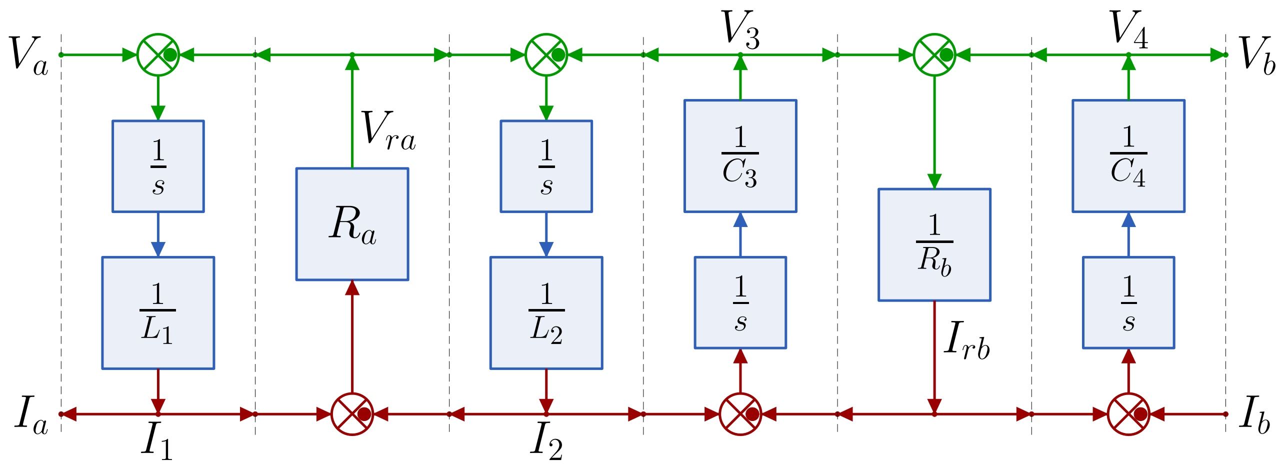
1.b) Draw the series/parallel structure of the POG block scheme:



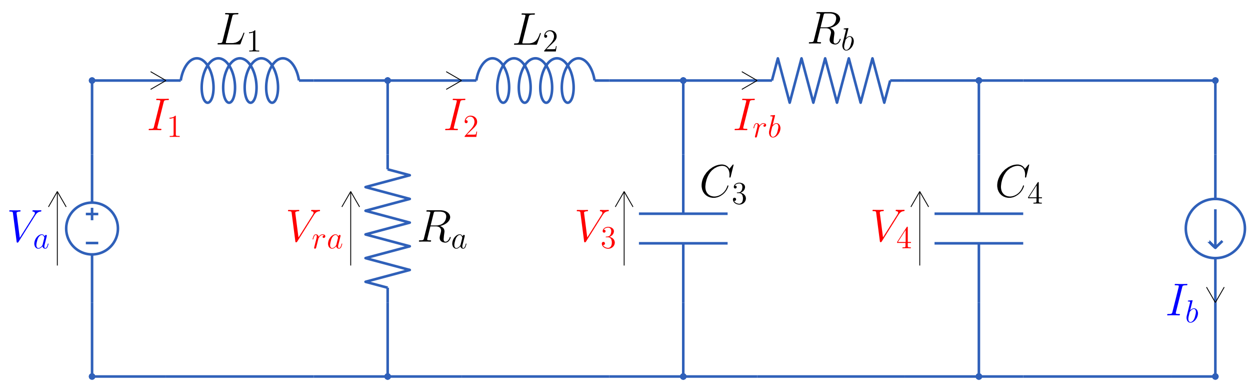
1.c) Add the dynamic blocks to the POG scheme:



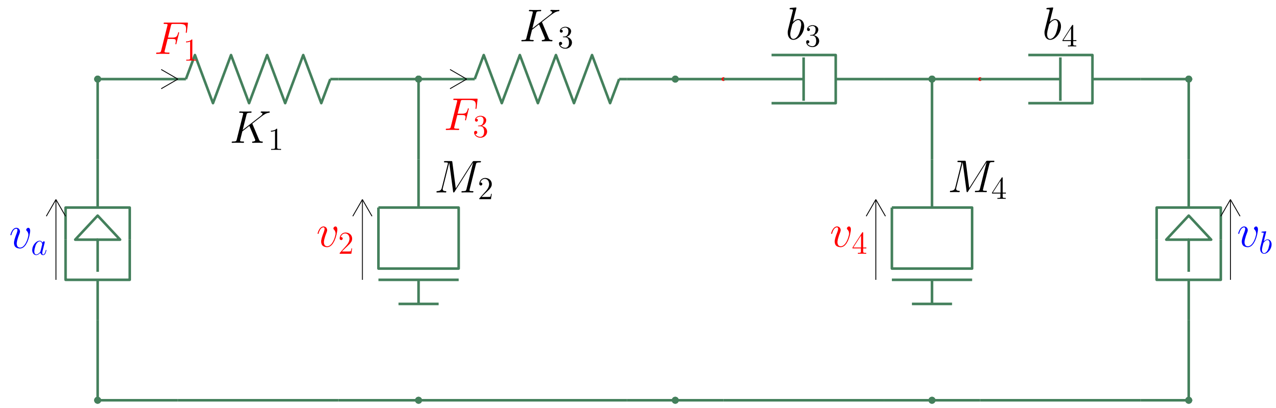
1.d) Add the dissipative blocks and the summation signs to the POG scheme:



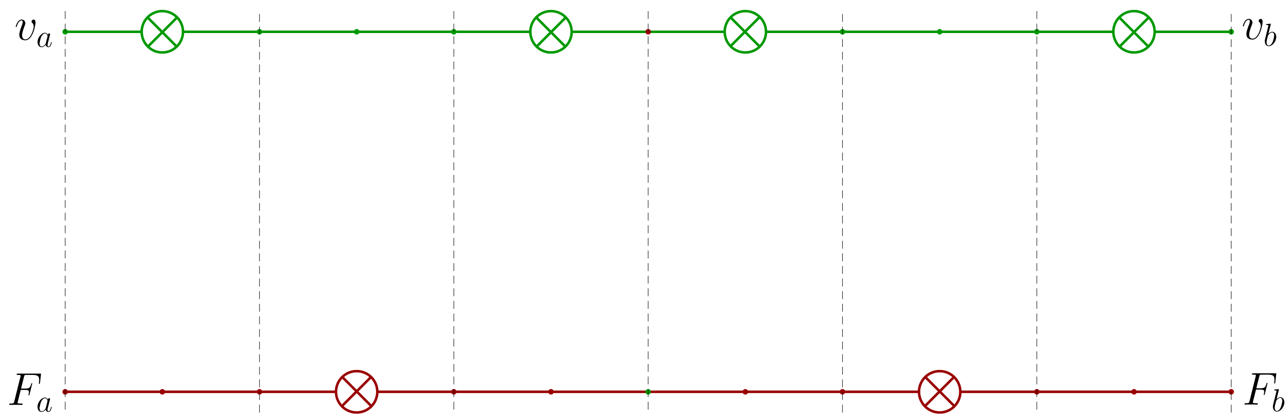
1.e) Check the signs by comparing with the considered physical system:



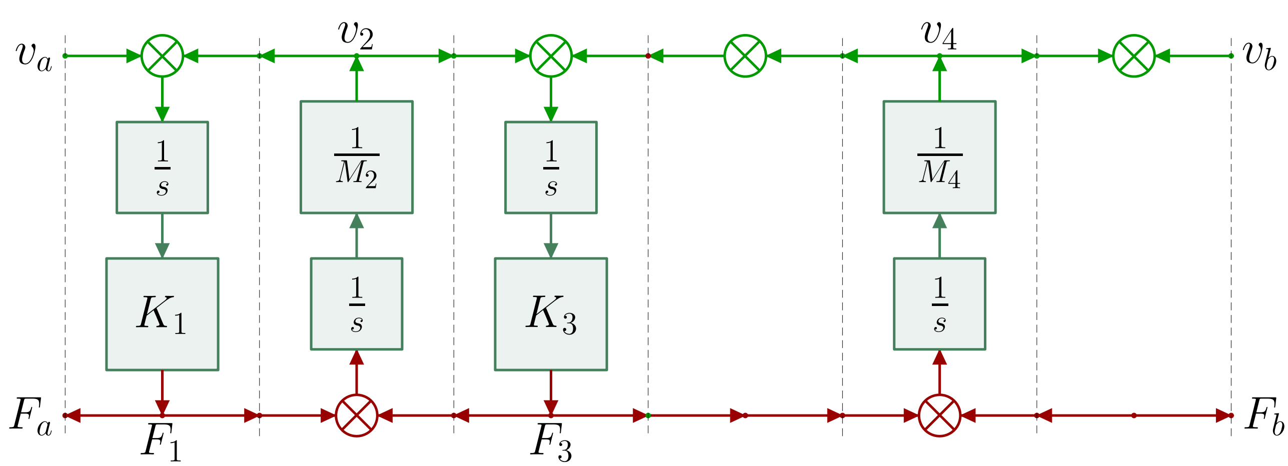
2.a) Choose the positive directions of the state and input variables of the considered physical system:



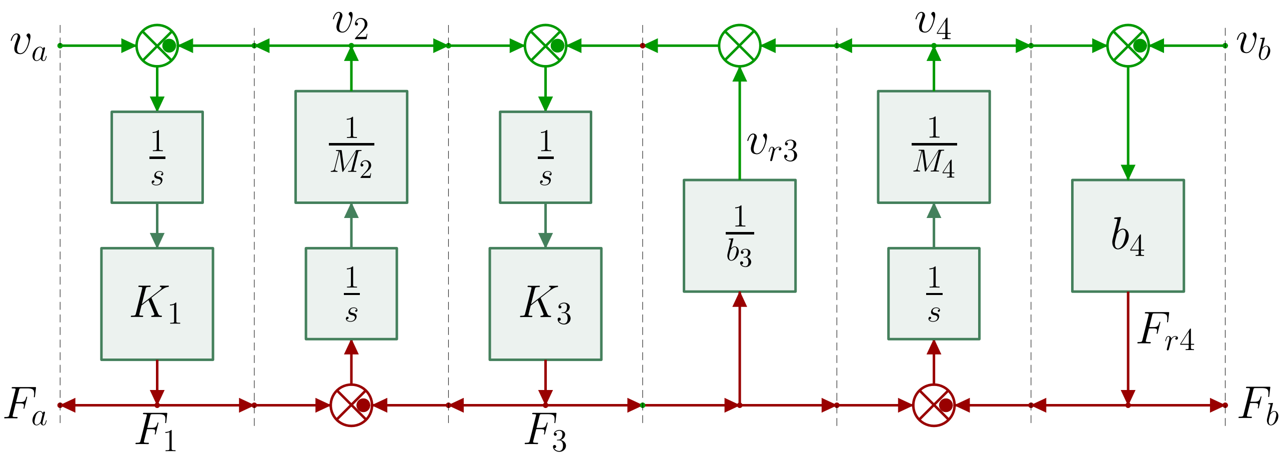
2.b) Draw the series/parallel structure of the POG block scheme:



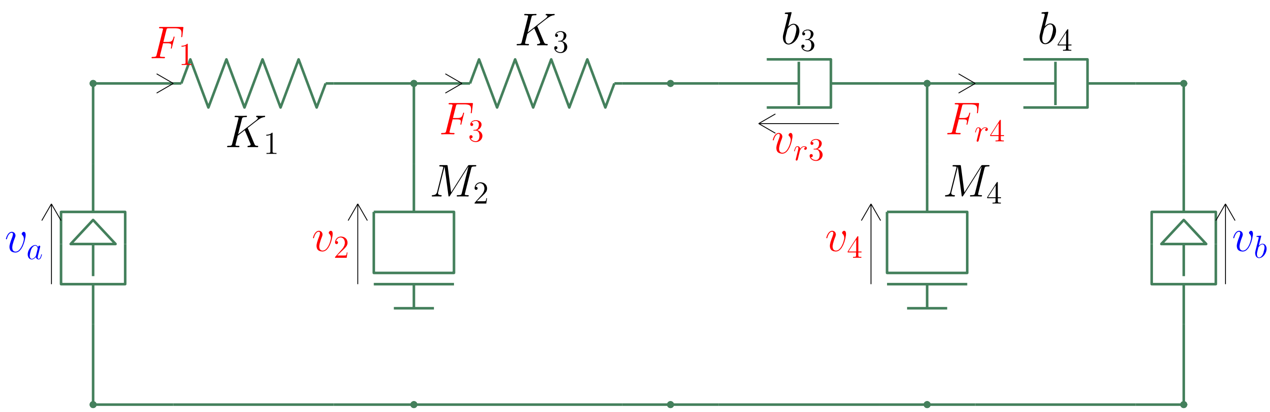
2.c) Add the dynamic blocks to the POG scheme:



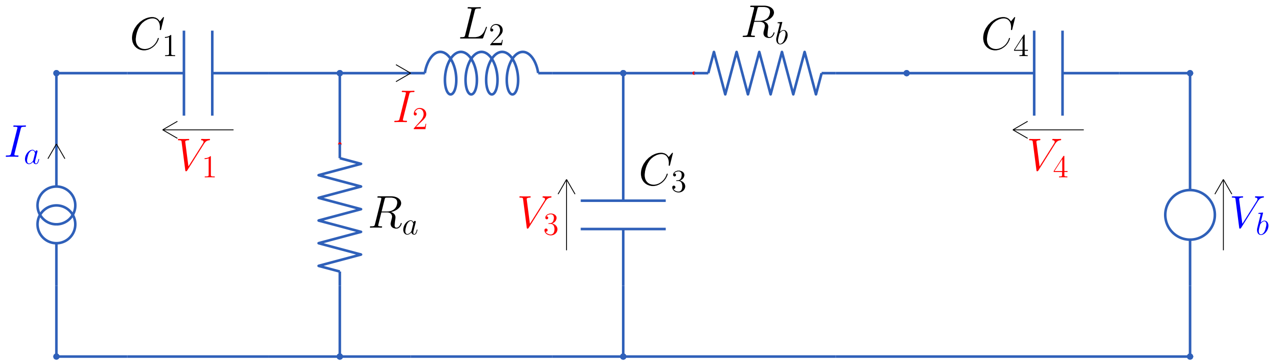
2.d) Add the dissipative blocks and the summation signs to the POG scheme:



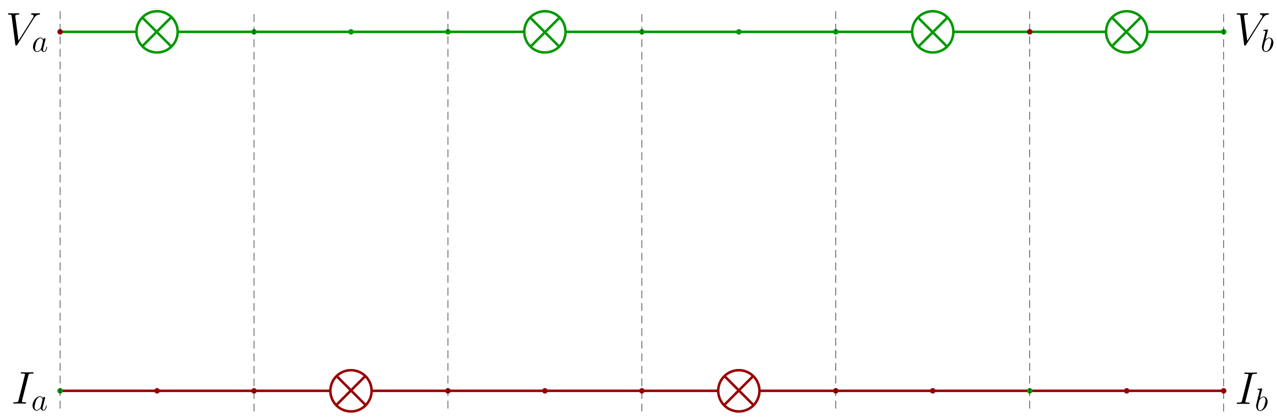
2.e) Check the signs by comparing with the considered physical system:



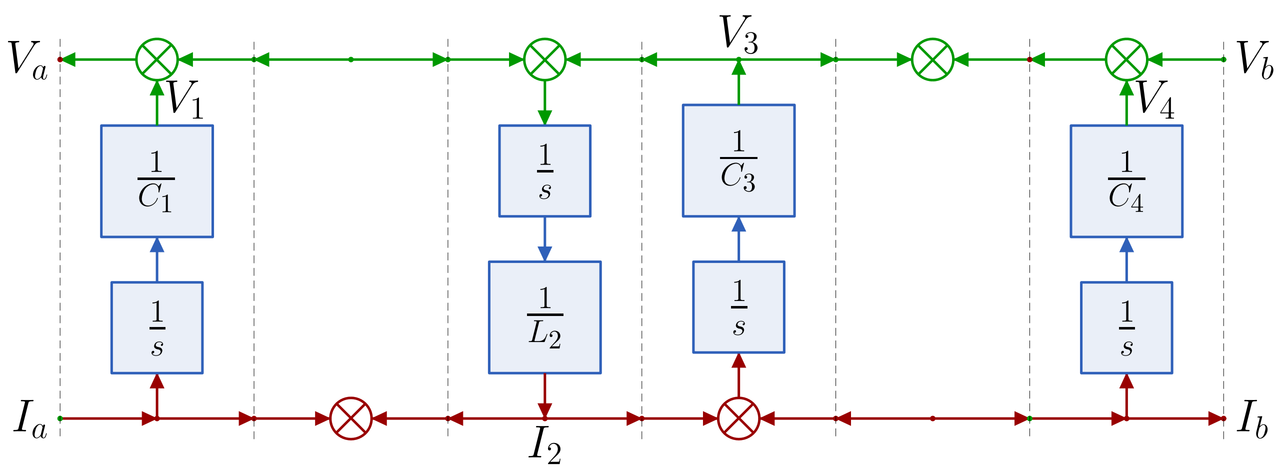
3.a) Choose the positive directions of the state and input variables of the considered physical system:



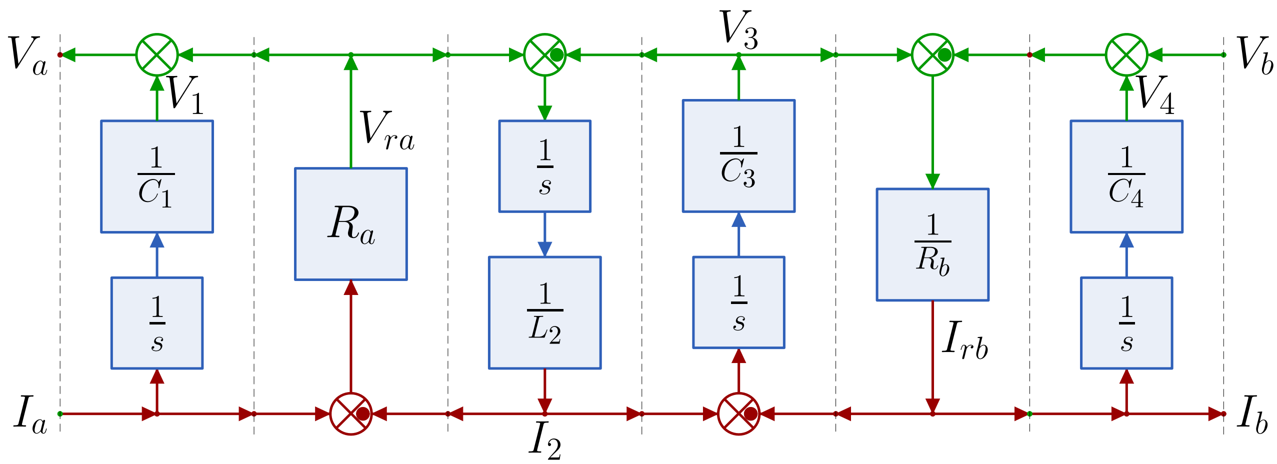
3.b) Draw the series/parallel structure of the POG block scheme:



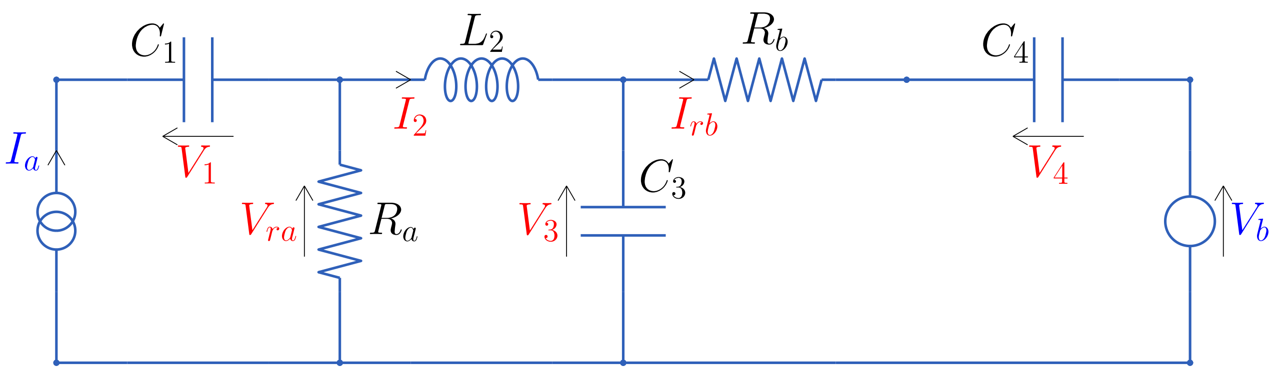
3.c) Add the dynamic blocks to the POG scheme:



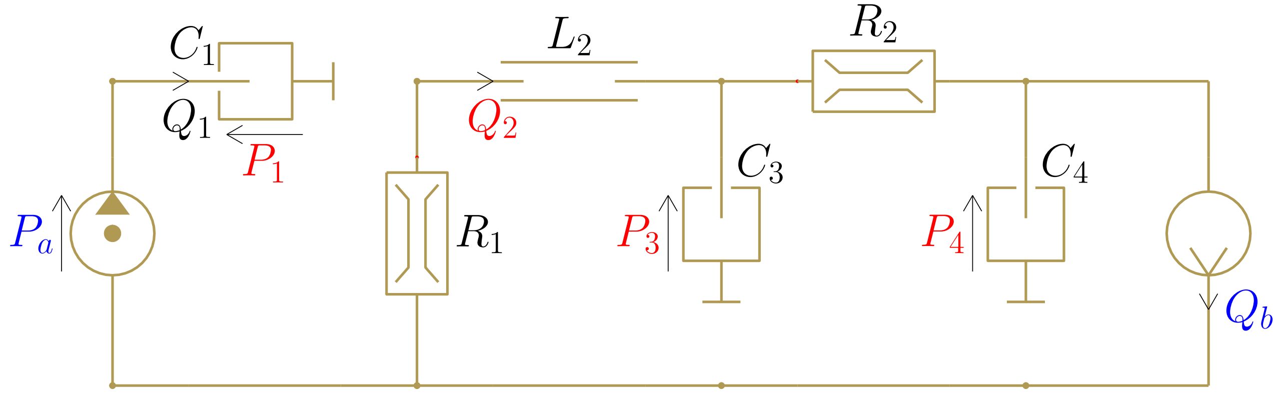
3.d) Add the dissipative blocks and the summation signs to the POG scheme:



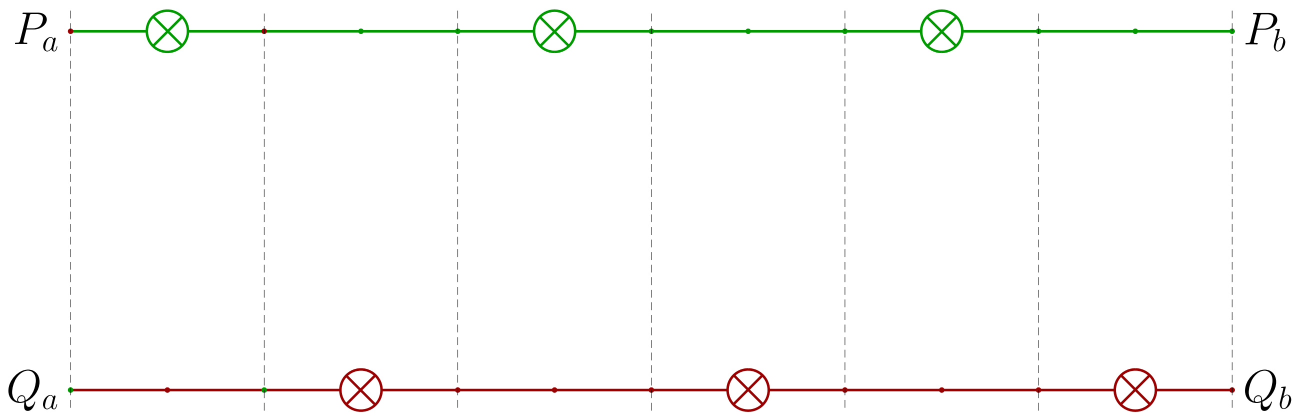
3.e) Check the signs by comparing with the considered physical system:



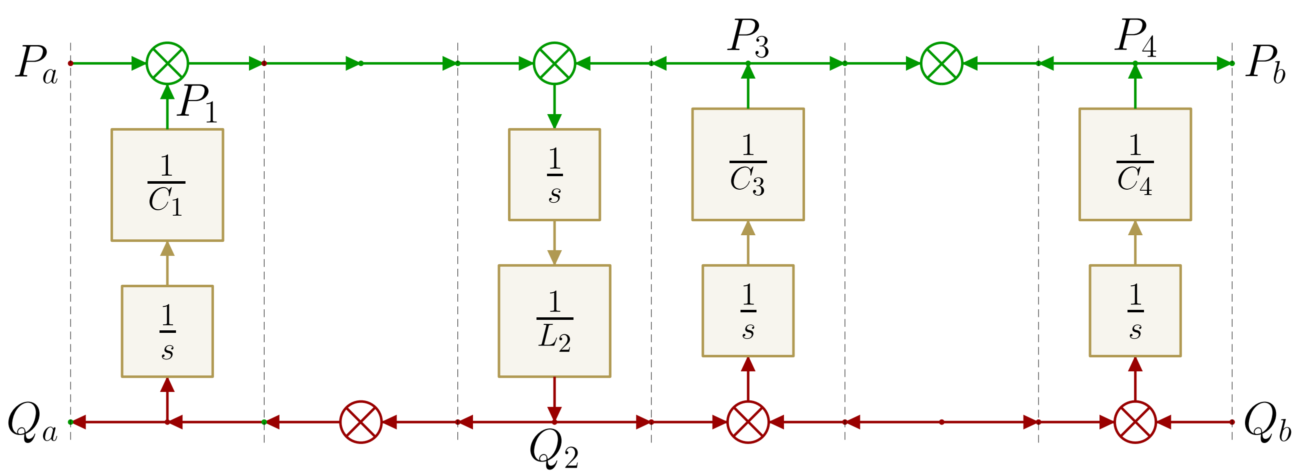
4.a) Choose the positive directions of the state and input variables of the considered physical system:



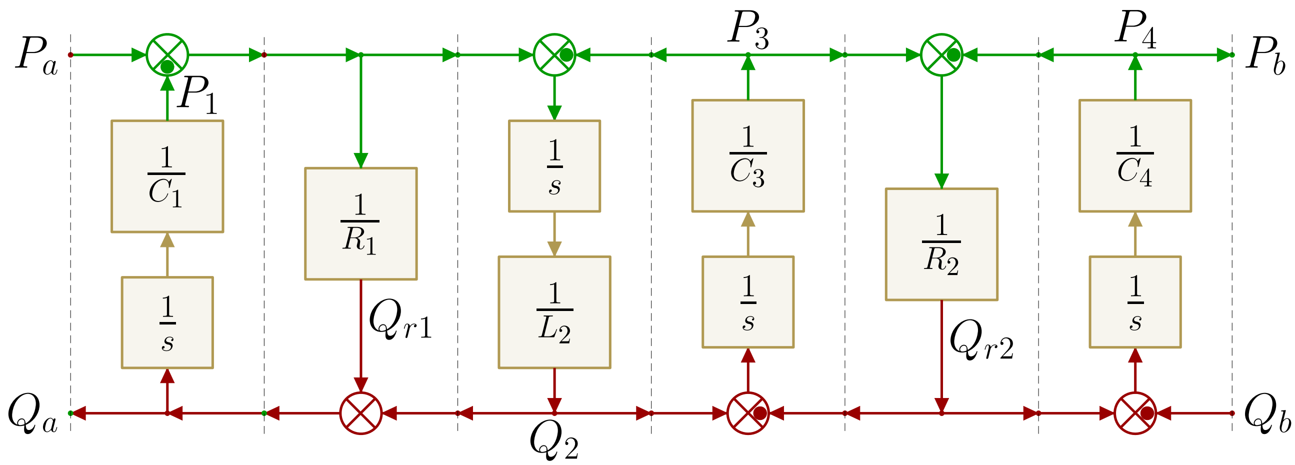
4.b) Draw the series/parallel structure of the POG block scheme:



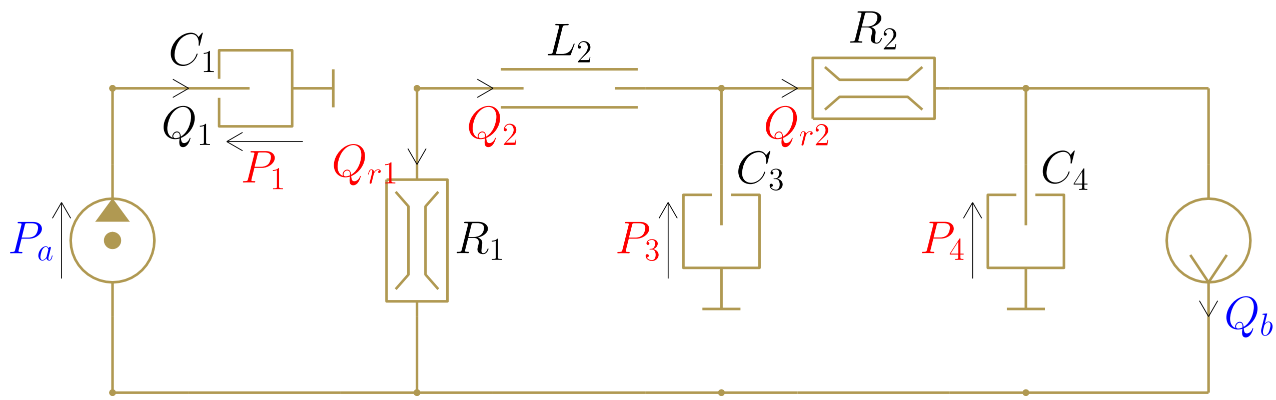
4.c) Add the dynamic blocks to the POG scheme:



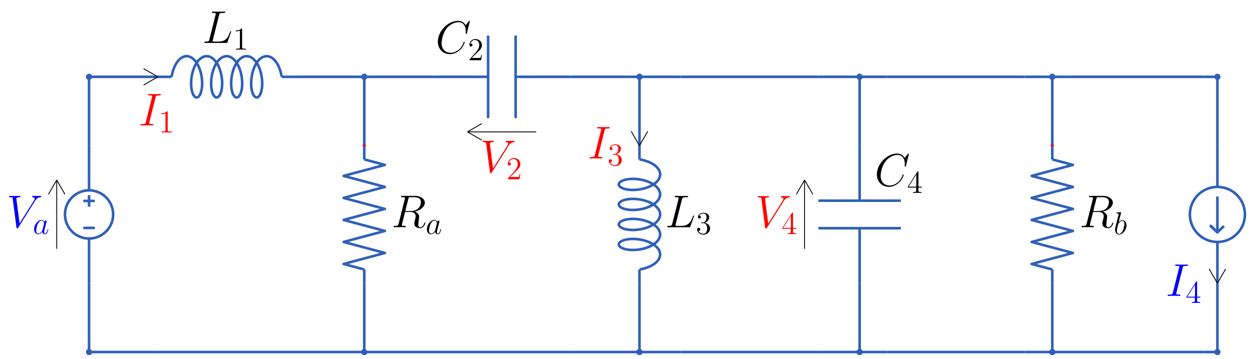
4.d) Add the dissipative blocks and the summation signs to the POG scheme:



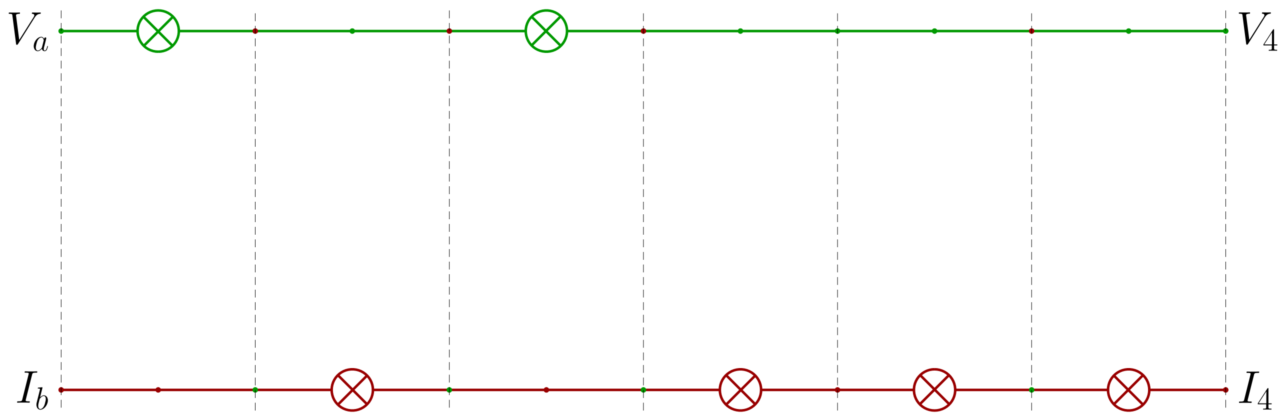
4.e) Check the signs by comparing with the considered physical system:



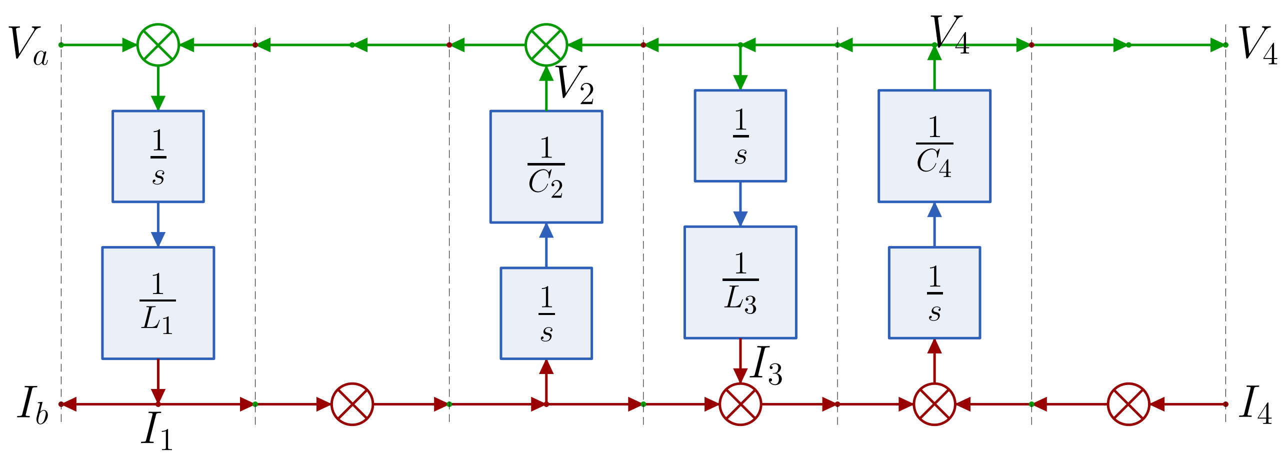
5.a) Choose the positive directions of the state and input variables of the considered physical system:



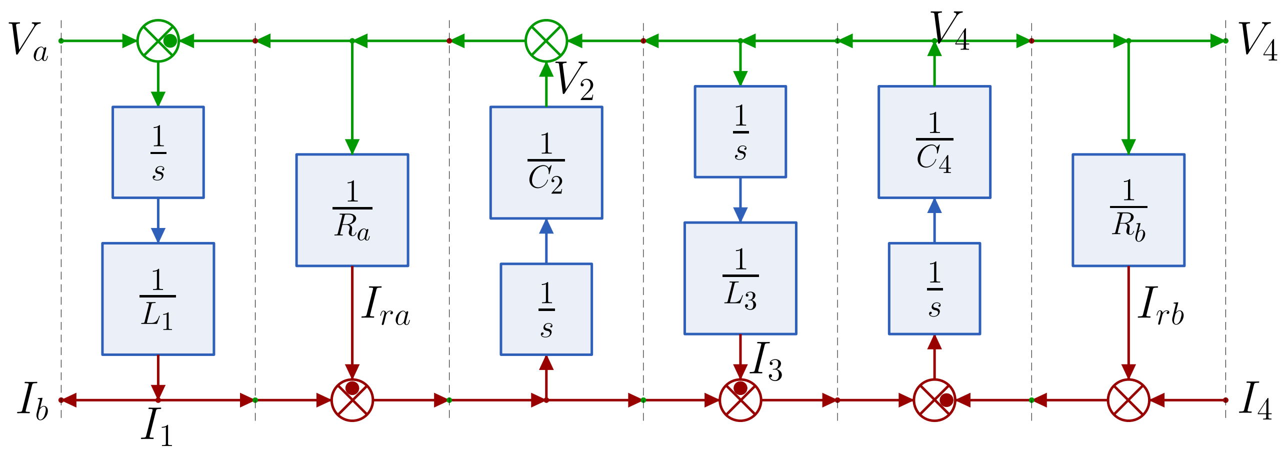
5.b) Draw the series/parallel structure of the POG block scheme:



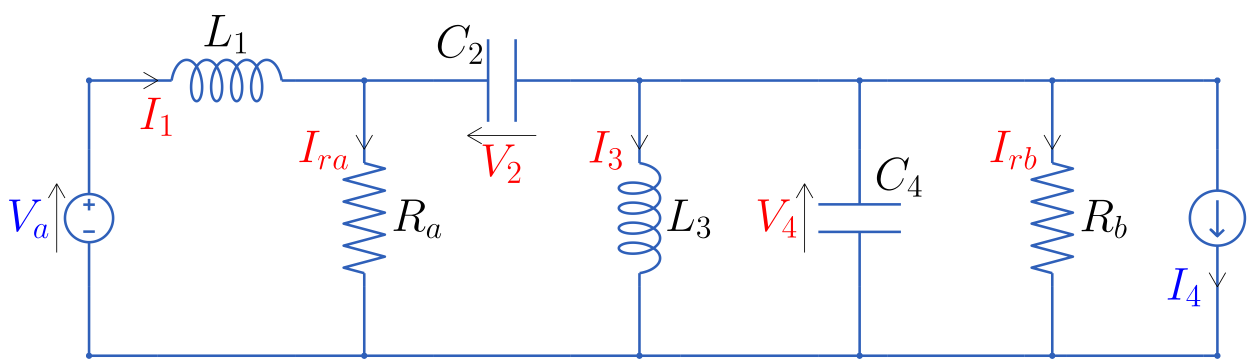
5.c) Add the dynamic blocks to the POG scheme:



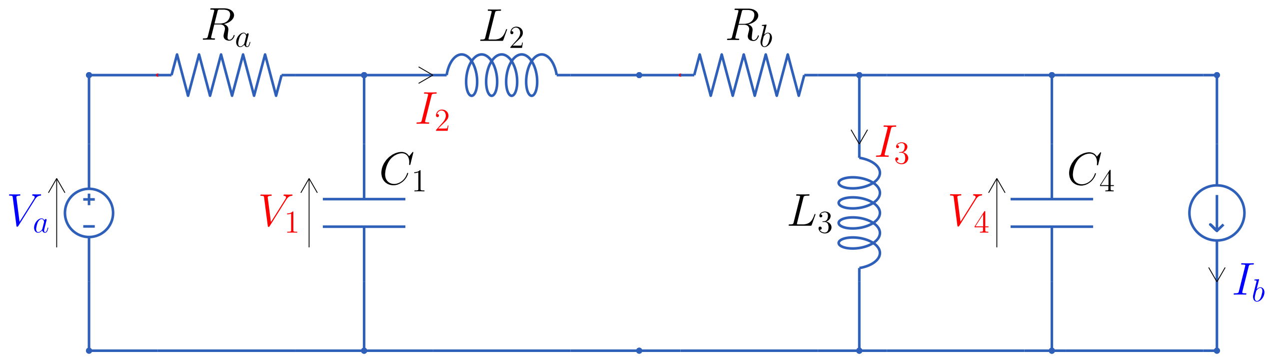
5.d) Add the dissipative blocks and the summation signs to the POG scheme:



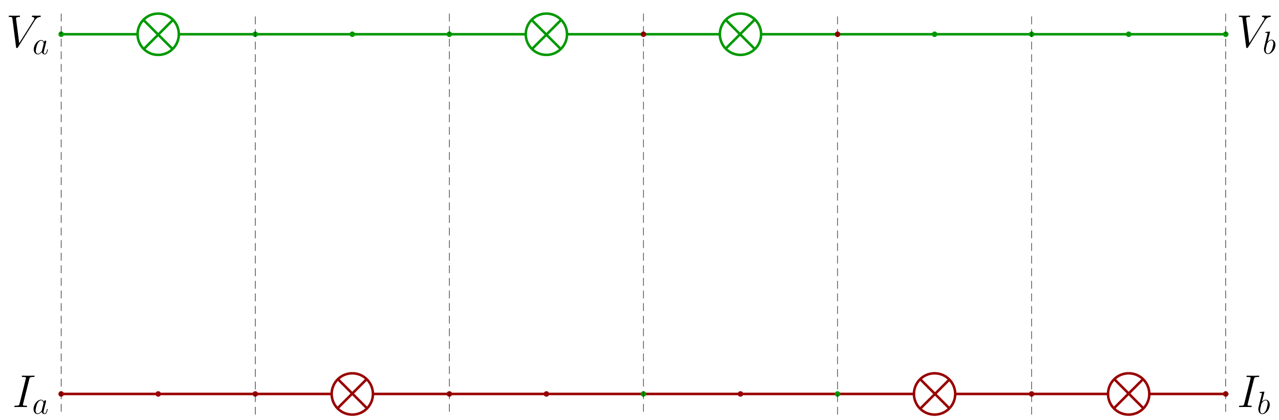
5.e) Check the signs by comparing with the considered physical system:



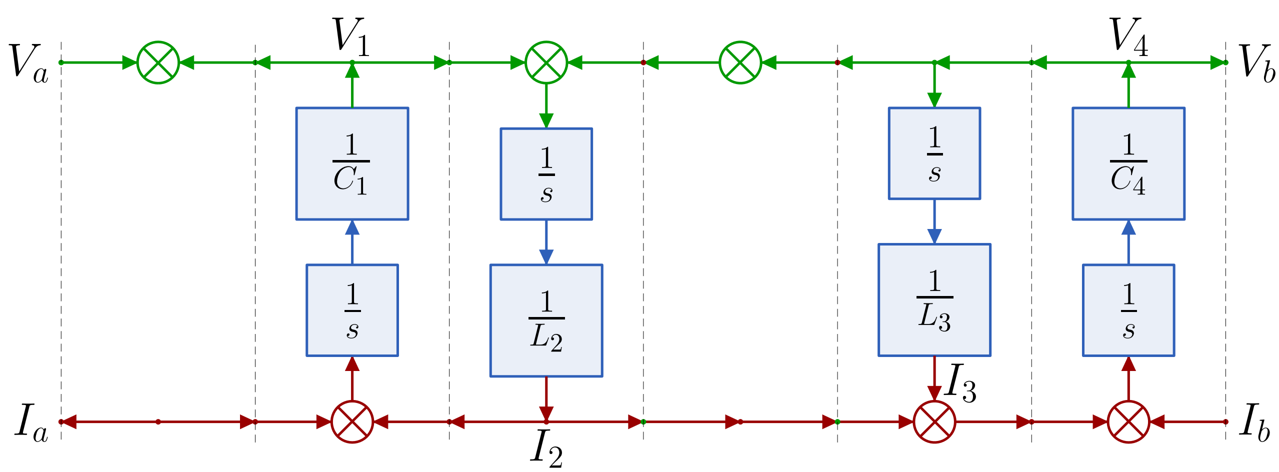
6.a) Choose the positive directions of the state and input variables of the considered physical system:



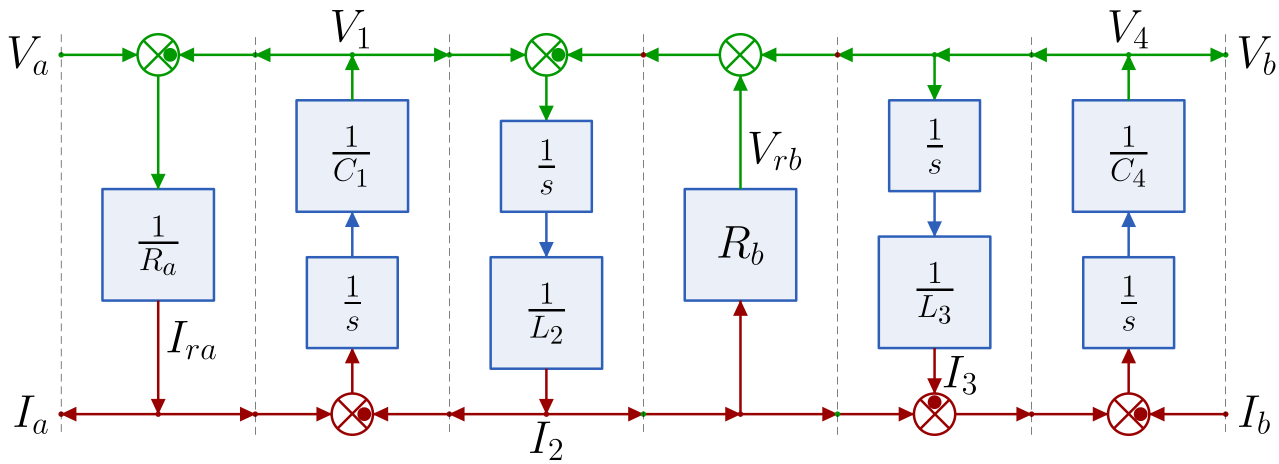
6.b) Draw the series/parallel structure of the POG block scheme:



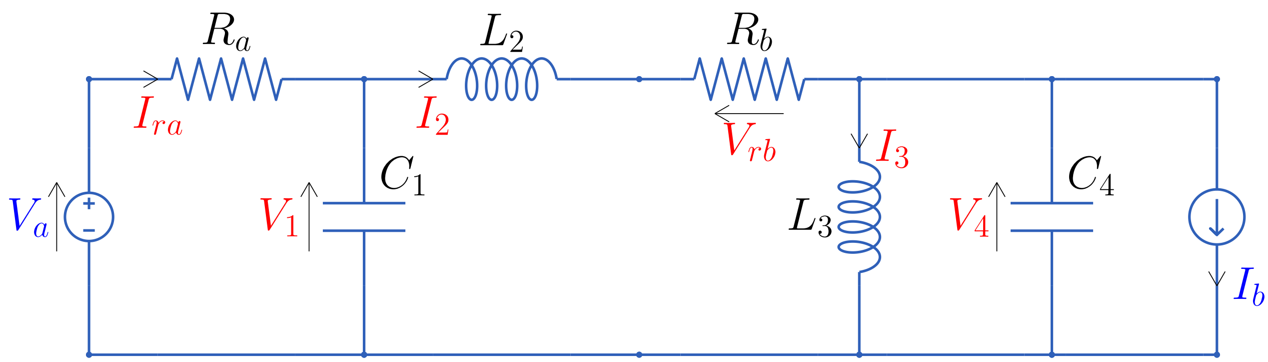
6.c) Add the dynamic blocks to the POG scheme:



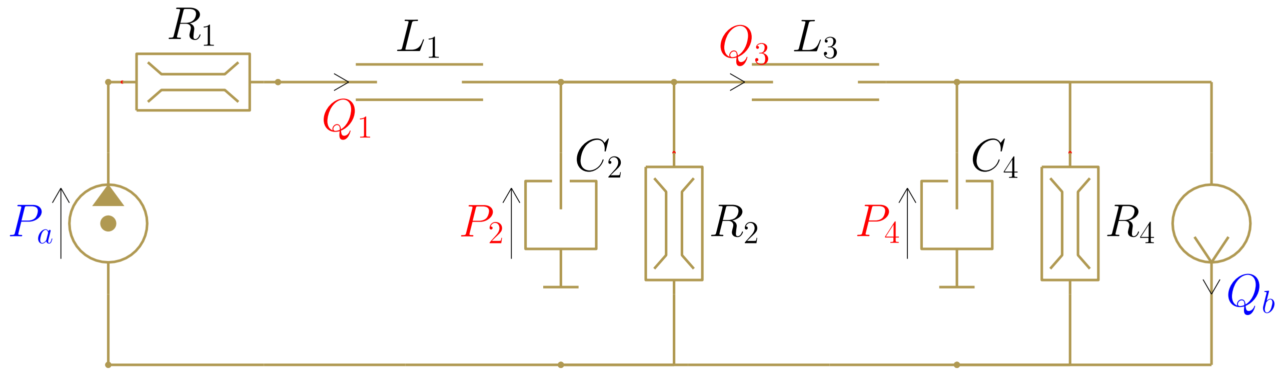
6.d) Add the dissipative blocks and the summation signs to the POG scheme:



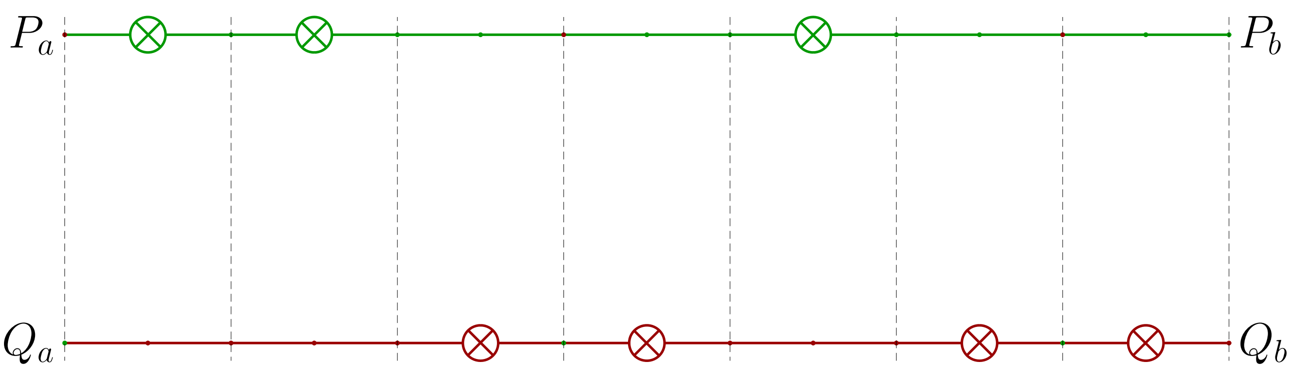
6.e) Check the signs by comparing with the considered physical system:



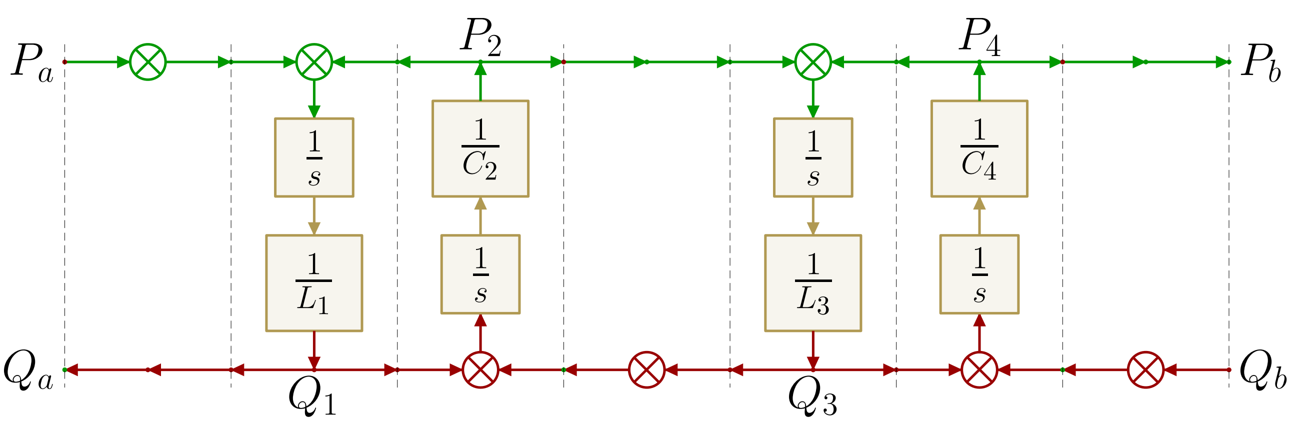
7.a) Choose the positive directions of the state and input variables of the considered physical system:



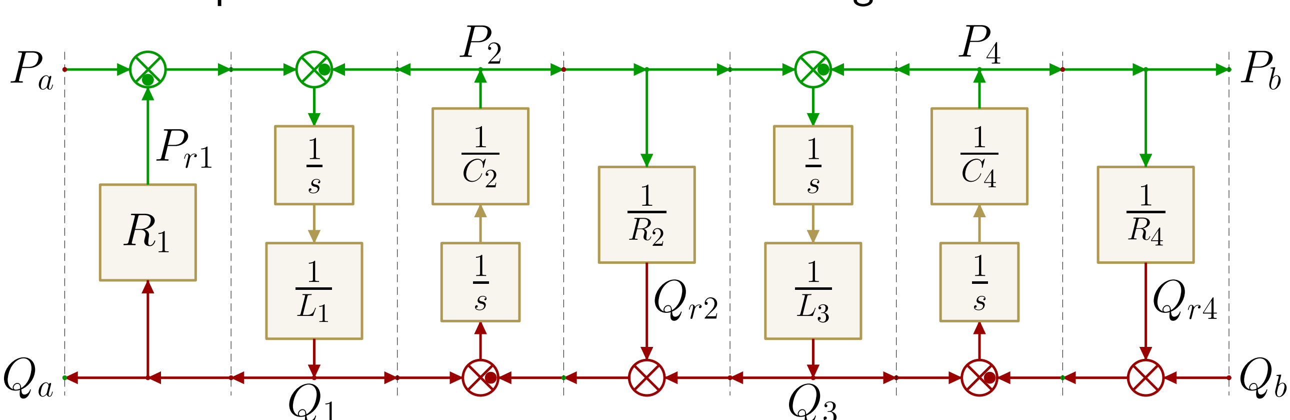
7.b) Draw the series/parallel structure of the POG block scheme:



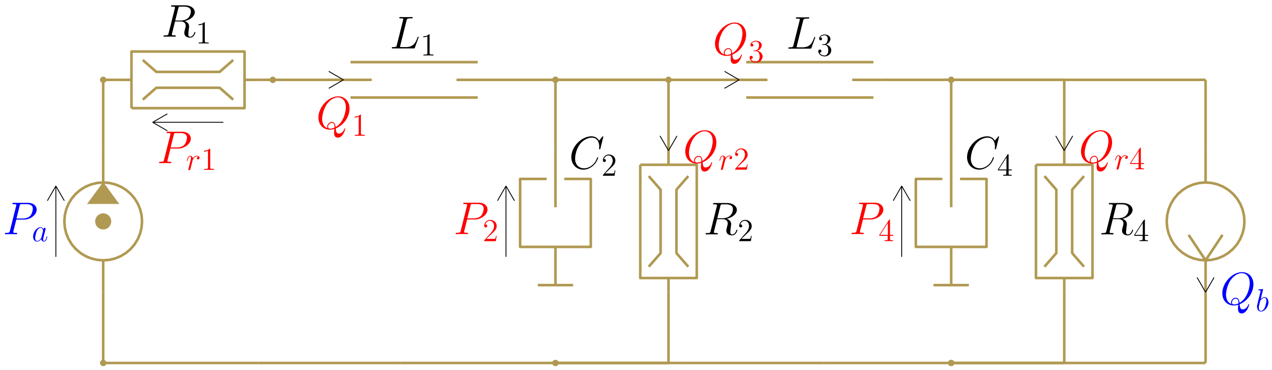
7.c) Add the dynamic blocks to the POG scheme:



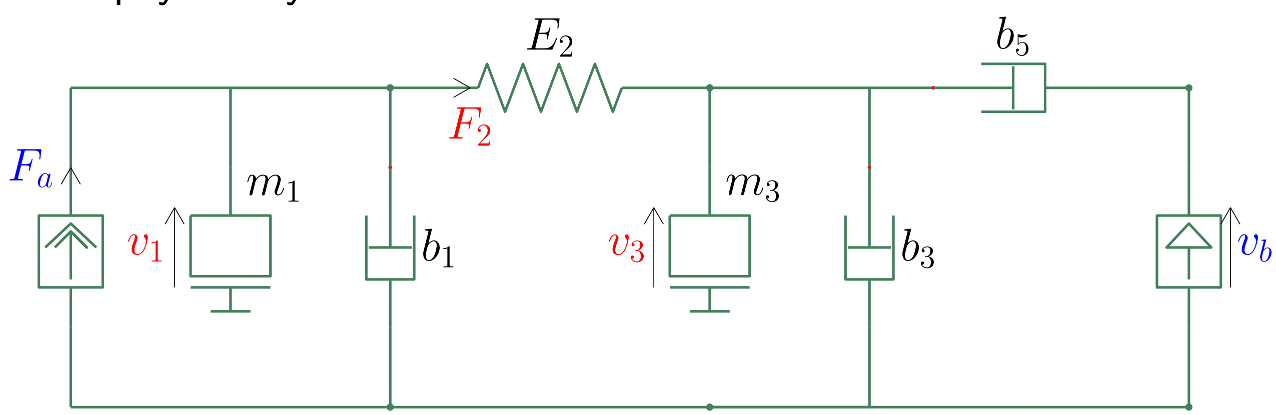
7.d) Add the dissipative blocks and the summation signs to the POG scheme:



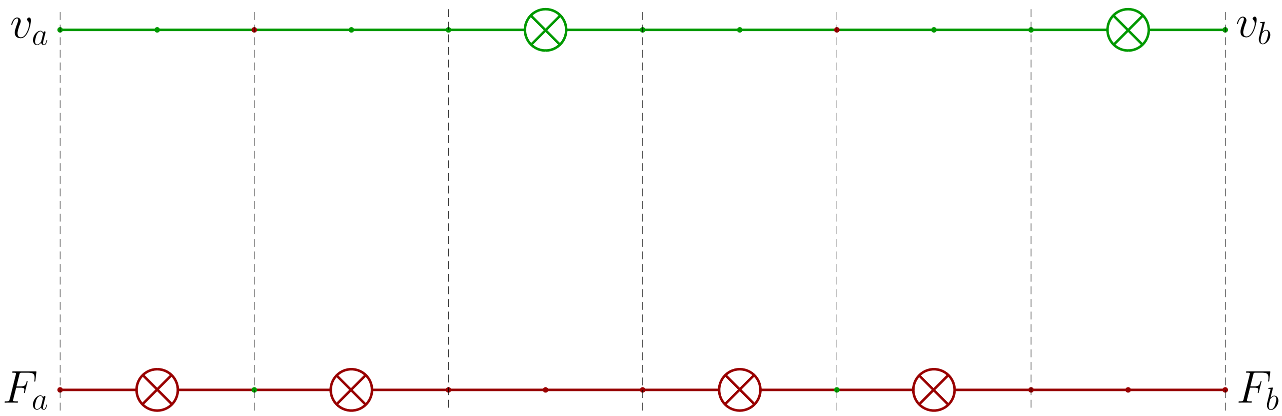
7.e) Check the signs by comparing with the considered physical system:



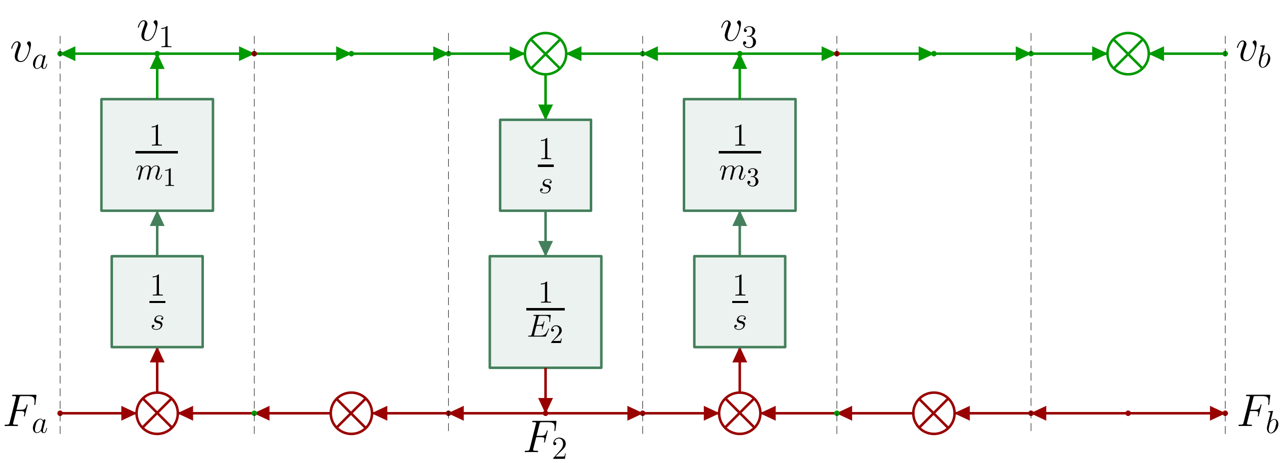
8.a) Choose the positive directions of the state and input variables of the considered physical system:



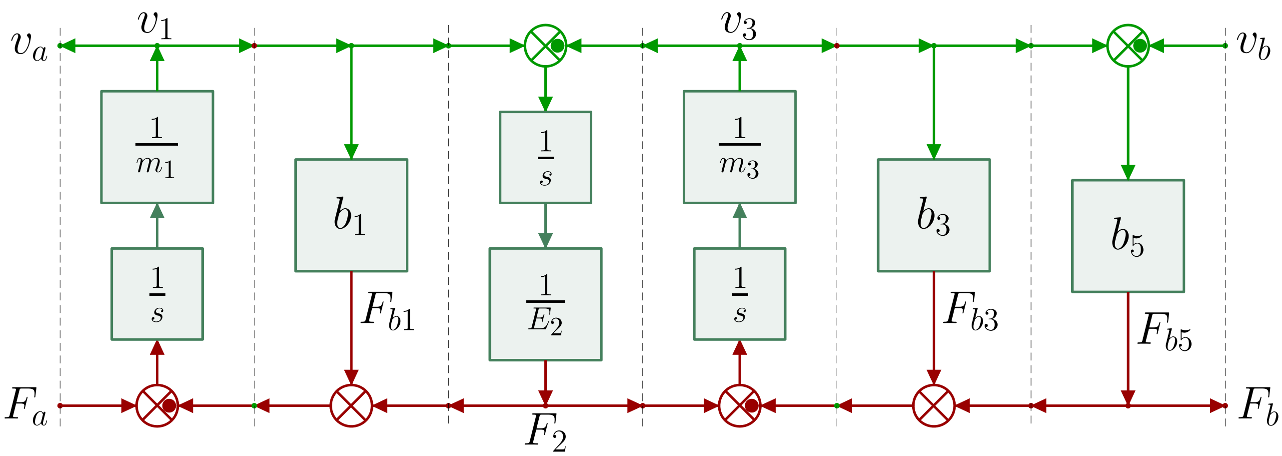
8.b) Draw the series/parallel structure of the POG block scheme:



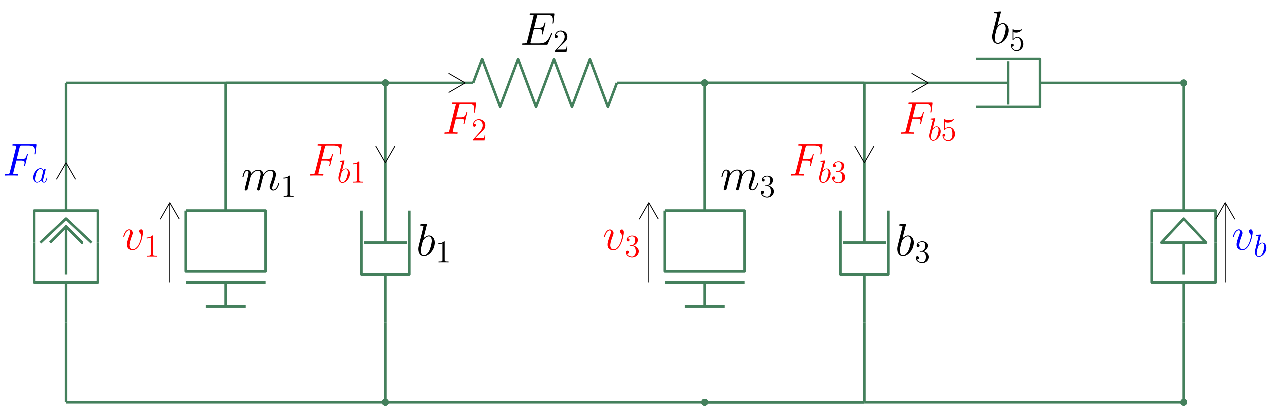
8.c) Add the dynamic blocks to the POG scheme:



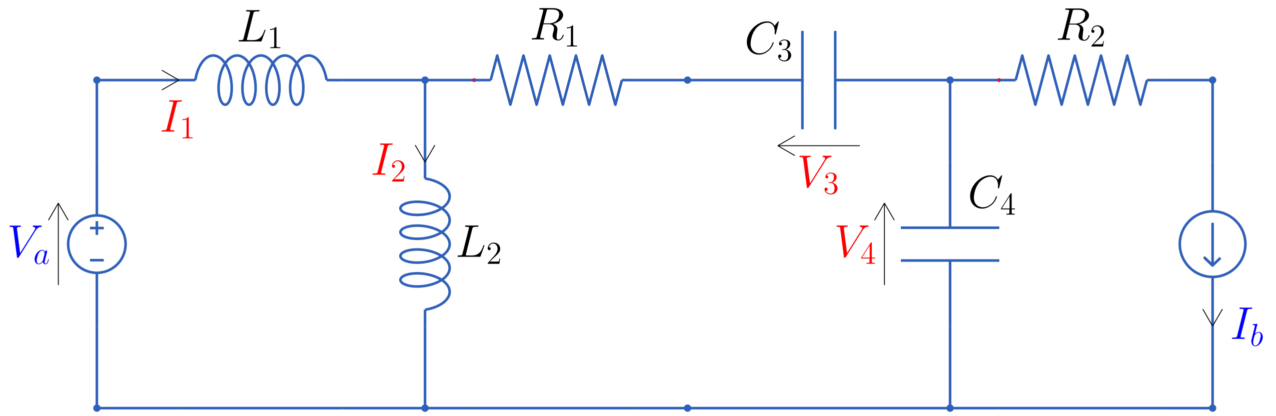
8.d) Add the dissipative blocks and the summation signs to the POG scheme:



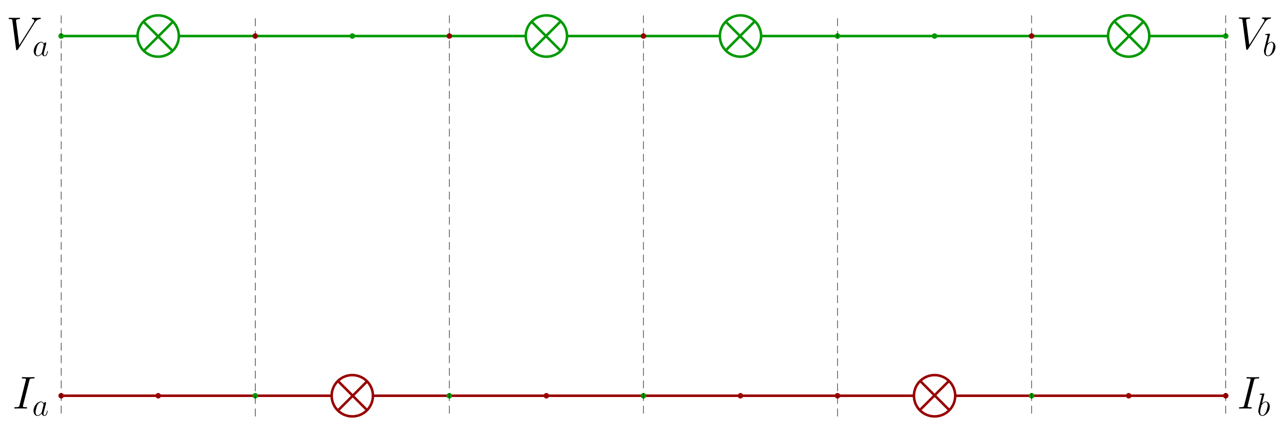
8.e) Check the signs by comparing with the considered physical system:



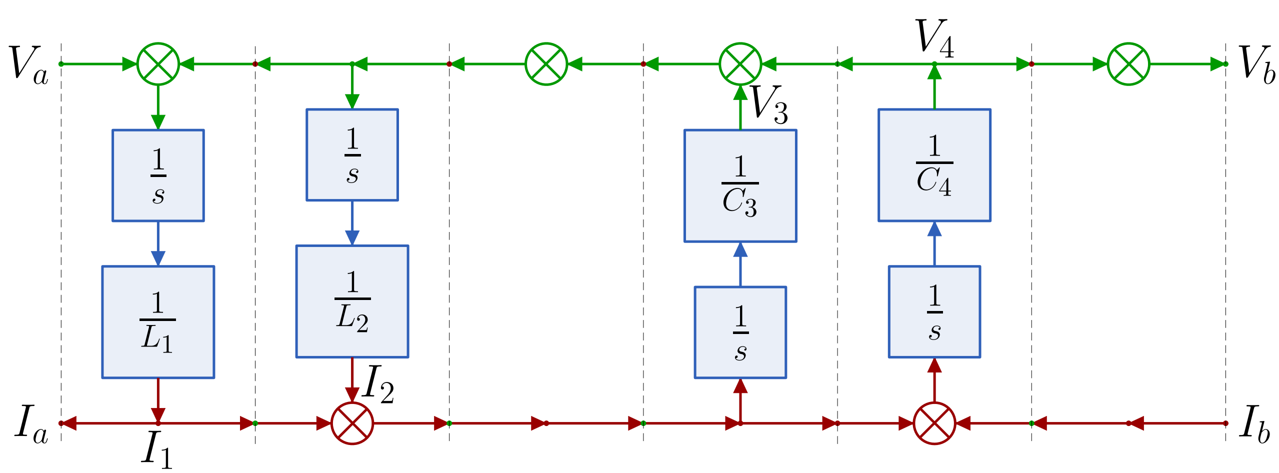
9.a) Choose the positive directions of the state and input variables of the considered physical system:



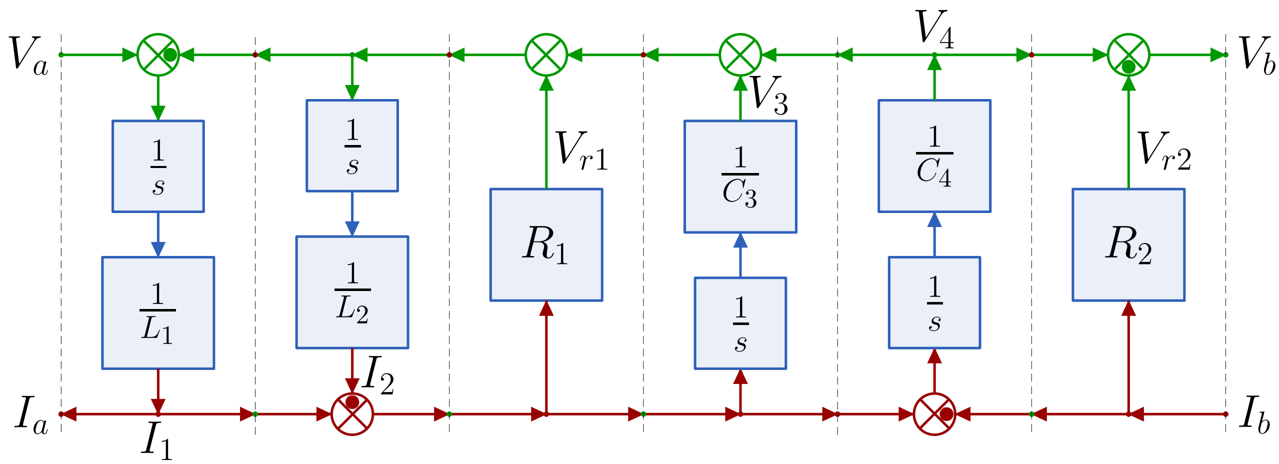
9.b) Draw the series/parallel structure of the POG block scheme:



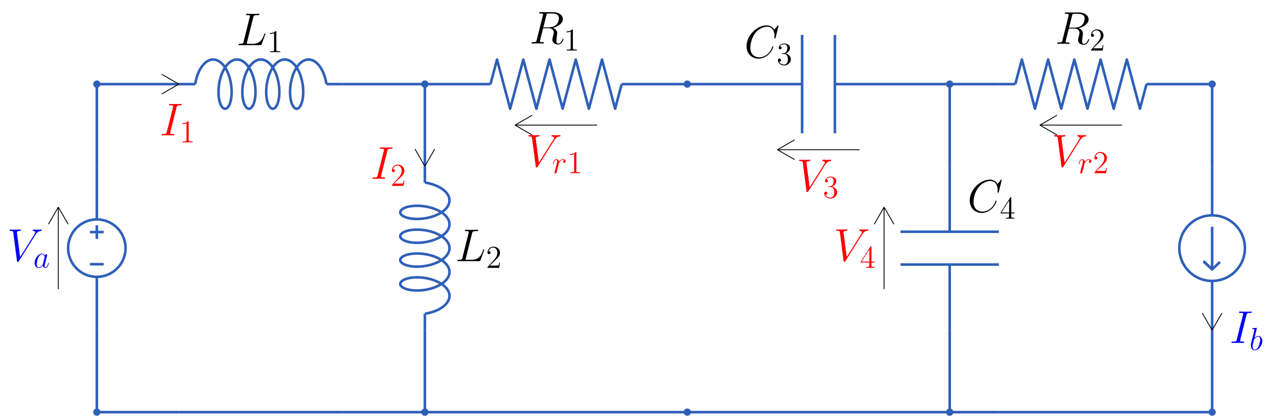
9.c) Add the dynamic blocks to the POG scheme:



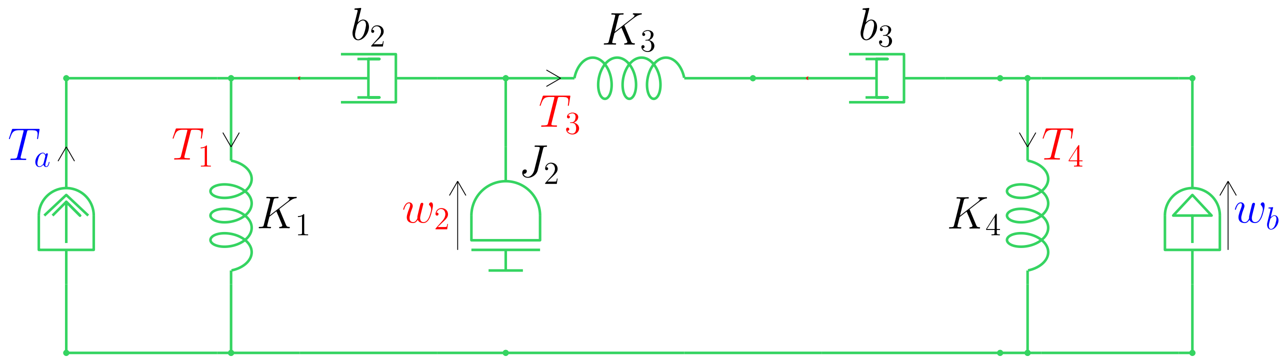
9.d) Add the dissipative blocks and the summation signs to the POG scheme:



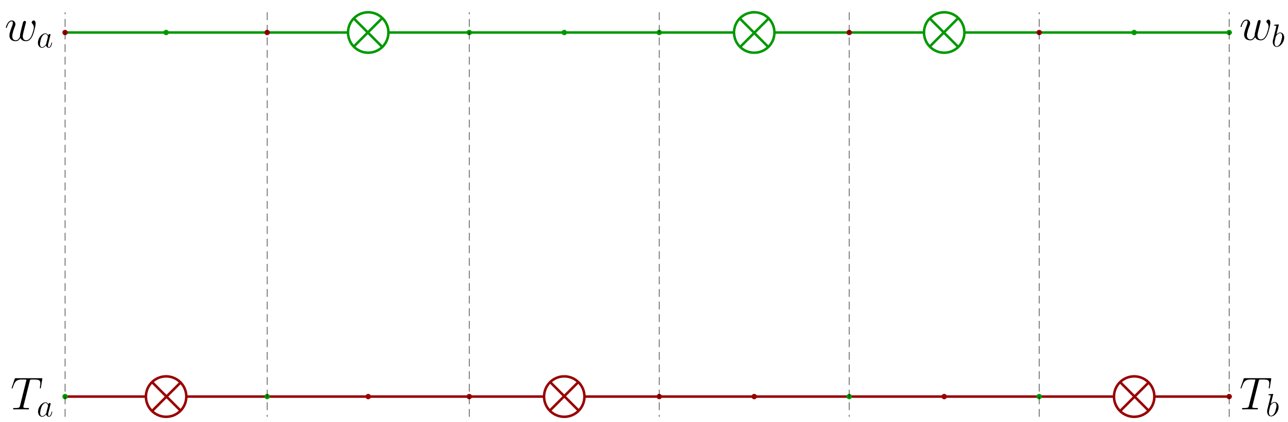
9.e) Check the signs by comparing with the considered physical system:



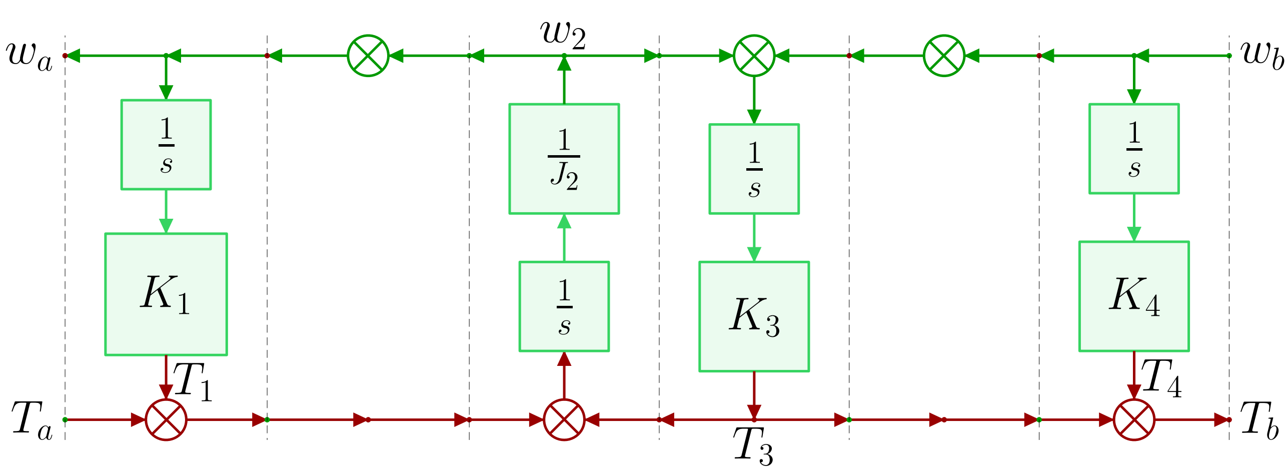
10.a) Choose the positive directions of the state and input variables of the considered physical system:



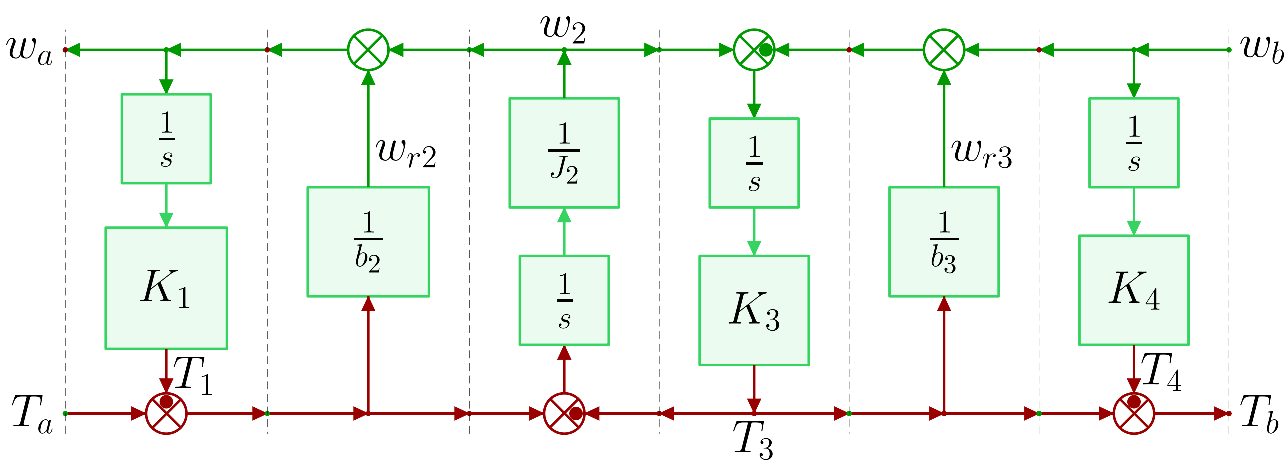
10.b) Draw the series/parallel structure of the POG block scheme:



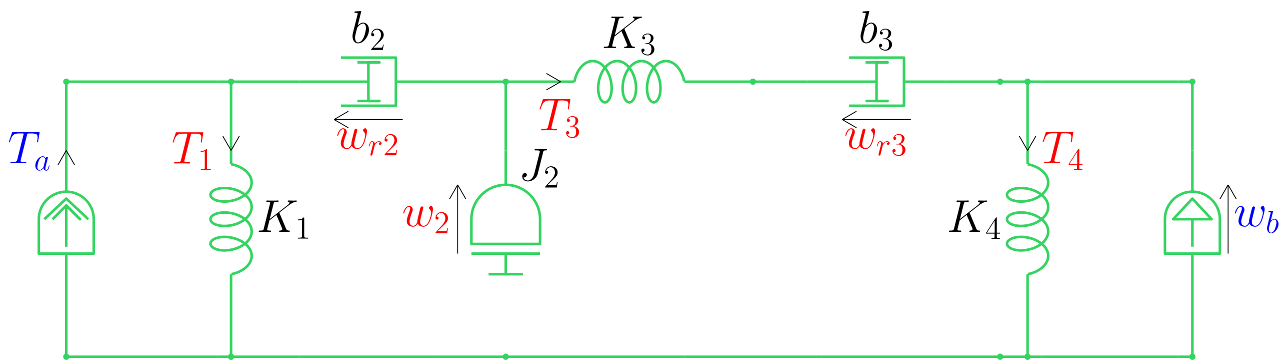
10.c) Add the dynamic blocks to the POG scheme:



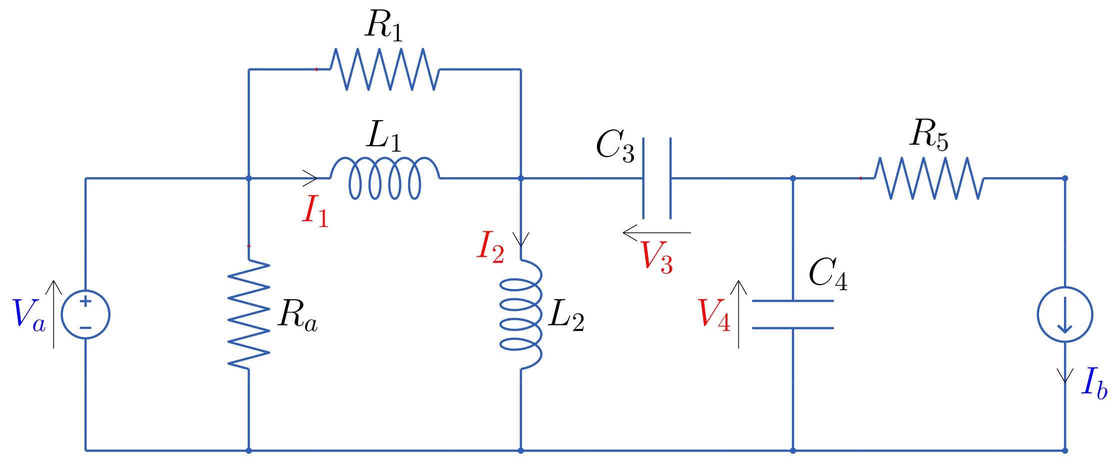
10.d) Add the dissipative blocks and the summation signs to the POG scheme:



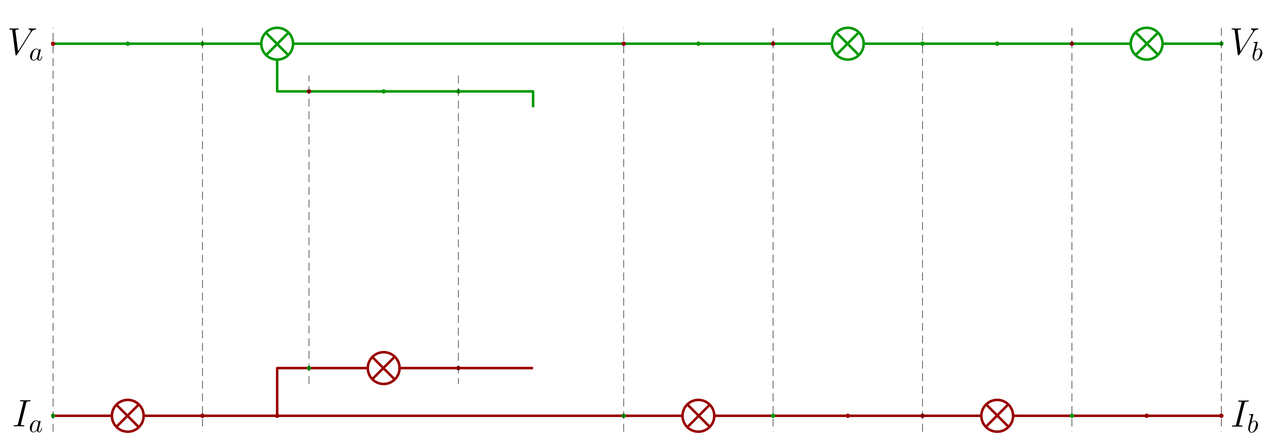
10.e) Check the signs by comparing with the considered physical system:



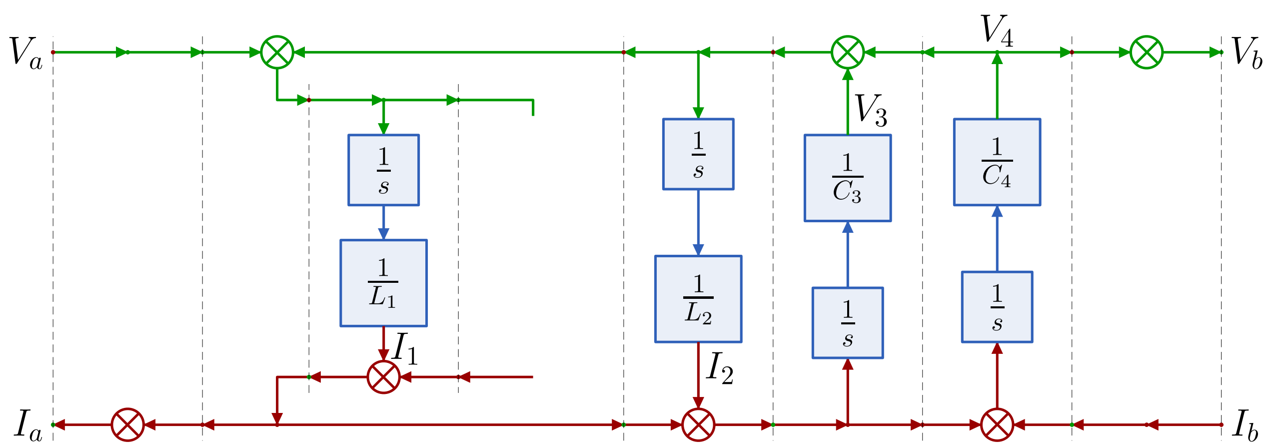
11.a) Choose the positive directions of the state and input variables of the considered physical system:



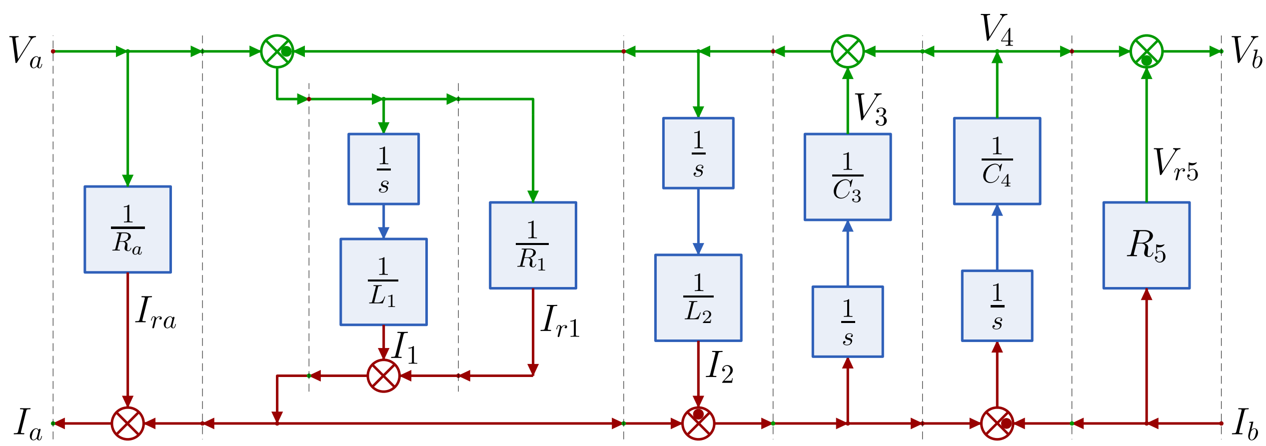
11.b) Draw the series/parallel structure of the POG block scheme:



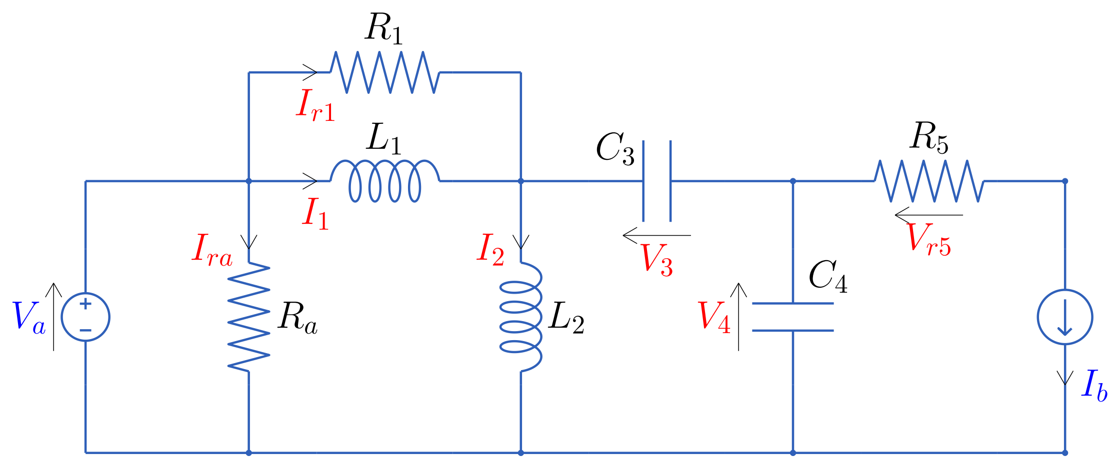
11.c) Add the dynamic blocks to the POG scheme:



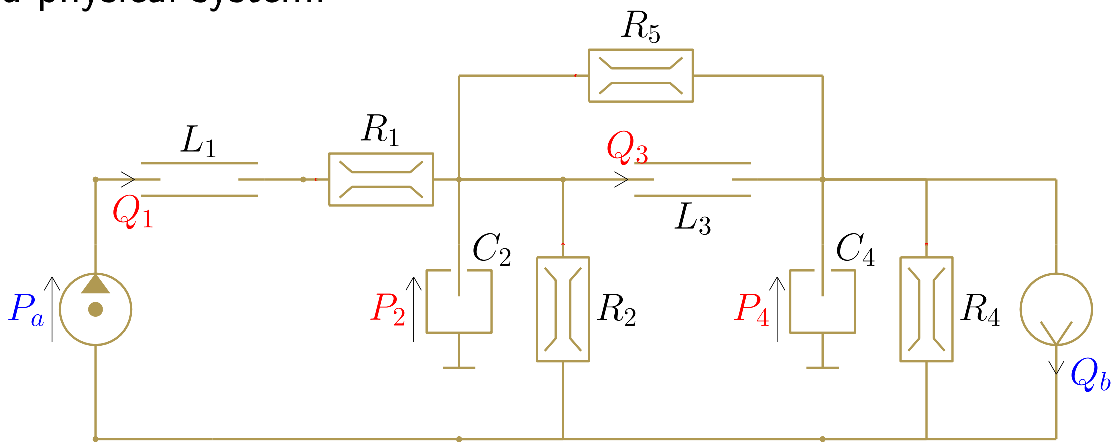
11.d) Add the dissipative blocks and the summation signs to the POG scheme:



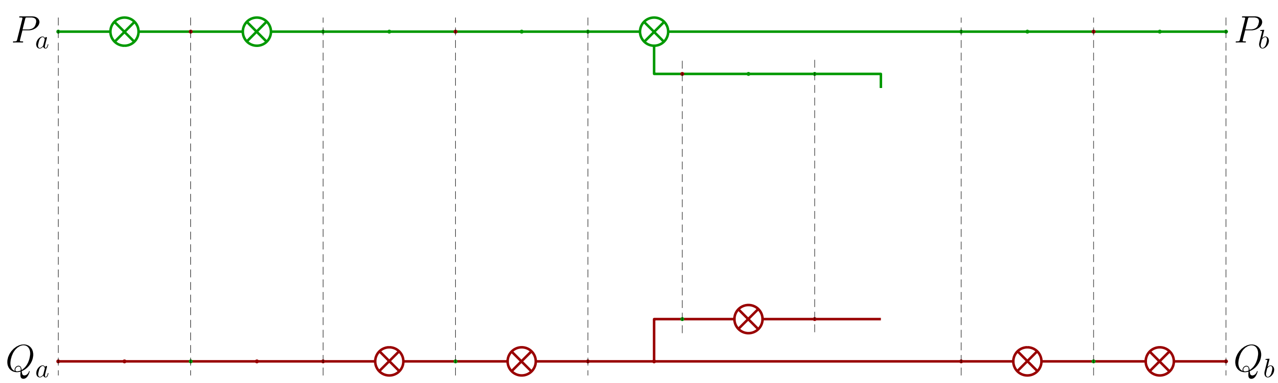
11.e) Check the signs by comparing with the considered physical system:



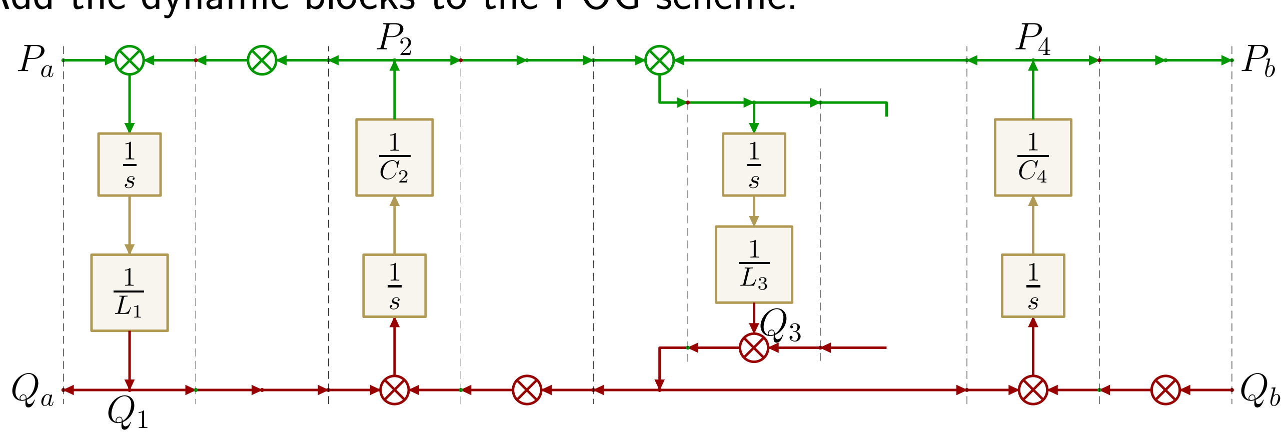
12.a) Choose the positive directions of the state and input variables of the considered physical system:



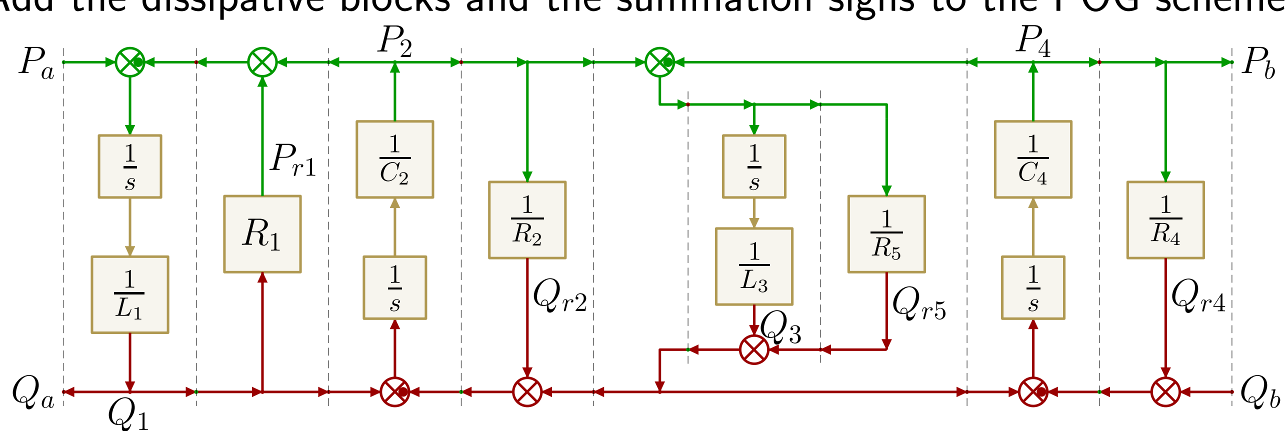
12.b) Draw the series/parallel structure of the POG block scheme:



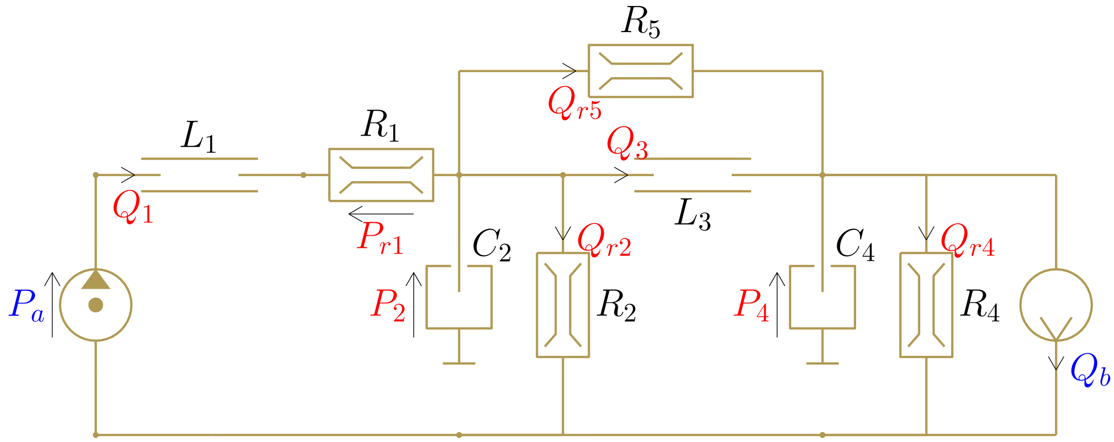
12.c) Add the dynamic blocks to the POG scheme:



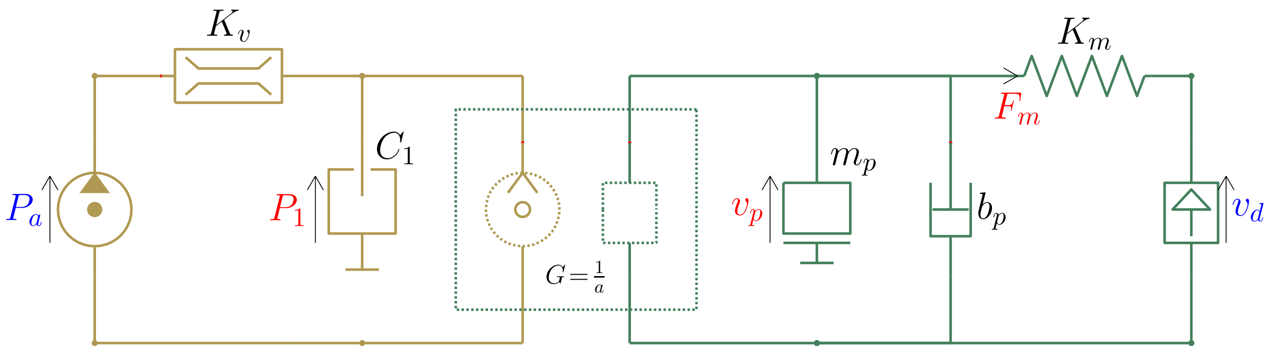
12.d) Add the dissipative blocks and the summation signs to the POG scheme:



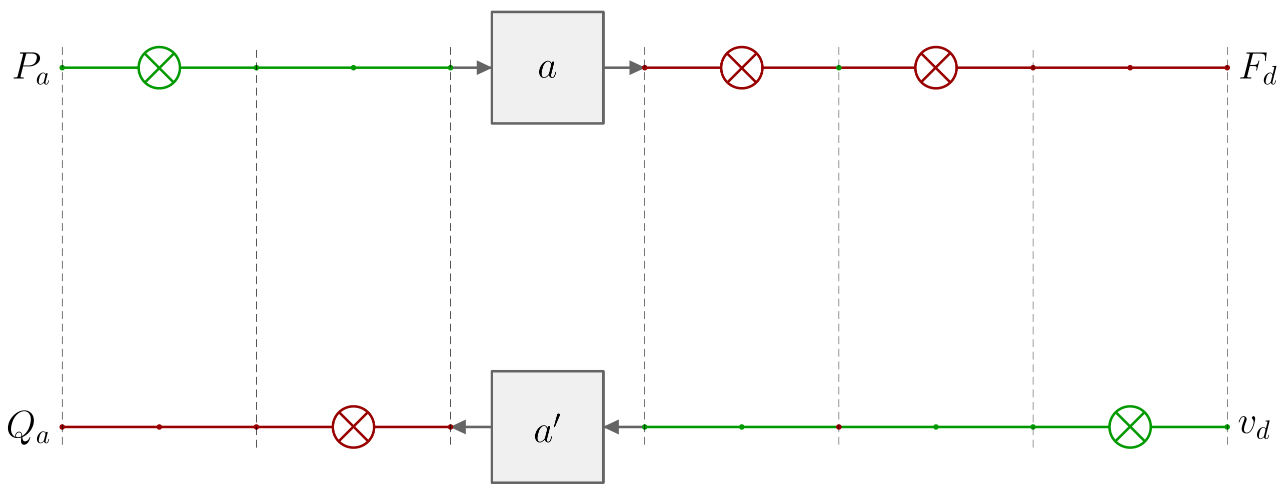
12.e) Check the signs by comparing with the considered physical system:



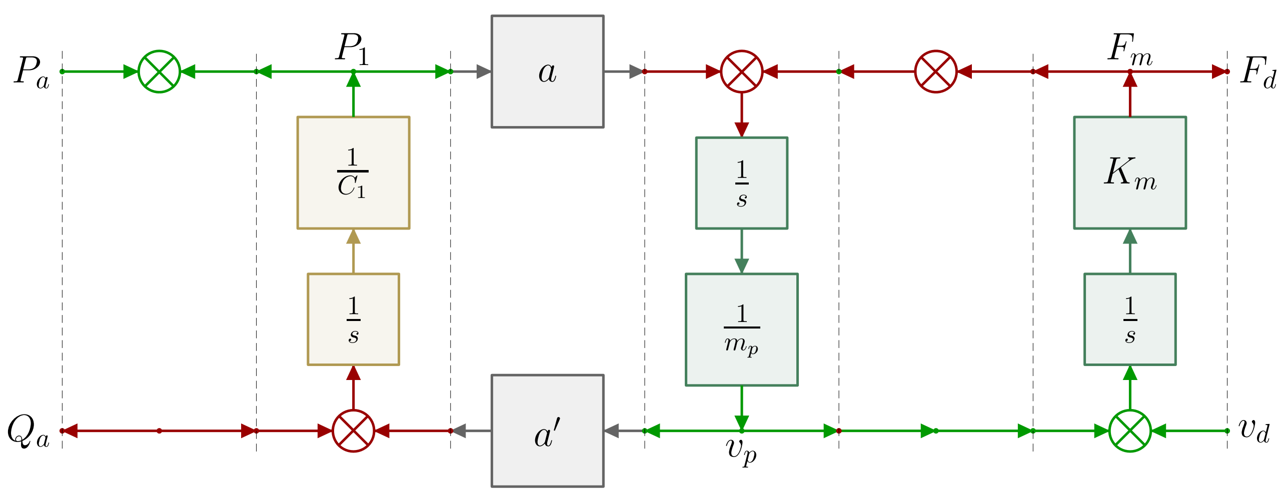
13.a) Choose the positive directions of the state and input variables of the considered physical system:



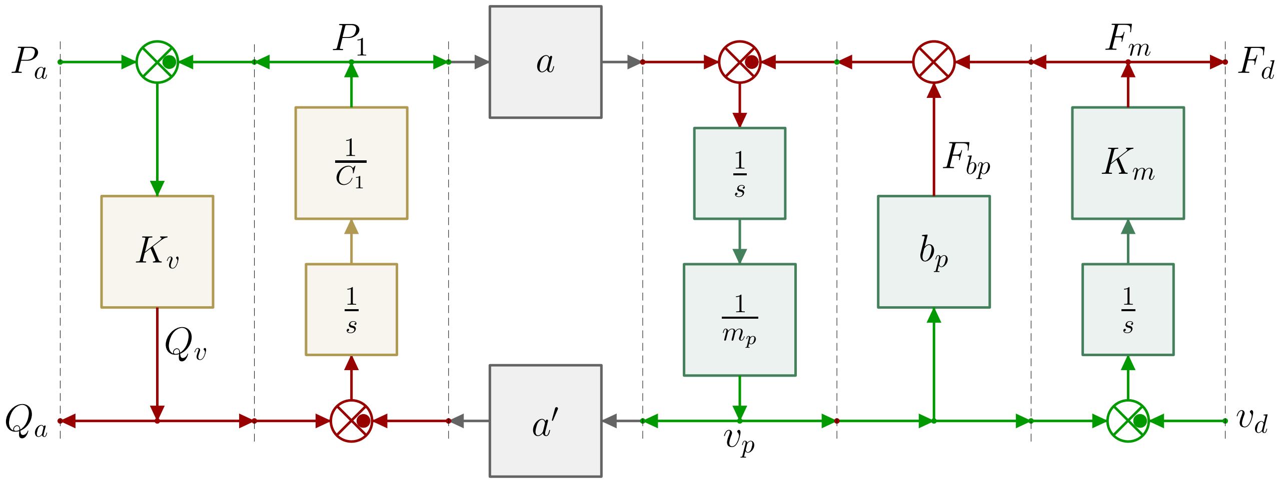
13.b) Draw the series/parallel structure of the POG block scheme:



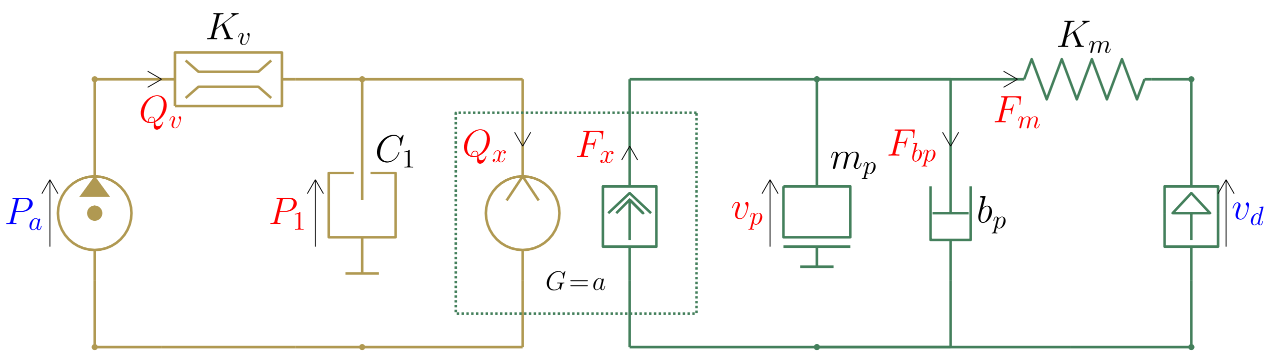
13.c) Add the dynamic blocks to the POG scheme:



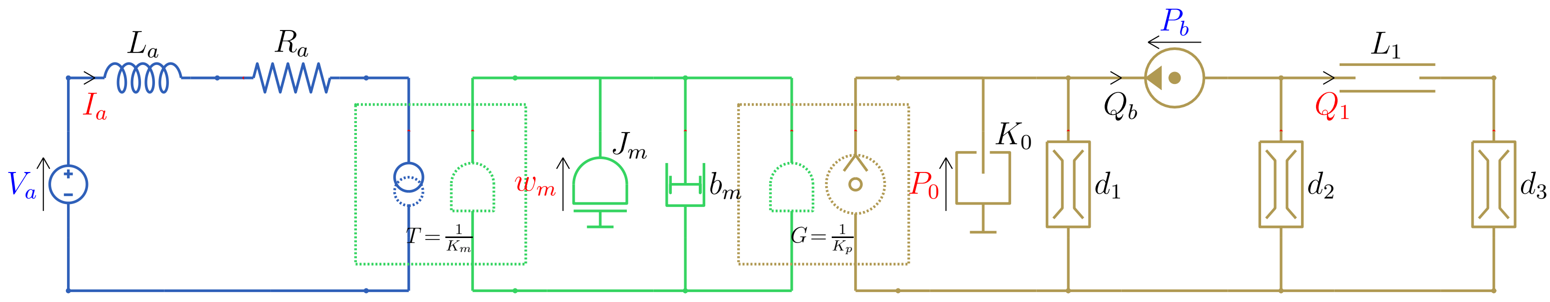
13.d) Add the dissipative blocks and the summation signs to the POG scheme:



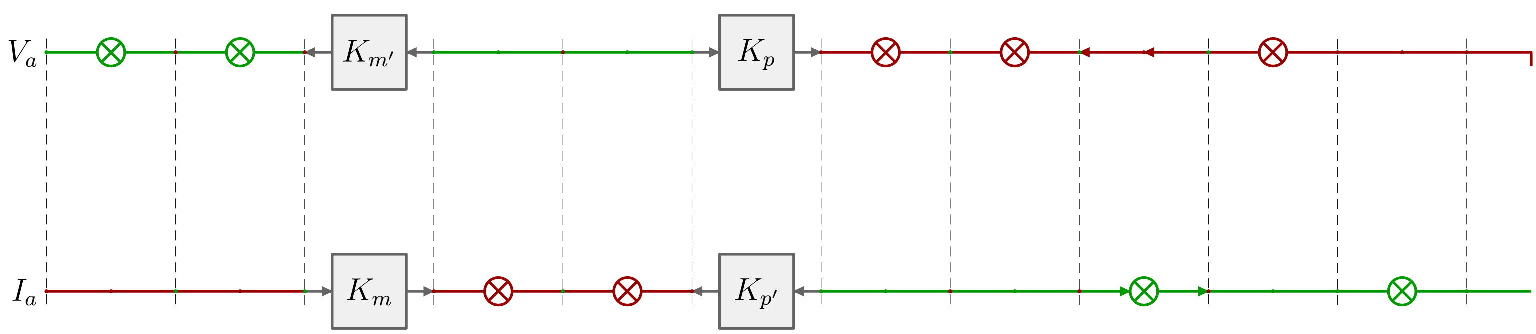
13.e) Check the signs by comparing with the considered physical system:



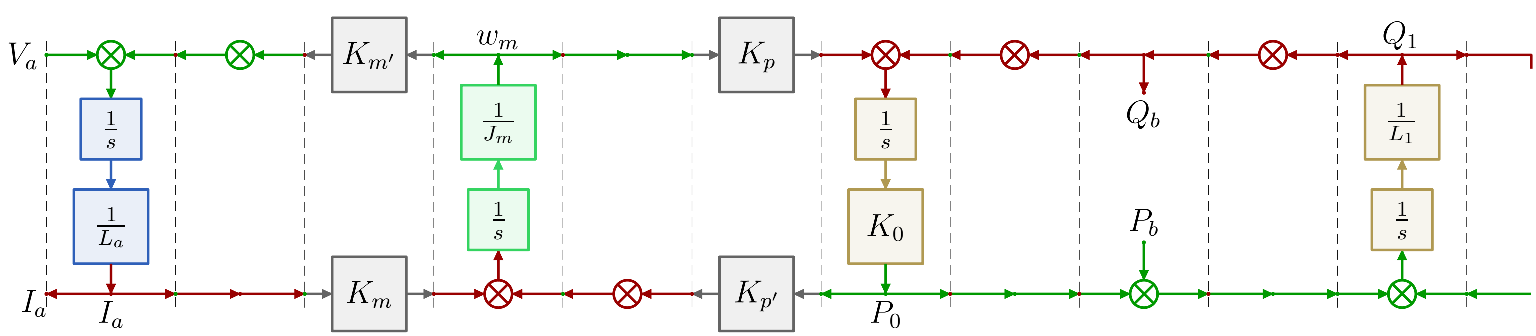
14.a) Choose the positive directions of the state and input variables of the considered physical system:



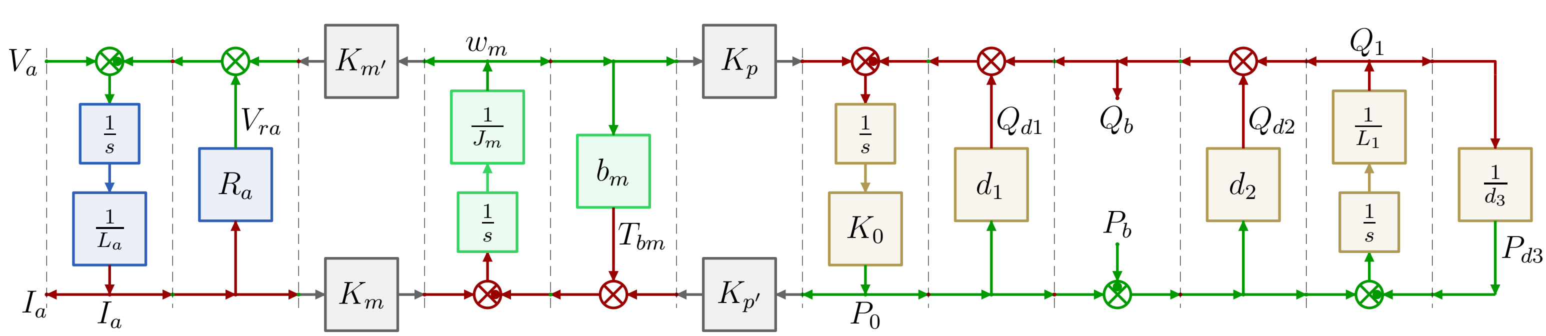
14.b) Draw the series/parallel structure of the POG block scheme:



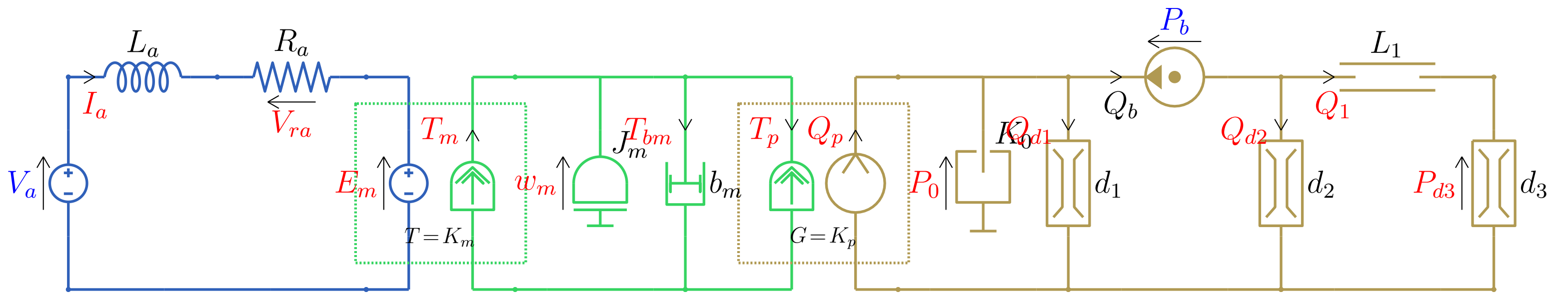
14.c) Add the dynamic blocks to the POG scheme:



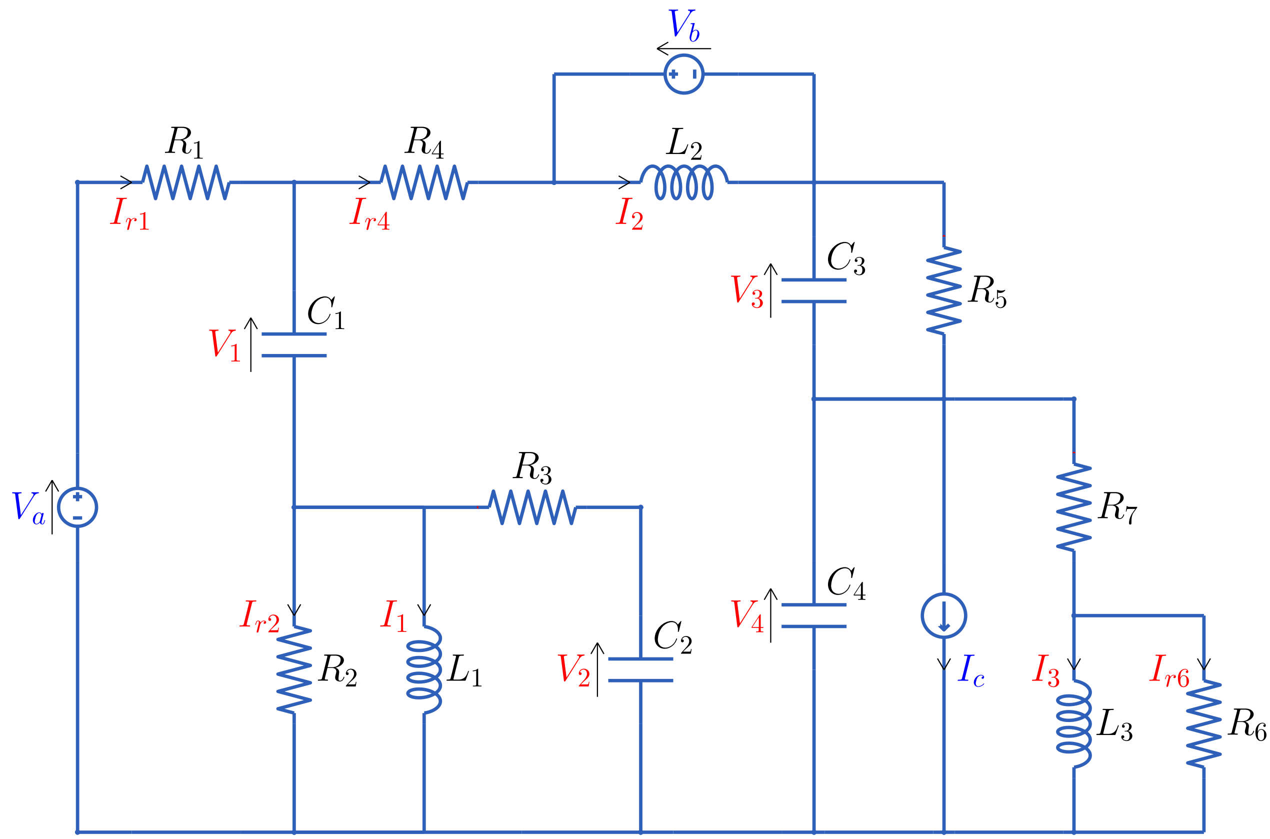
14.d) Add the dissipative blocks and the summation signs to the POG scheme:



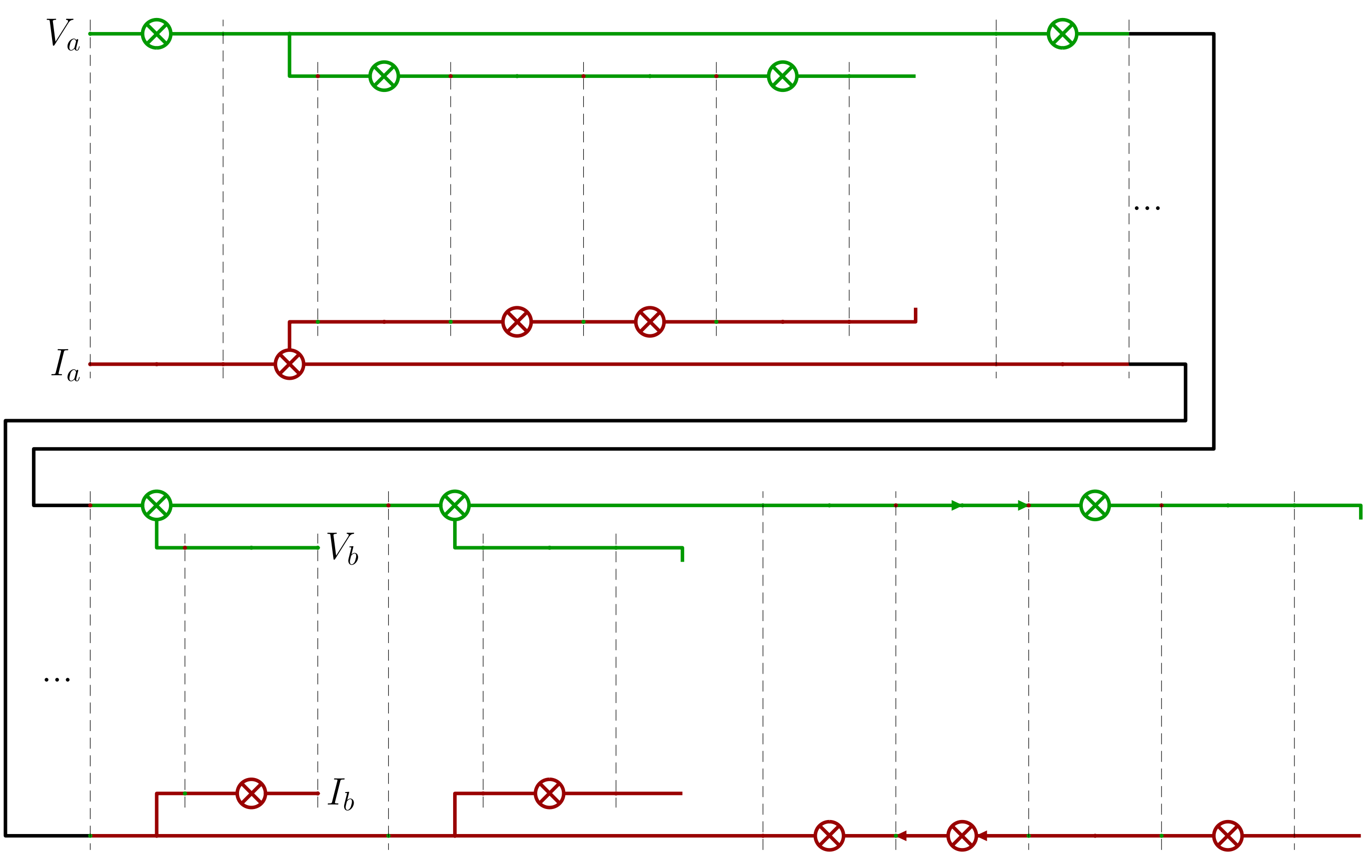
14.e) Check the signs by comparing with the considered physical system:



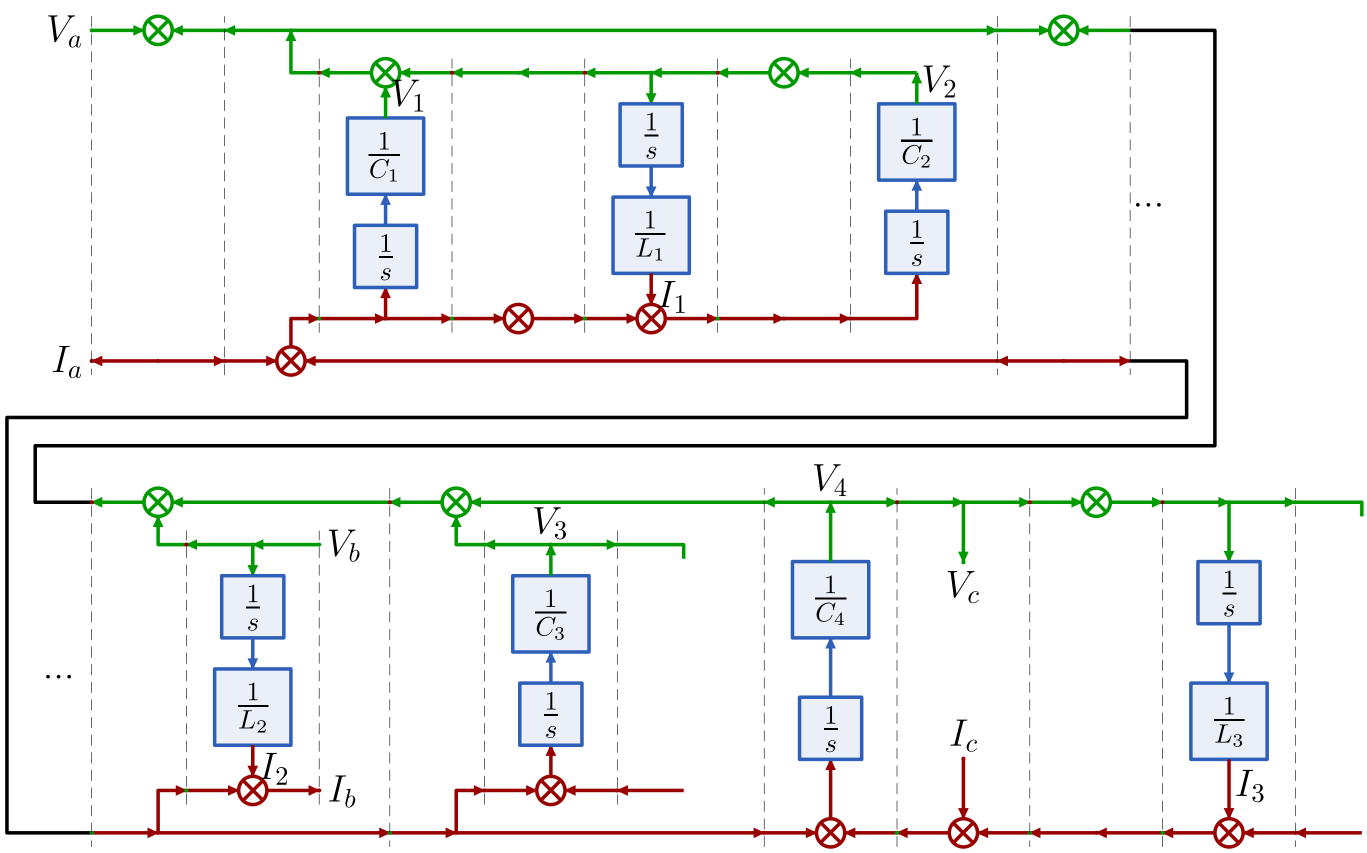
15.a) Choose the positive directions of the state and input variables of the considered physical system:



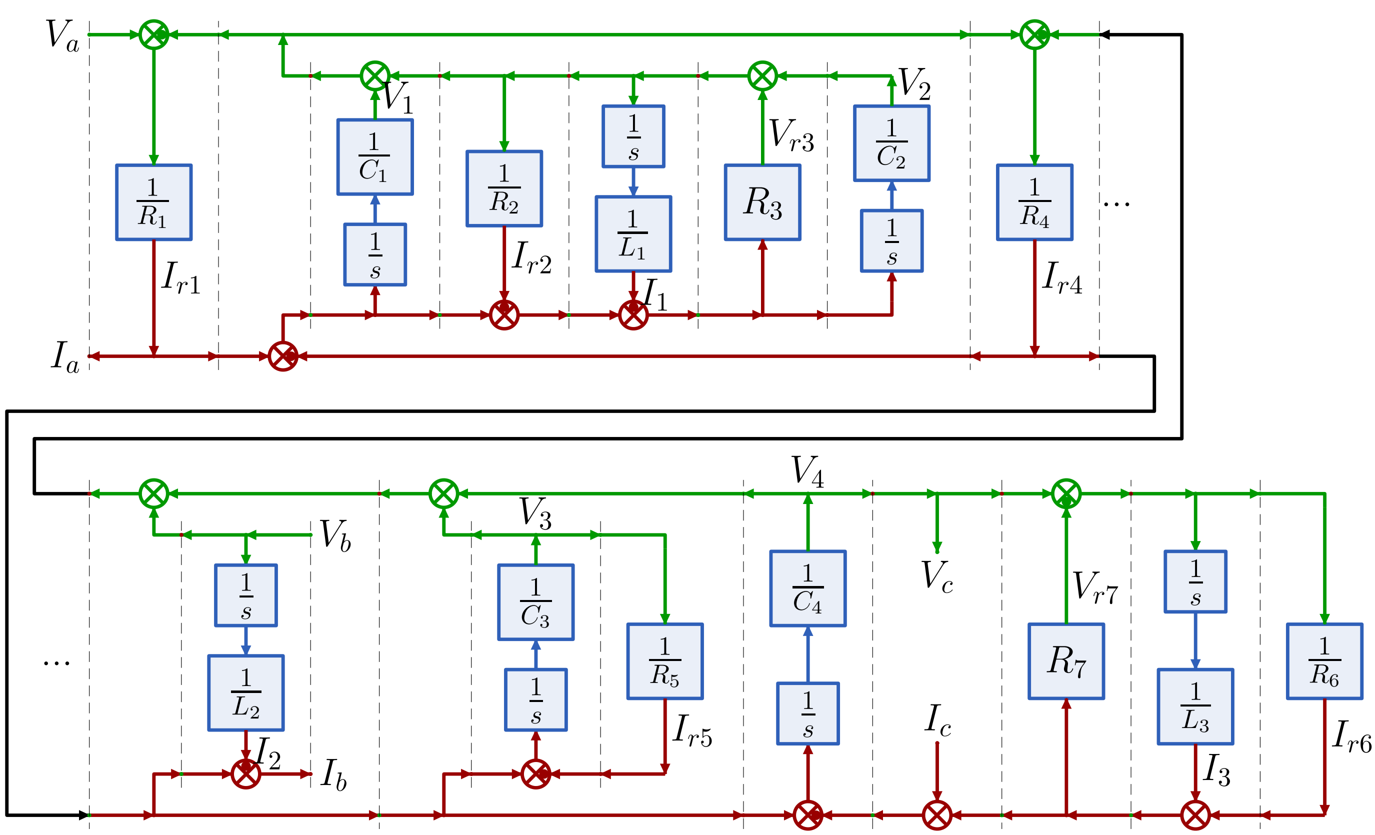
15.b) Draw the series/parallel structure of the POG block scheme:



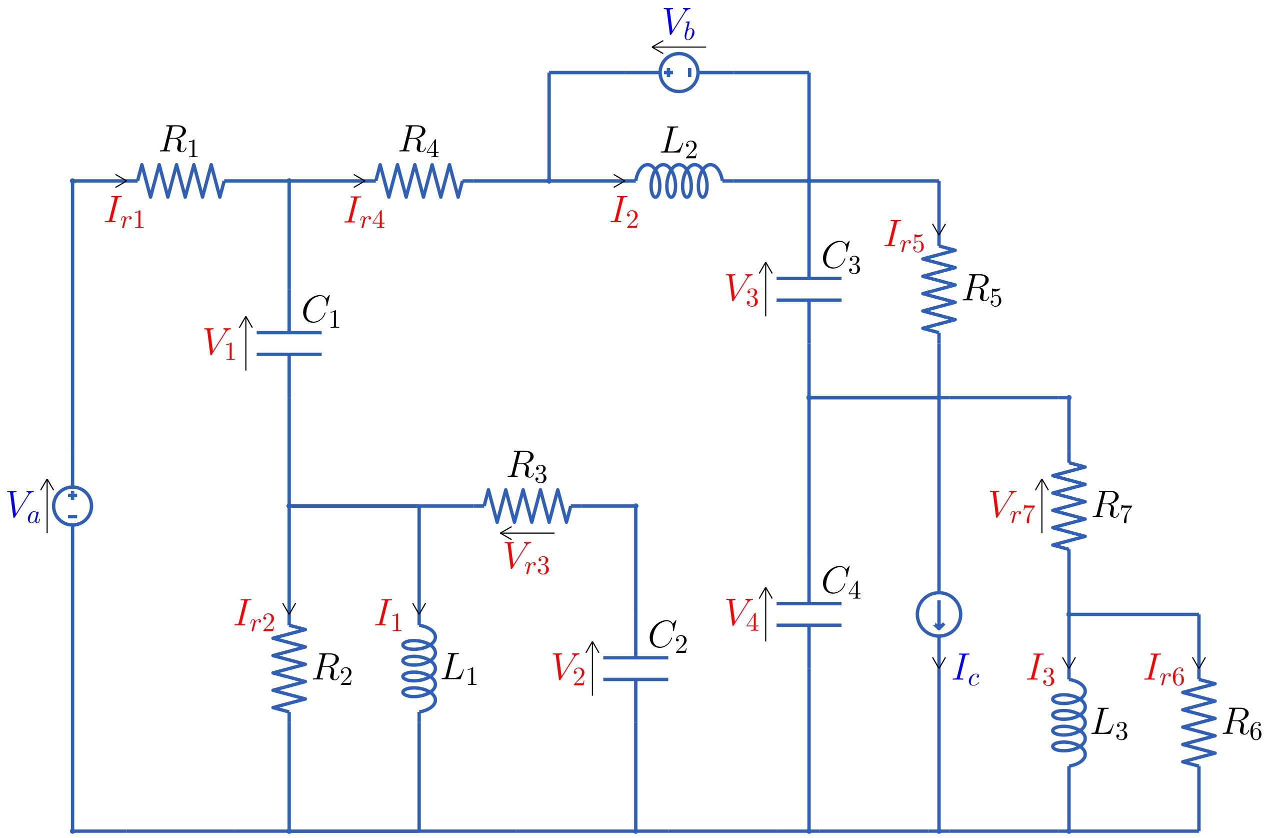
15.c) Add the dynamic blocks to the POG scheme:



15.d) Add the dissipative blocks and the summation signs to the POG scheme:



15.e) Check the signs by comparing with the considered physical system:

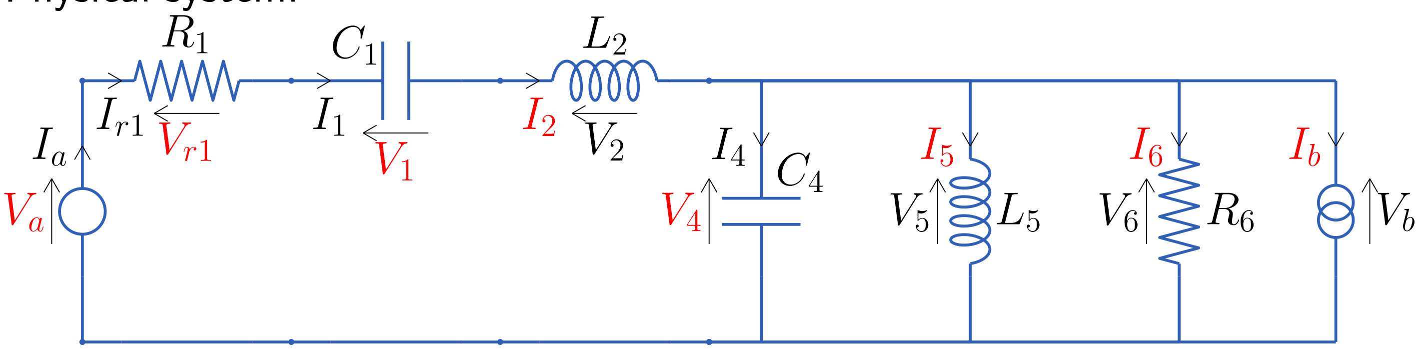


POG modeling of linear systems: useful rules

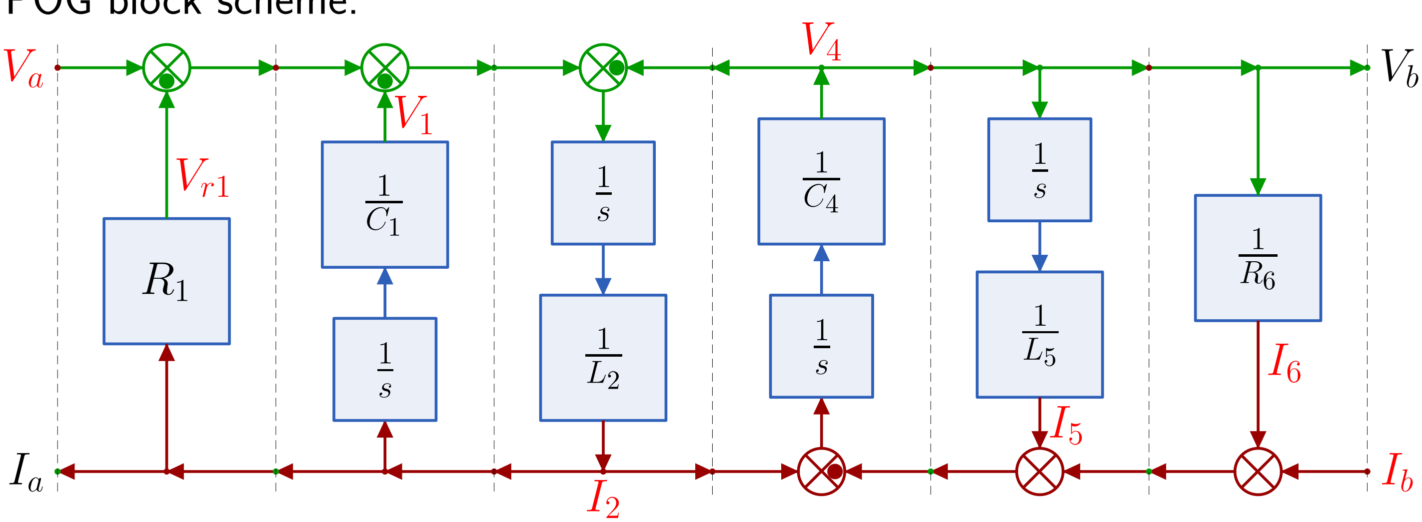
Rule 1: the positive directions of the two power variables (v_e, v_f) of all the physical elements of the considered system (except the generators) must be chosen such that the the positive direction of the corresponding power $P = v_e v_f$ enters into the physical element.

Rule 2: the power variables v_e and v_f of all the physical elements connected in series or connected in parallel (except the generators) must share the same positive direction.

Physical system:

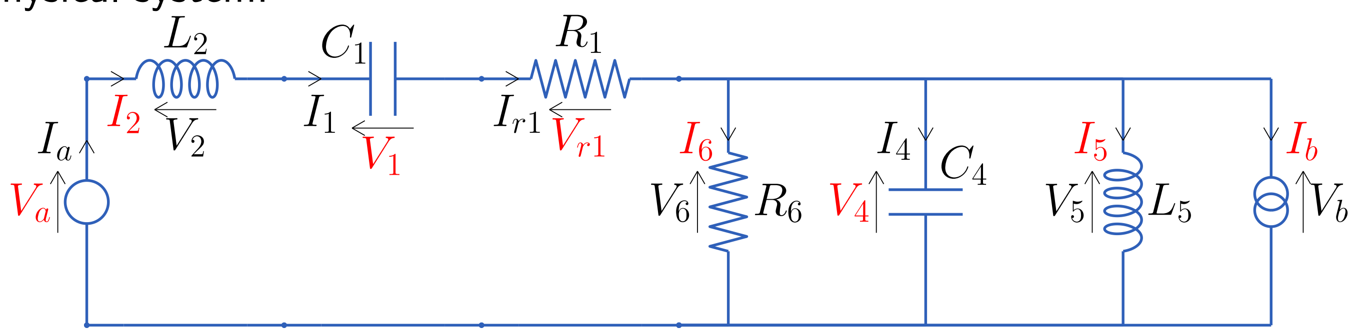


POG block scheme:

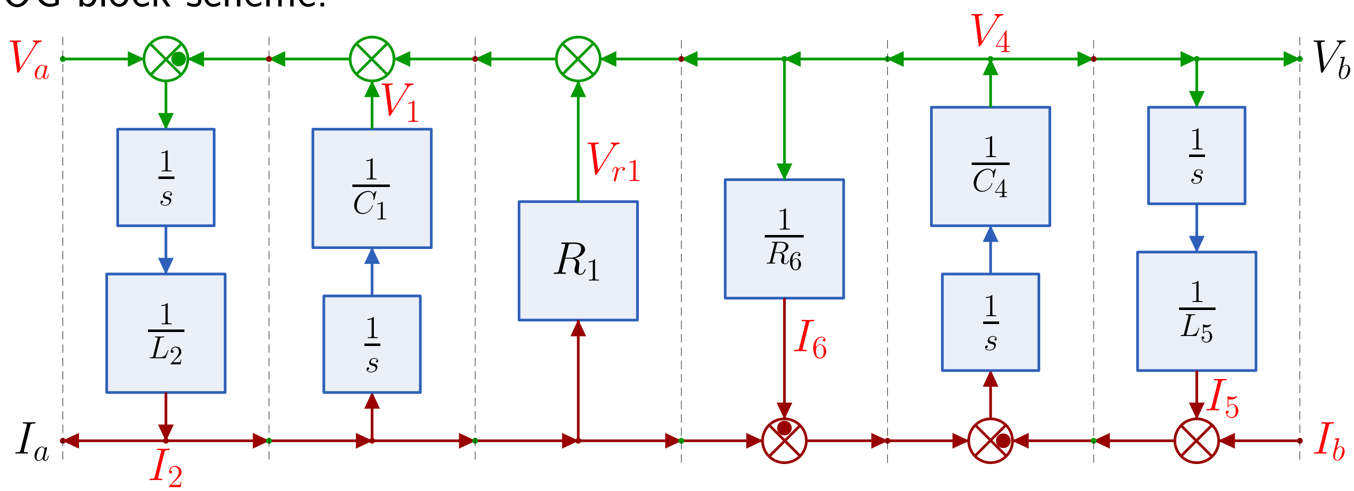


Rule 3: the position of all the physical elements connected in series or connected in parallel can be changed at will: the corresponding POG schemes are different from a graphical point of view, but equivalent from a mathematical point of view.

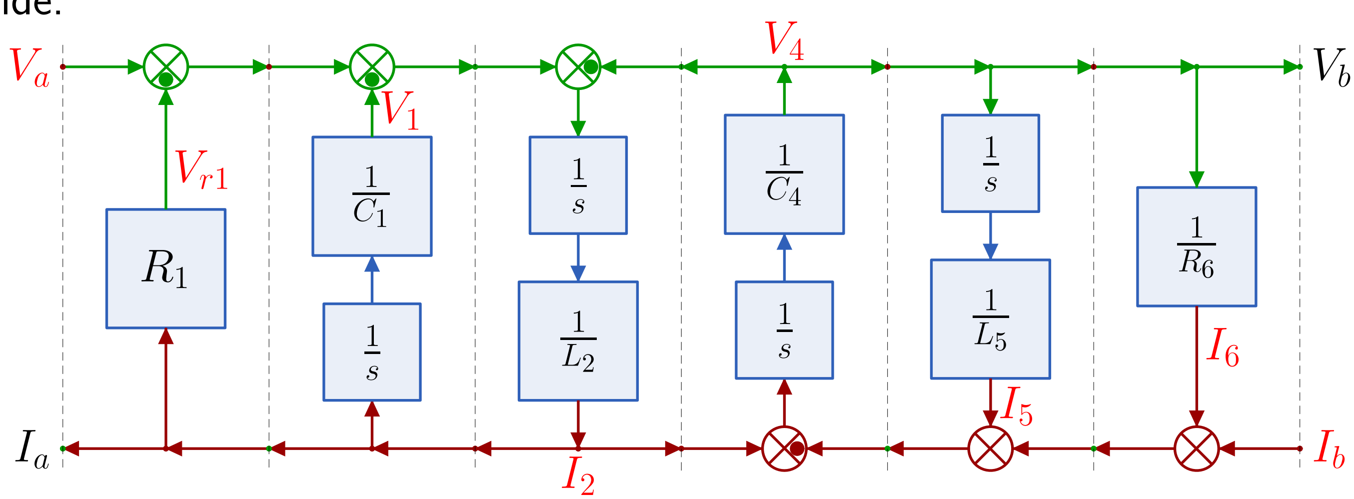
Physical system:



POG block scheme:

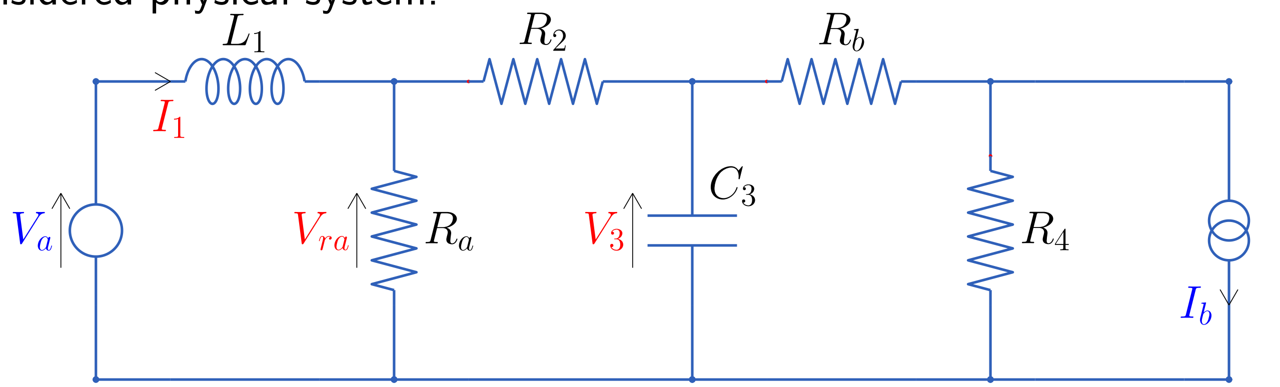


Note that, from a mathematical point of view, the obtained POG block scheme is equivalent to the POG block scheme of the system considered in the previous slide:

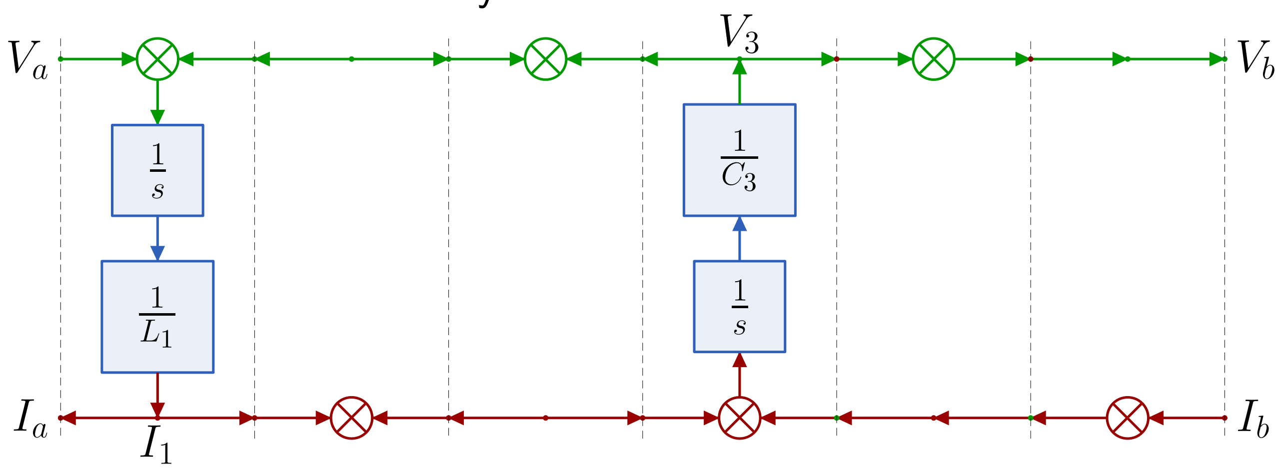


Rule 4: when modeling physical systems, different POG block schemes can be obtained only if **algebraic loops** are present within the system.

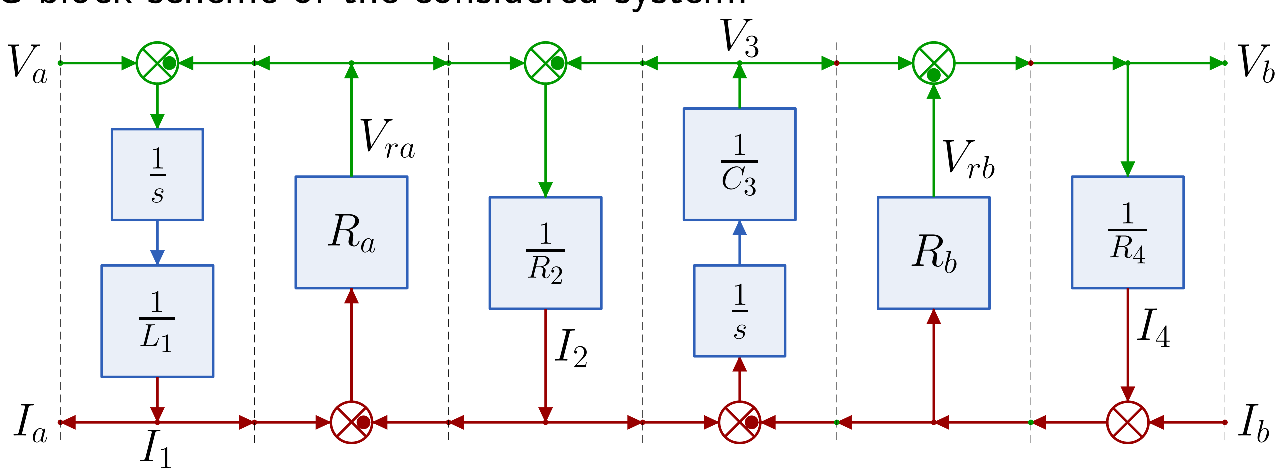
Considered physical system:



Basic POG scheme with the dynamic blocks:

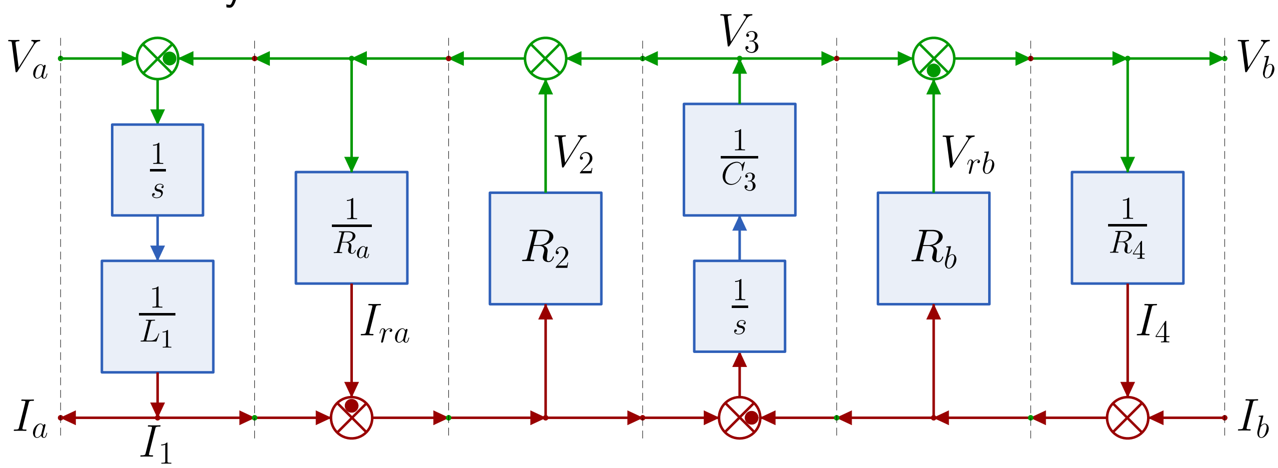


POG block scheme of the considered system:

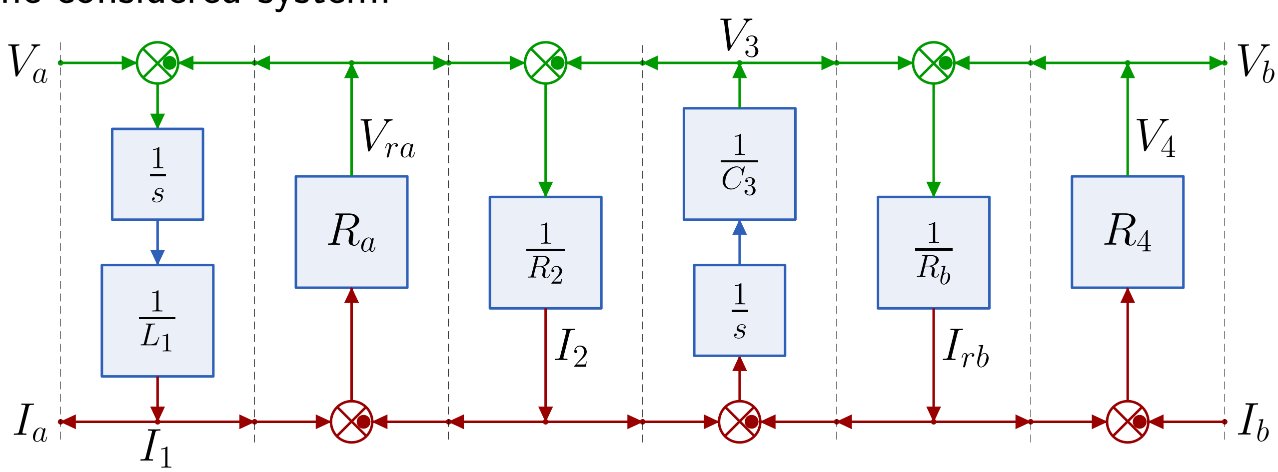


- An **algebraic loop** is a closed path which involves only dissipative elements. Two algebraic loops are present in the above POG scheme.
- Each algebraic loop can be inverted. The number m of possible POG schemes associated to a physical system is $m = 2^n$, where n the number of the algebraic loops present within the block scheme.

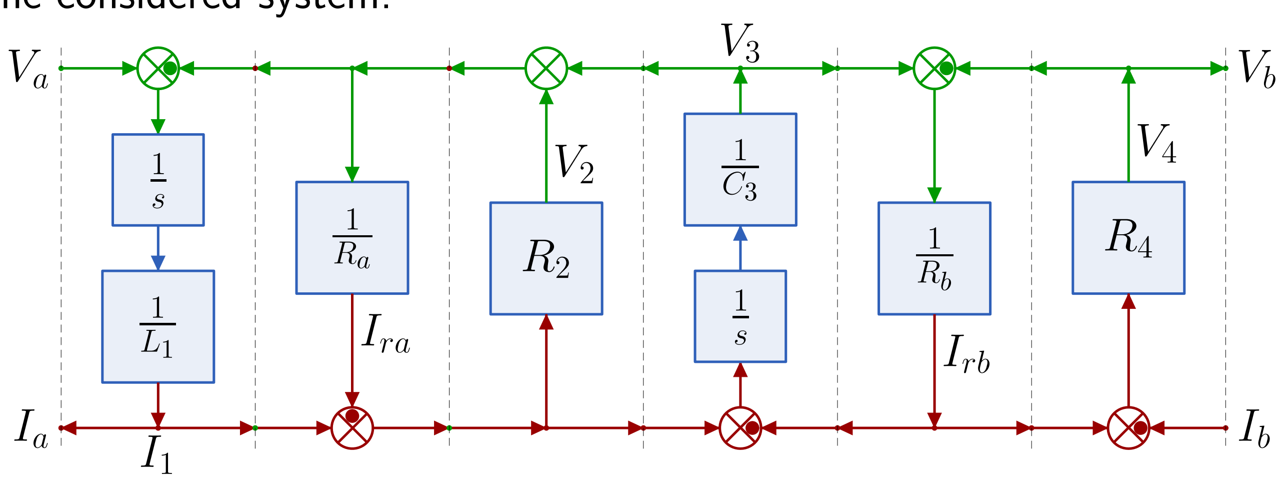
POG scheme obtained inverting the first algebraic loop: POG block scheme of the considered system:



POG scheme obtained inverting the second algebraic loop: POG block scheme of the considered system:



POG scheme obtained inverting both the algebraic loops: POG block scheme of the considered system:

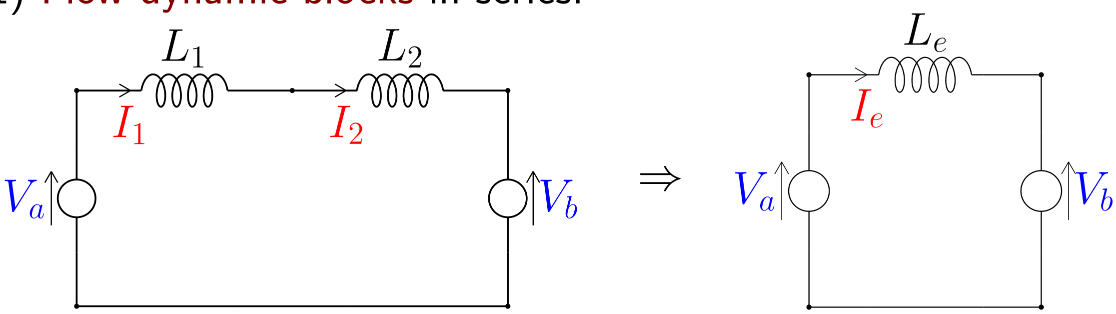


- The four POG block schemes are equivalent from a mathematical point of view.
- Block schemes with algebraic loops are not suitable to be simulated directly in Simulink. For linear systems, the algebraic loops can be easily solved.

Rule 5: two physical elements of the same type connected in series or connected in parallel must be substituted by the corresponding equivalent element before applying the POG modeling.

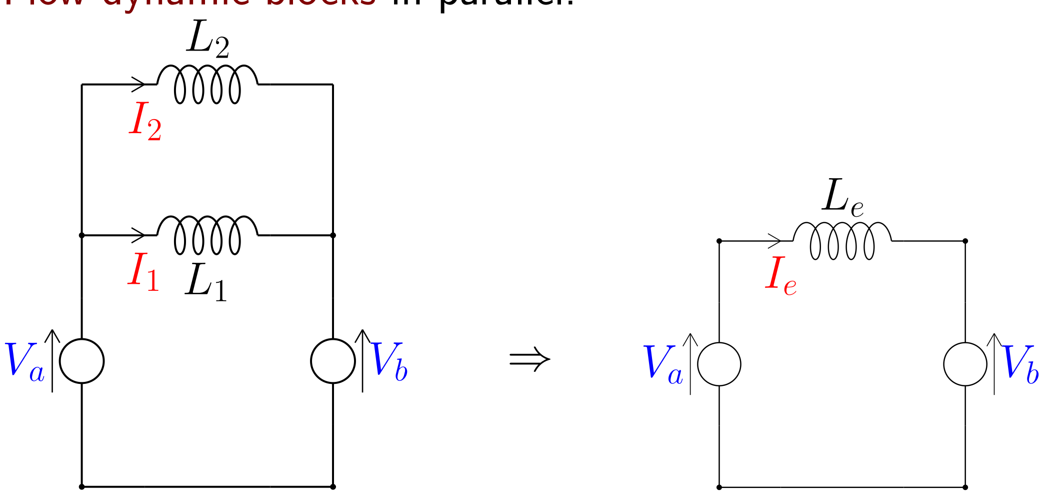
In the electrical domain the following simplifications can be done (external connections in series):

1) Flow dynamic blocks in series:



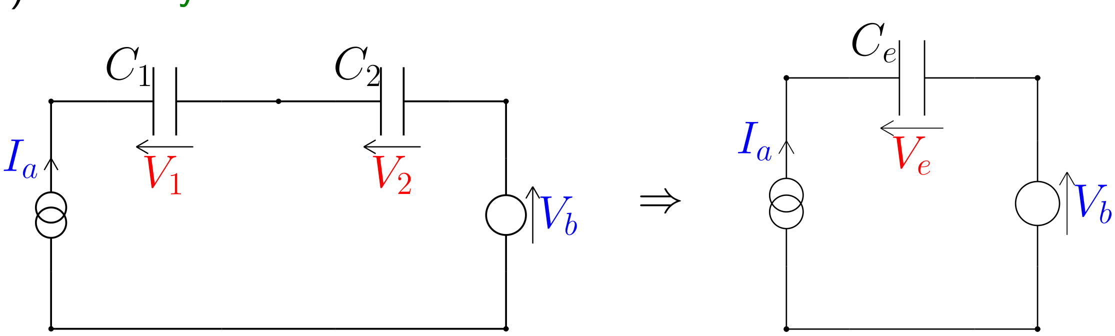
$$L_e = L_1 + L_2$$

2) Flow dynamic blocks in parallel:



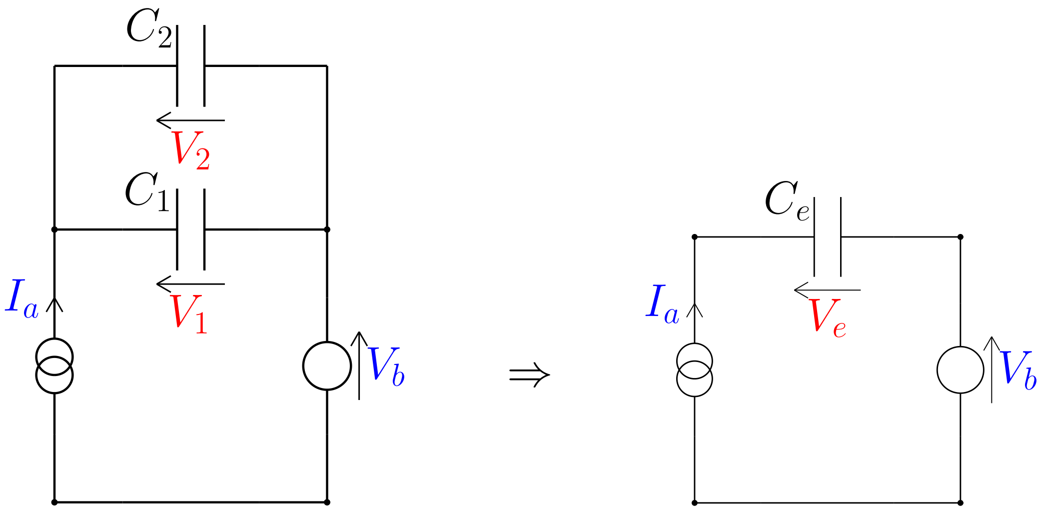
$$L_e = \frac{L_1 L_2}{L_1 + L_2}$$

3) Effort dynamic blocks in series:



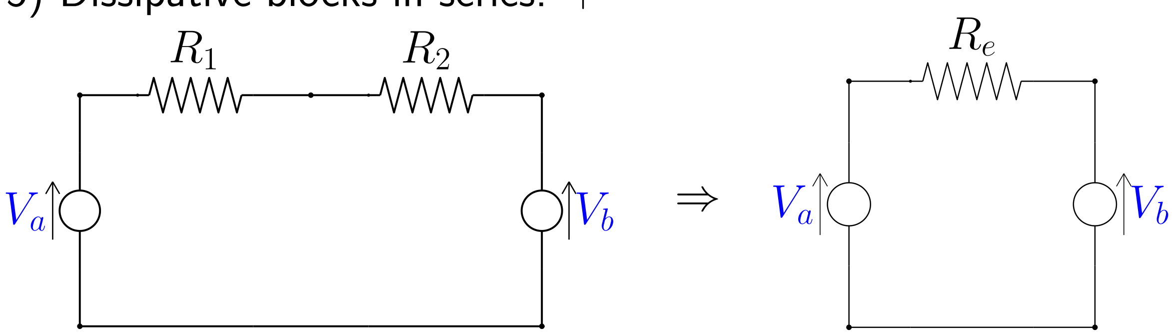
$$C_e = \frac{C_1 C_2}{C_1 + C_2}$$

4) Effort dynamic blocks in parallel:



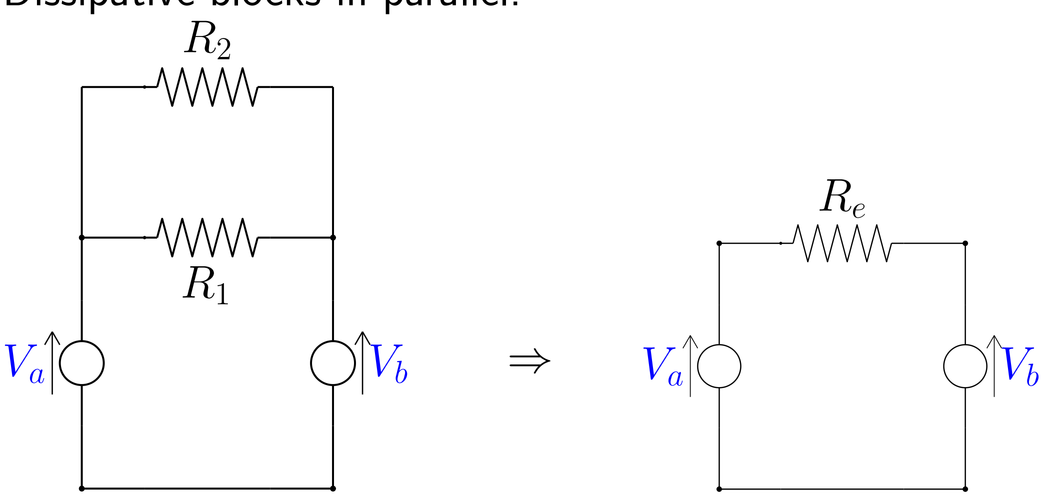
$$C_e = C_1 + C_2$$

5) Dissipative blocks in series: $+$



$$R_e = R_1 + R_2$$

6) Dissipative blocks in parallel:

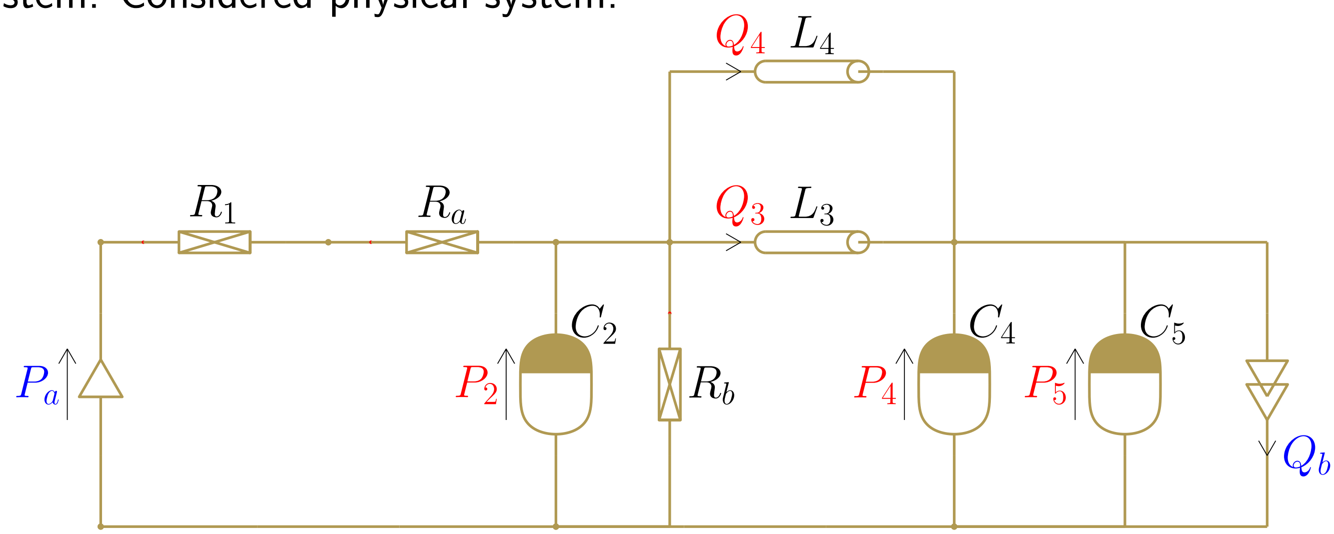


$$R_e = \frac{R_1 R_2}{R_1 + R_2}$$

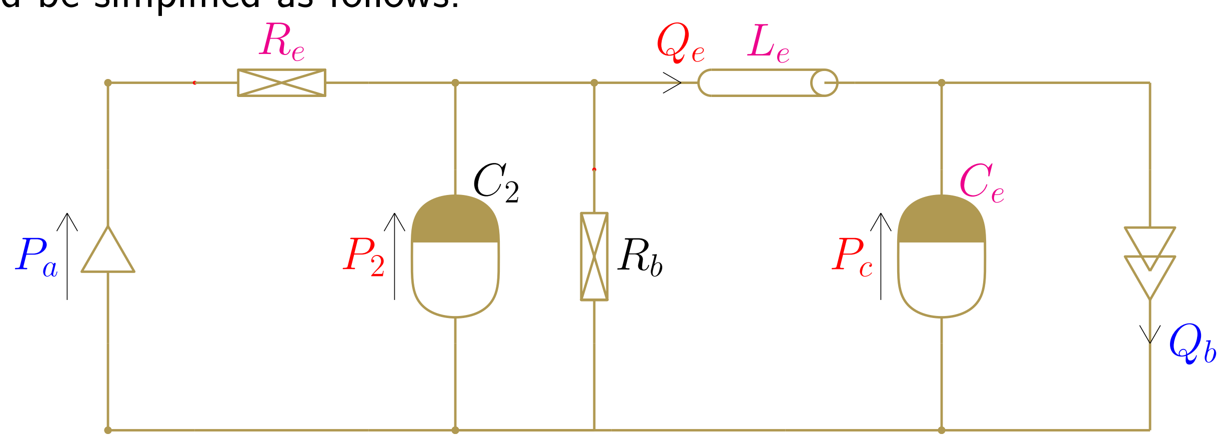
- All the previous simplifications hold also if the external connection of the considered elements is in parallel.

• Similar simplifications can be done also in the mechanical domains (translational and rotational) and in the hydraulic domain.

Example. Before applying the POG modeling approach, the following hydraulic system: Considered physical system:



should be simplified as follows:

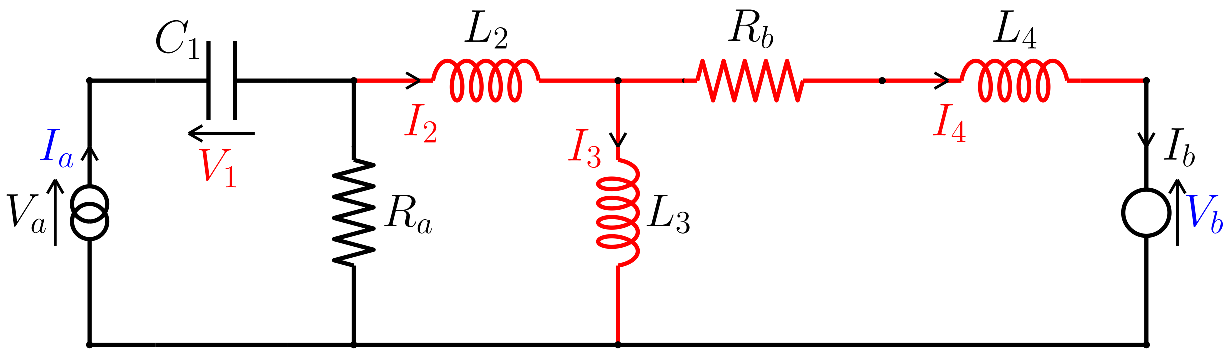


where:

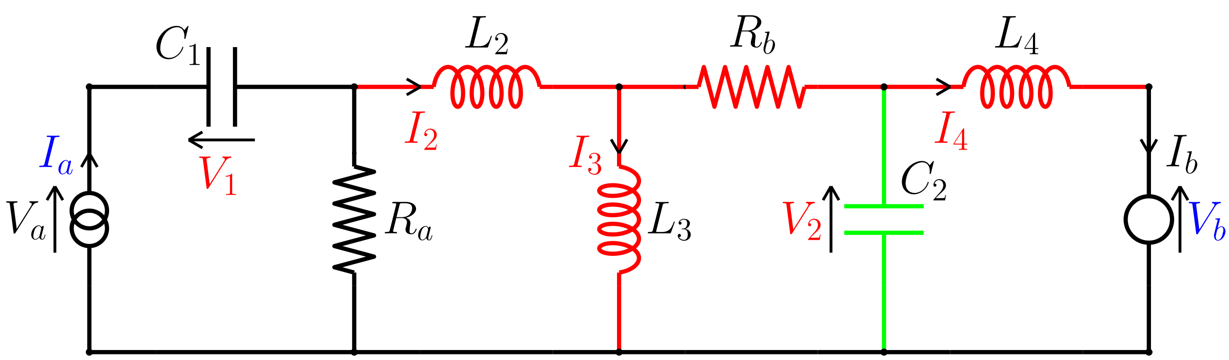
$$R_e = R_1 + R_a, \qquad L_e = \frac{L_3 L_4}{L_3 + L_4}, \qquad C_e = C_4 + C_5$$

Rule 6: the POG modeling approach cannot be used if the physical system contains nodes connected only to branches with flow dynamic blocks in series.

Example. The following electrical system cannot be directly POG modeled because it contains a node connected only to branches with inductors in series:



The correct dynamic model can be obtained by adding a spurious physical element (for example the capacitance C_2)



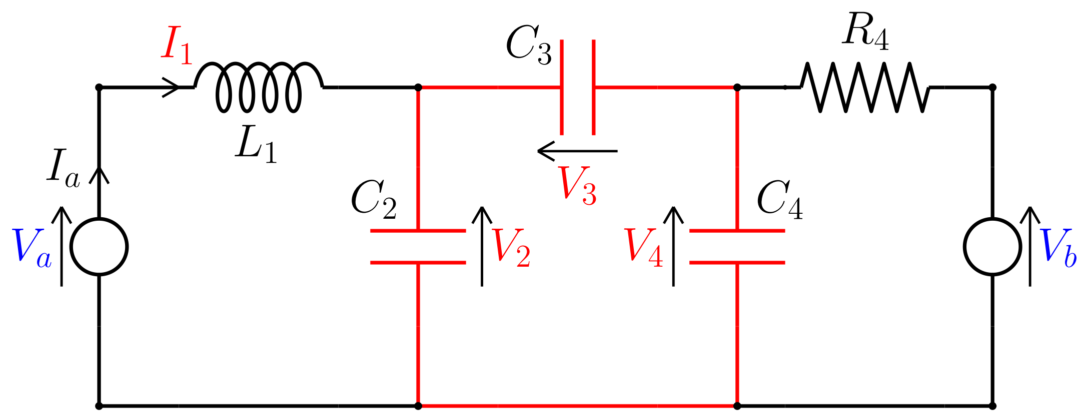
then

a) obtain of the augmented system;

b) read from the POG block scheme the state space dynamic model of the of the augmented system;

c) apply a congruent transformation to the state space model in order to obtain the dynamic model of the reduced system when the spurious element is neglected: $C_2 = 0$.

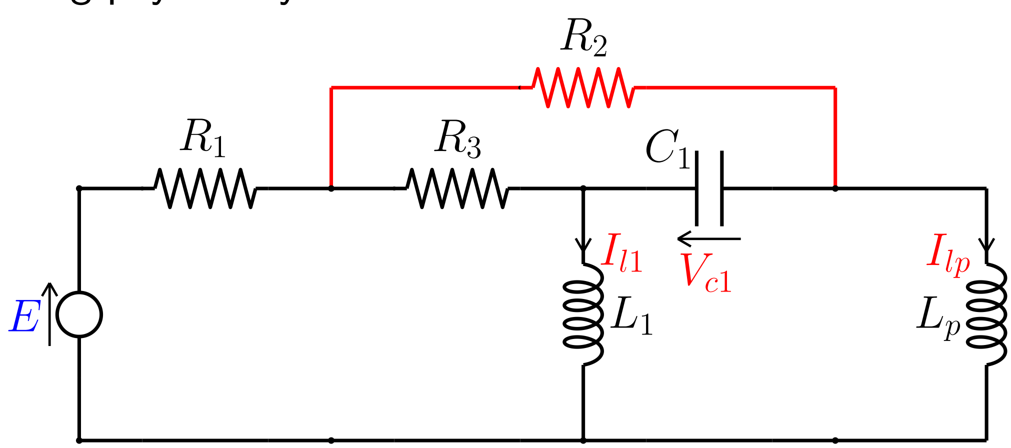
Rule 7: the POG modeling approach cannot be used if the physical system contains closed paths characterized only by effort dynamic blocks .



In this case the correct dynamic model can be obtained using an approach similar to the one described in the previous case. In this a spurious element (for example an inductance L_2) mast be added in series with one of the three capacitors.

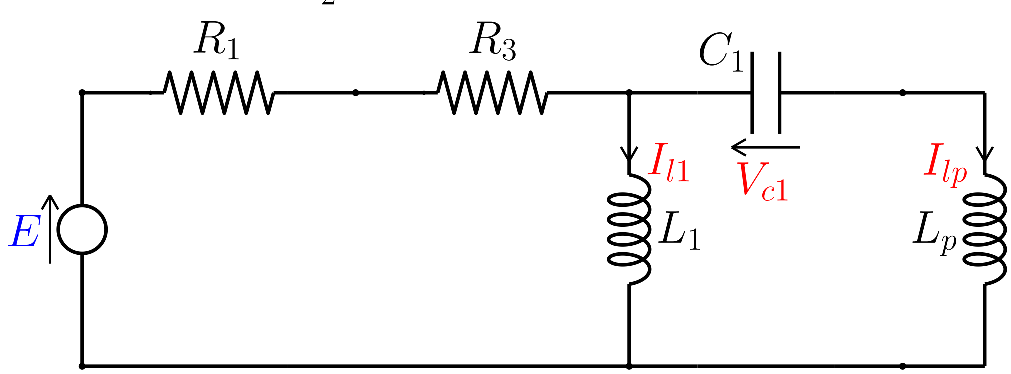
POG modeling: a particular case

• The following physical system:

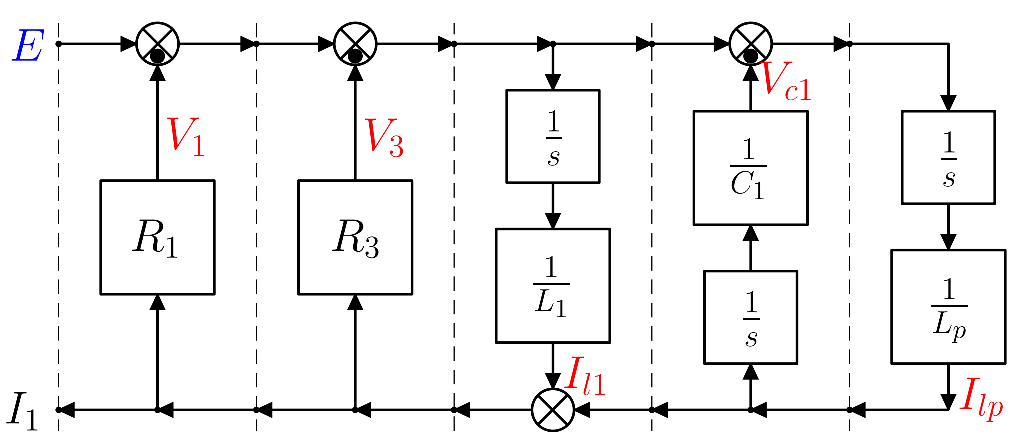


cannot be modeled using the "POG direct approch" because, due to the presence of resistance R_2, the system cannot be described by a sequence of series and parallel elements.

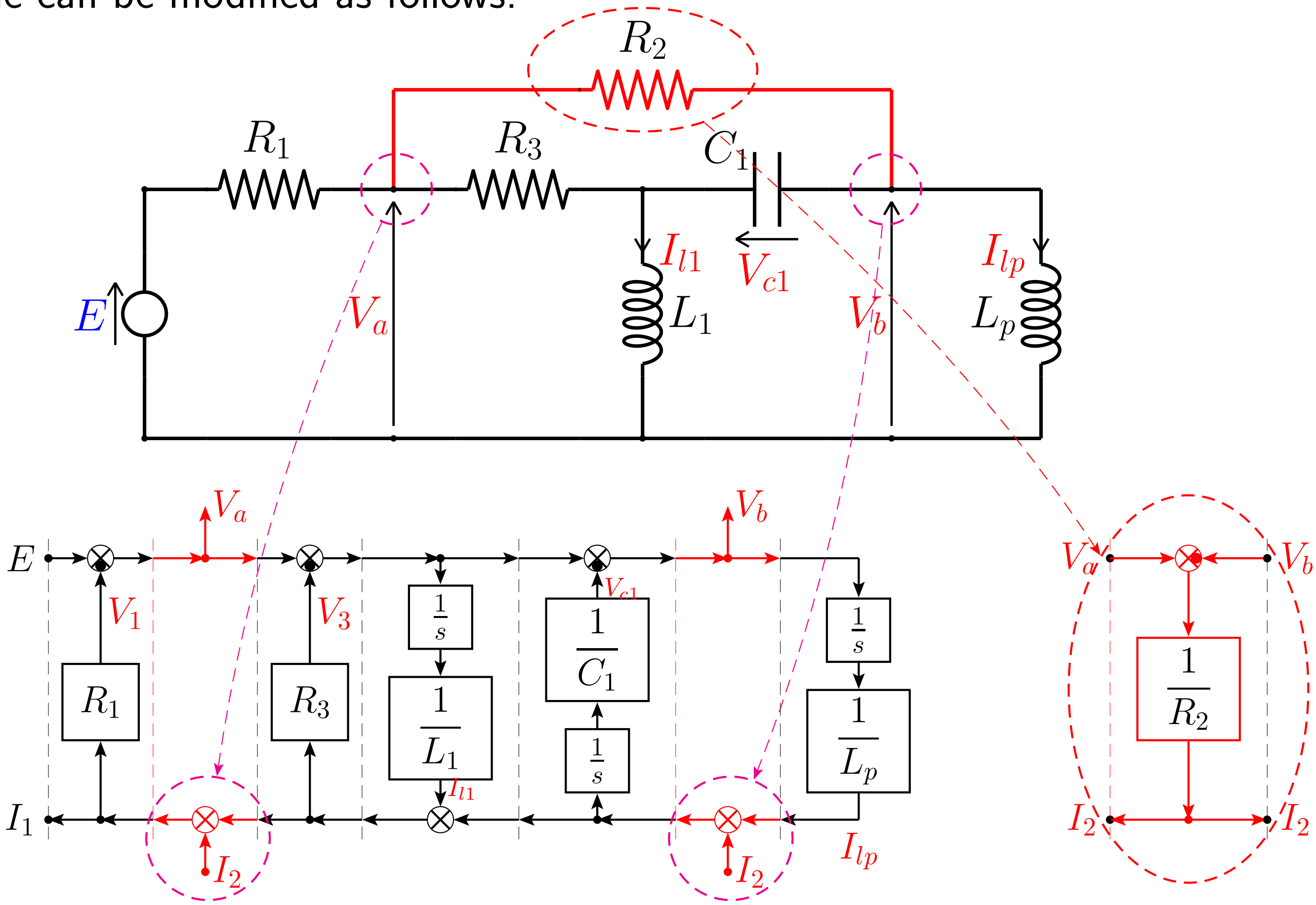
• This problem can be solved obtaining the POG block scheme of the original system without resistance R_2:



The corresponding POG block scheme is:



• If resistance R_2 is added to the original system, the corresponding POG block scheme can be modified as follows:

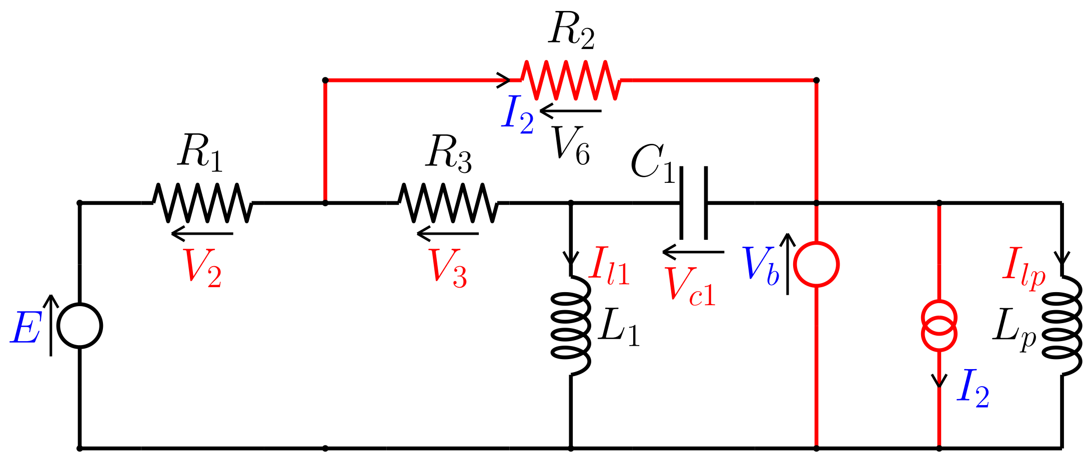


• The first two red elements represent the two points where resistance R_2 is attached the rest of the system. The last red element is the POG clock scheme of the resistance R_2.

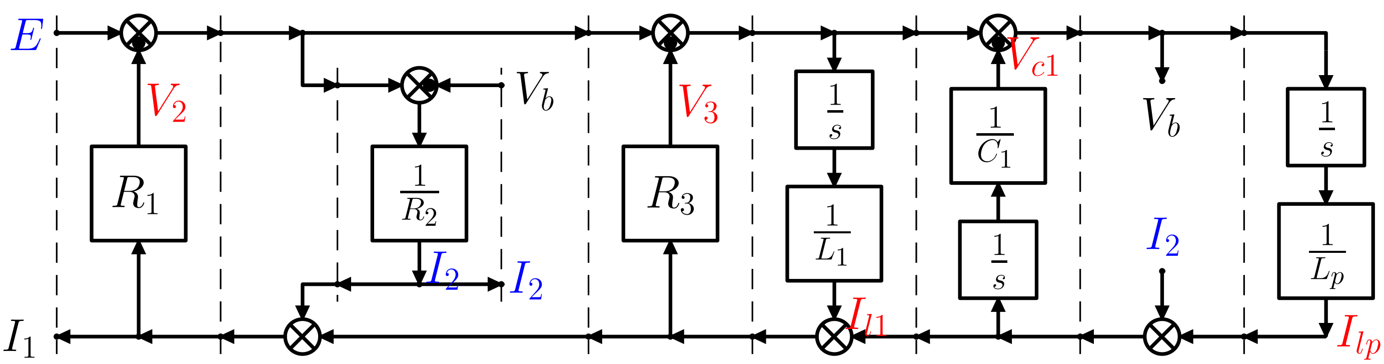
• The two red summation blocks which involve the currents represent the Kirchhoff's current law applied to the two points where the resistance R_2 is attached to the other elements.

• The red summation block in the right POG block scheme represents the Kirchhoff's voltage law applied to the new closed path due to the connection of the resistance R_2 to the other elements.

• The modeling problem can also be solved adding two generators (V_b generator and I_2 generator) in correspondence of one of the two terminals of the resistance R_2:



• The corresponding POG block scheme is:



• The correct POG block scheme of the original system is obtained connecting the outputs of the two generator to the inputs of the same generators.

State Space Model in the "Control System Theory"

- ***Classical State Space Model***. In the classical "Control System Theory" a *linear time-invariant dynamic system* $\tilde{\mathrm{S}}$ is usually described by using state space equations having the following structure:

$$\tilde{\mathbf{S}} = \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

  where $\mathbf{A}$ is the system matrix, $\mathbf{B}$ is the input matrix, $\mathbf{C}$ is the output matrix and $\mathbf{D}$ is the input-output matrix.

- ***Transformed Systems***. The Classical State Space Model $\tilde{\mathrm{S}}$ is usually transformed using a ***"similitude"*** transformation $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$:

$$\tilde{\mathbf{S}} = \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases} \quad \overset{\mathbf{x}=\mathbf{T}\,\overline{\mathbf{x}}}{\Longrightarrow} \quad \hat{\mathbf{S}} = \begin{cases} \dot{\overline{\mathbf{x}}} = \hat{\mathbf{A}}\,\overline{\mathbf{x}} + \hat{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \hat{\mathbf{C}}\,\overline{\mathbf{x}} + \mathbf{D}\,\mathbf{u} \end{cases}$$

  where $\hat{\mathrm{S}}$ is the transformed system and

$$\hat{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \qquad \hat{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \qquad \hat{\mathbf{C}} = \mathbf{C}\mathbf{T}.$$

  The ***"similitude"*** transformation $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$ can be applied only if matrix $\mathbf{T}$ is "squared" and "non-singular".

- ***Transfer Matrix***. *For a Classical State Space Model* $\tilde{\mathrm{S}}$:

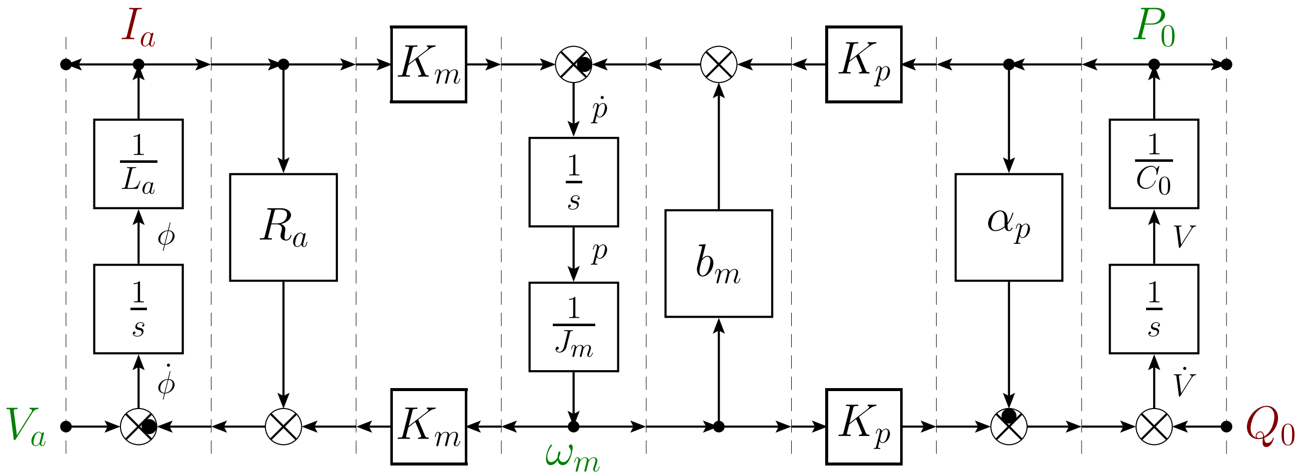
$$\tilde{\mathbf{S}} = \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

  the input-output transfer matrix $\mathbf{H}(s)$ *is obtained as follows*:

$$\mathbf{H}(s) = \mathbf{C}(\mathbf{I}\,s - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

- The transfer matrix $\mathbf{H}(s)$ does not change if a "similitude" transformation $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$ is applied to system $\tilde{\mathrm{S}}$.

Example. Block scheme:



State space equations:

$$\begin{cases} L_a\,\dot{I}_a = -R_a\,I_a - K_m\,\omega_m + V_a \\ J_m\,\dot{\omega}_m = K_m\,I_a - b_m\,\omega_m - K_p\,P_0 \\ C_0\,\dot{P}_0 = K_p\,\omega_m - \alpha_p\,P_0 + Q_0 \end{cases}$$

"Classical" state space model ($\dot{\mathbf{x}} = \overline{\mathbf{A}}\,\mathbf{x} + \overline{\mathbf{B}}\,\mathbf{u}$):

$$\underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_m}{L_a} & 0 \\ \frac{K_m}{J_m} & -\frac{b_m}{J_m} & -\frac{K_p}{J_m} \\ 0 & \frac{K_p}{C_0} & -\frac{\alpha_p}{C_0} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C_0} \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ Q_0 \end{bmatrix}}_{\mathbf{u}}$$

POG state space model ($\mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$):

$$\underbrace{\begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ Q_0 \end{bmatrix}}_{\mathbf{u}}$$

From the POG scheme to the State Space Model

- ***POG State Space Model.*** A *linear* *time-invariant* *POG dynamic system* $\mathbf{S}$ is characterized by state space differential equations having the following structure:

$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

where $\mathbf{L}$ is the *energy matrix*, $\mathbf{A}$ is the *power matrix*, $\mathbf{B}$ is the *input power matrix*, $\mathbf{C}$ is the *output matrix* and $\mathbf{D}$ is the *input-output matrix*.

- A POG state space model satisfies the following properties:

  a) $\mathbf{L} = \mathbf{L}^T \geq 0$ is a symmetric semidefinite matrix;

  b) the "*energy E_s stored in the system*" can be expressed as follows:

$$E_s = \frac{1}{2}\mathbf{x}^T\mathbf{L}\mathbf{x} \geq 0$$

  c) the "*power P_d dissipated in the system*" can be expressed as follows:

$$P_d = \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}_s\mathbf{x} \qquad \text{where} \qquad \mathbf{A}_s = \frac{\mathbf{A}+\mathbf{A}^T}{2}.$$

- ***POG Transformed Systems.*** A POG dynamic system $\mathbf{S}$ can be transformed using a ***"congruent"*** transformation $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$:

$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases} \quad \overset{\mathbf{x}=\mathbf{T}\,\overline{\mathbf{x}}}{\Longrightarrow} \quad \overline{\mathbf{S}} = \begin{cases} \overline{\mathbf{L}}\,\dot{\overline{\mathbf{x}}} = \overline{\mathbf{A}}\,\overline{\mathbf{x}} + \overline{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \overline{\mathbf{C}}\,\overline{\mathbf{x}} + \mathbf{D}\,\mathbf{u} \end{cases}$$

where $\overline{\mathbf{S}}$ is the transformed system and (if matrix $\mathbf{T}$ is constant)

$$\overline{\mathbf{L}} = \mathbf{T}^T\mathbf{L}\mathbf{T}, \qquad \overline{\mathbf{A}} = \mathbf{T}^T\mathbf{A}\mathbf{T}, \qquad \overline{\mathbf{B}} = \mathbf{T}^T\mathbf{B}, \qquad \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}.$$

- The transformed POG system $\overline{\mathbf{S}}$ maintains the same properties of the original POG system $\mathbf{S}$.

- A ***"congruent"*** transformation $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$ does not require the calculation of the the inverse of matrix $\mathbf{T}$ and therefore it can be applied also when $\mathbf{T}$ is "singular" or "rectangular".

- ***POG Transfer Matrix***. *For linear time-invariant POG systems*:

$$\mathbf{S} = \begin{cases} \mathbf{L}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

the input-output transfer matrix $\mathbf{H}(s)$ *can be obtained as follows*:

$$\mathbf{H}(s) = \mathbf{C}(\mathbf{L}\, s - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

- In fact, the POG dynamic system $\mathbf{S}$ can be rewritten as follows:

$$\begin{cases} \dot{\mathbf{x}} = \underbrace{\mathbf{L}^{-1}\mathbf{A}}_{\overline{\mathbf{A}}}\mathbf{x} + \underbrace{\mathbf{L}^{-1}\mathbf{B}}_{\overline{\mathbf{B}}}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

and the transfer matrix $\mathbf{H}(s)$ can be computed as follows:

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(\mathbf{I}\, s - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} + \mathbf{D} \\ &= \mathbf{C}(\mathbf{I}\, s - (\mathbf{L}^{-1}\mathbf{A}))^{-1}(\mathbf{L}^{-1}\mathbf{B}) + \mathbf{D} \\ &= \mathbf{C}(\mathbf{L}^{-1}(\mathbf{L}\, s - \mathbf{A}))^{-1}(\mathbf{L}^{-1}\mathbf{B}) + \mathbf{D} \\ &= \mathbf{C}(\mathbf{L}\, s - \mathbf{A})^{-1}\,\mathbf{L}\,(\mathbf{L}^{-1}\mathbf{B}) + \mathbf{D} \\ &= \mathbf{C}(\mathbf{L}\, s - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \end{aligned}$$

- *A time-invariant "congruent" transformation* $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$:

$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases} \quad \xRightarrow{\mathbf{x}=\mathbf{T}\,\overline{\mathbf{x}}} \quad \overline{\mathbf{S}} = \begin{cases} \overline{\mathbf{L}}\,\dot{\overline{\mathbf{x}}} = \overline{\mathbf{A}}\,\overline{\mathbf{x}} + \overline{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \overline{\mathbf{C}}\,\overline{\mathbf{x}} + \mathbf{D}\,\mathbf{u} \end{cases}$$

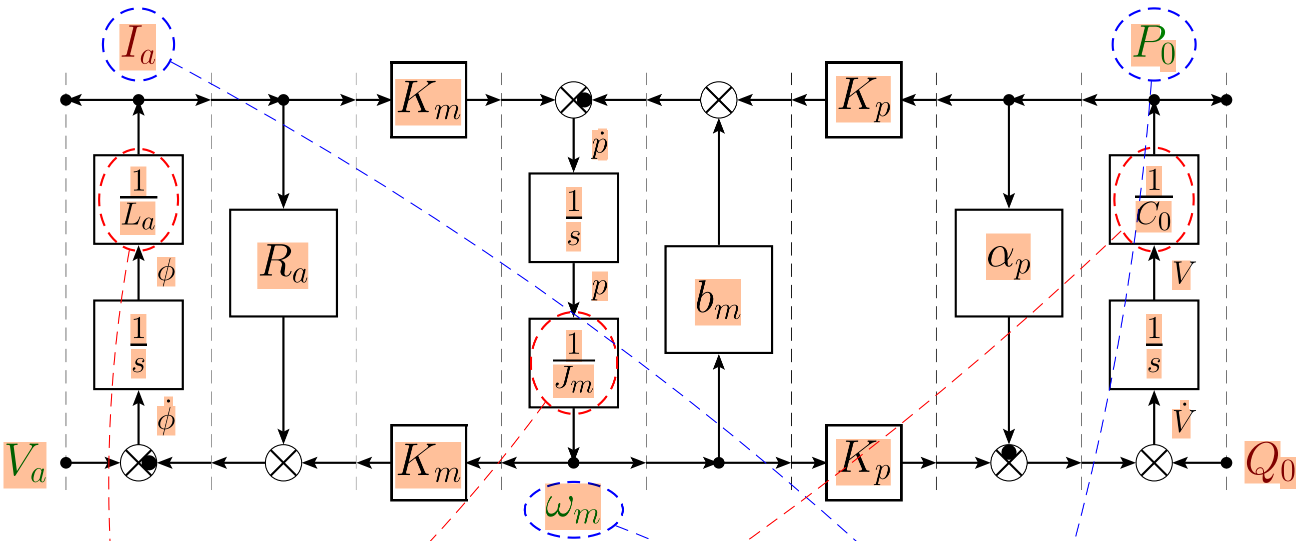
does not change the transfer matrix $\mathbf{H}(s)$ of the considered POG system:

$$\overline{\mathbf{H}}(s) = \overline{\mathbf{C}}(\overline{\mathbf{L}}\, s - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} + \mathbf{D} = \mathbf{C}(\mathbf{L}\, s - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \mathbf{H}(s).$$

- In fact, the transfer matrix $\overline{\mathbf{H}}(s)$ of system $\overline{\mathbf{S}}$ can be computed as follows:

$$\begin{aligned} \overline{\mathbf{H}}(s) &= \overline{\mathbf{C}}(\overline{\mathbf{L}}\, s - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} + \mathbf{D} \\ &= \mathbf{C}\mathbf{T}(\mathbf{T}^T\mathbf{L}\mathbf{T}s - \mathbf{T}^T\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^T\mathbf{B} + \mathbf{D} \\ &= \mathbf{C}\mathbf{T}(\mathbf{T}^T(\mathbf{L}s - \mathbf{A})\mathbf{T})^{-1}\mathbf{T}^T\mathbf{B} + \mathbf{D} \\ &= \mathbf{C}\mathbf{T}\mathbf{T}^{-1}(\mathbf{L}s - \mathbf{A})^{-1}\mathbf{T}^{-T}\mathbf{T}^T\mathbf{B} + \mathbf{D} \\ &= \mathbf{C}(\mathbf{L}\, s - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \mathbf{H}(s). \end{aligned}$$

- *The POG state space equations can be read directly from the POG scheme.* Example: from the following POG scheme



one obtains the following POG state space equations:

$$\underbrace{\begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ Q_0 \end{bmatrix}}_{\mathbf{u}}$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \mathbf{u}$$

- The POG state space equations are obtained using the following procedure:

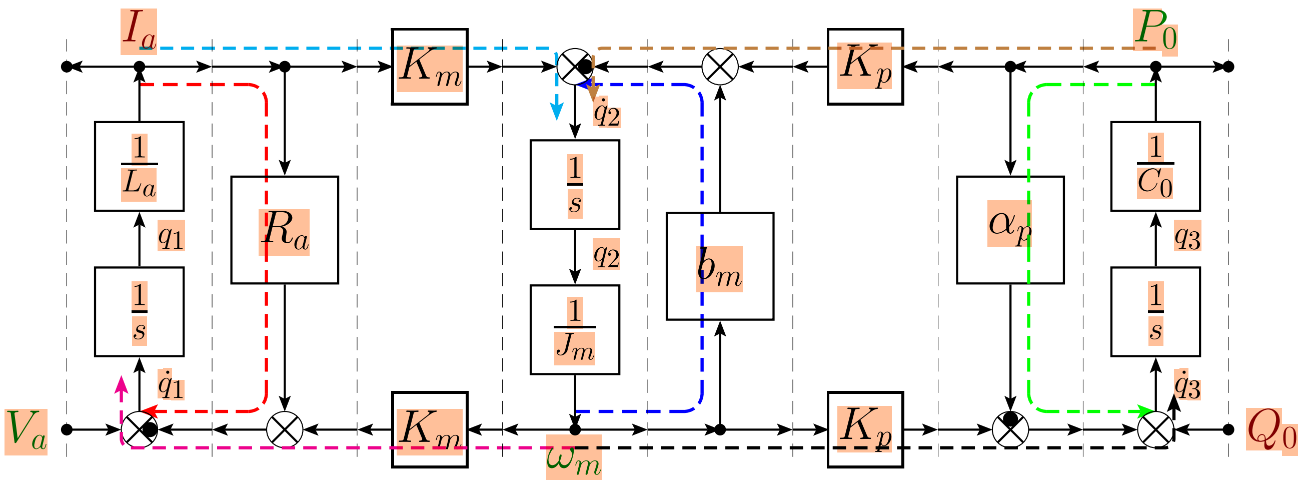
  1) The components of the state vector $\mathbf{x}$ must be chosen equal to the "*output power variables*" of the dynamic elements. In this case it is:

$$\mathbf{x} = \begin{bmatrix} I_a & \omega_m & P_0 \end{bmatrix}^T$$

  2) The diagonal coefficients of the energy matrix $\mathbf{L}$ are the coefficients of the *constitutive relations* that link the "*output power variables*" $\mathbf{x}$ to the "*internal energy variables*" $\mathbf{q} = \mathbf{L}\mathbf{x}$:

$$\begin{aligned} \phi &= L_a\, I_a \\ p &= J_m\, \omega_m \\ V &= C_0\, P_0 \end{aligned} \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix} \phi \\ p \\ V \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}}$$

3) The coefficients A_{ij} of matrix $\mathbf{A}$ are the gains of all the paths that link the j-th state variables x_j to the i-th input $\dot{q}_i$ of the integrators.



$$\underbrace{\begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ Q_0 \end{bmatrix}}_{\mathbf{u}}$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \mathbf{u}$$

The element $A_{23} = -K_p$ of matrix $\mathbf{A}$, for example, is the gain of the path that links the third state variable $x_3 = P_0$ to the input $\dot{q}_2 = \dot{p}$ of the second integrator.

4) The coefficients B_{ij}, C_{ij} and D_{ij} of matrices $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ can be determined in a similar way:

a) coefficients "B_{ij}" are the gains of the paths that link the *j-th input* u_j to the *input $\dot{q}_i$ of the i-th integrator*. The coefficient $B_{32} = 1$, for example, is the gain of the path that goes from input $u_2 = Q_0$ to the input $\dot{q}_3$ of the third integrator.

b) coefficients "C_{ij}" are the gains of the path that link the *j-th state variable* x_j to the *i-th output* y_i of the system;

c) coefficients "D_{ij}" are the gains of the paths that link the *j-th input* u_j to the *i-th output* y_i of the system.

- The properties of a POG dynamic system hold for the considered system:

1) The energy matrix $\mathbf{L}$ is *symmetric and positive definite*:

$$\mathbf{L} = \mathbf{L}^T > 0 \qquad \Rightarrow \qquad \mathbf{L} = \begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}$$

2) The energy E_s stored in the system can be expressed as follows:

$$E_s = \frac{1}{2}\mathbf{x}^T \mathbf{L}\, \mathbf{x} \geq 0 \qquad \Rightarrow \qquad E_s = \frac{1}{2} L_a I_a^2 + \frac{1}{2} J_m \omega_m^2 + \frac{1}{2} C_0 P_0^2.$$

Matrix $\mathbf{L}$ is characterized by the *coefficients of the constitutive relations* of the dynamic elements of the system.

3) The power P_d dissipated in the system can be expressed as follows:

$$P_d = \mathbf{x}^T \mathbf{A}_s\, \mathbf{x} \qquad \Rightarrow \qquad P_d = -R_a I_a^2 - b_m \omega_m^2 - \alpha_p P_0^2$$

where $\mathbf{A}_s$ is the symmetric part of the power matrix $\mathbf{A}$.

$$\mathbf{A}_s = \frac{(\mathbf{A} + \mathbf{A}^T)}{2} = \begin{bmatrix} -R_a & 0 & 0 \\ 0 & -b_m & 0 \\ 0 & 0 & -\alpha_p \end{bmatrix}$$

Matrix $\mathbf{A}_s$ is always characterized by the coefficients of all the *static dissipative* elements present in the system.

4) The *total power P_w redistributed within the system* is zero:

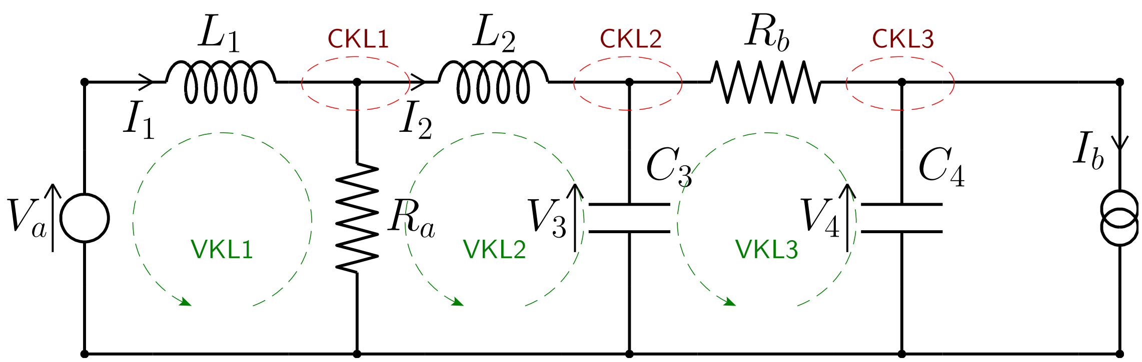
$$P_w = \mathbf{x}^T \mathbf{A}_w\, \mathbf{x} = 0$$

where $\mathbf{A}_w$ is the skew-symmetric part the power matrix $\mathbf{A}$:

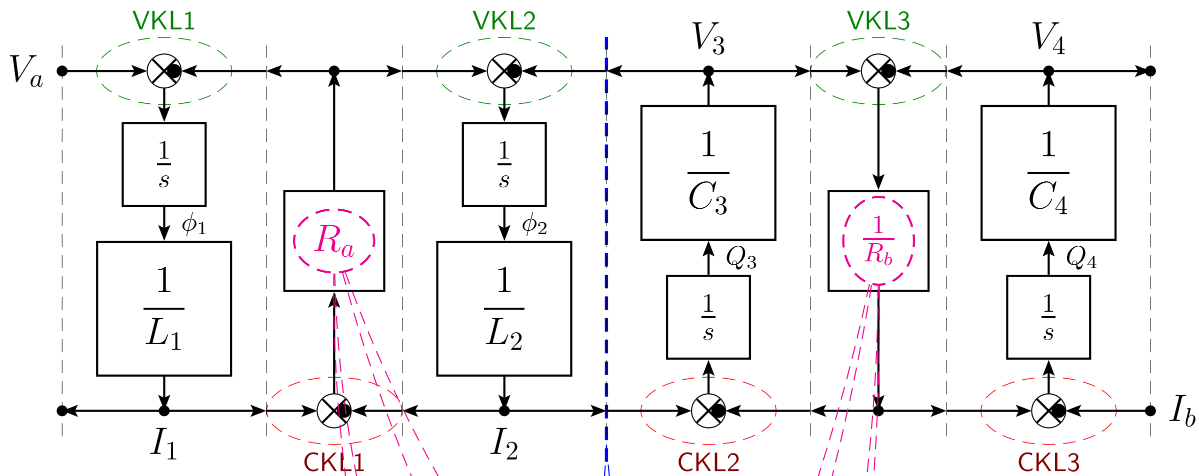
$$\mathbf{A}_w = \frac{(\mathbf{A} - \mathbf{A}^T)}{2} = \begin{bmatrix} 0 & -K_m & 0 \\ K_m & 0 & -K_p \\ 0 & K_p & 0 \end{bmatrix}$$

Matrix $\mathbf{A}_w$ is always characterized by the coefficients of all the *connecting blocks* present in the system.

- **Example**. Let us consider the following electric circuit:



The corresponding POG block scheme is:



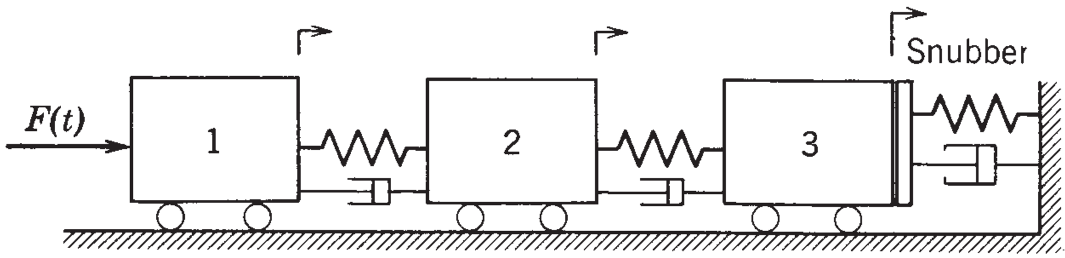
The POG state space dynamic model has the following structure:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & R_a & 0 & 0 \\ R_a & -R_a & -1 & 0 \\ 0 & 1 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

The connection coefficients appear within matrix $\mathbf{A}$ only when one type of energy (i.e. magnetic energy in L) is converted in another different type of energy (i.e. electrostatic energy in C).

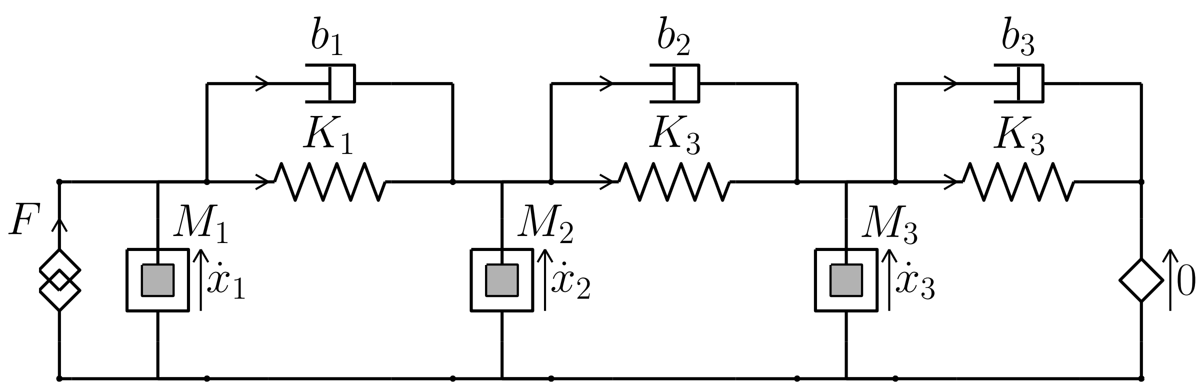
A dissipative parameter (i.e. R_a and R_b) appears 4 times, in a symmetric way, within matrix $\mathbf{A}$ when the corresponding physical element connects two dynamic elements of the same type.

- **Example**. A chain of carriages and snubbers:

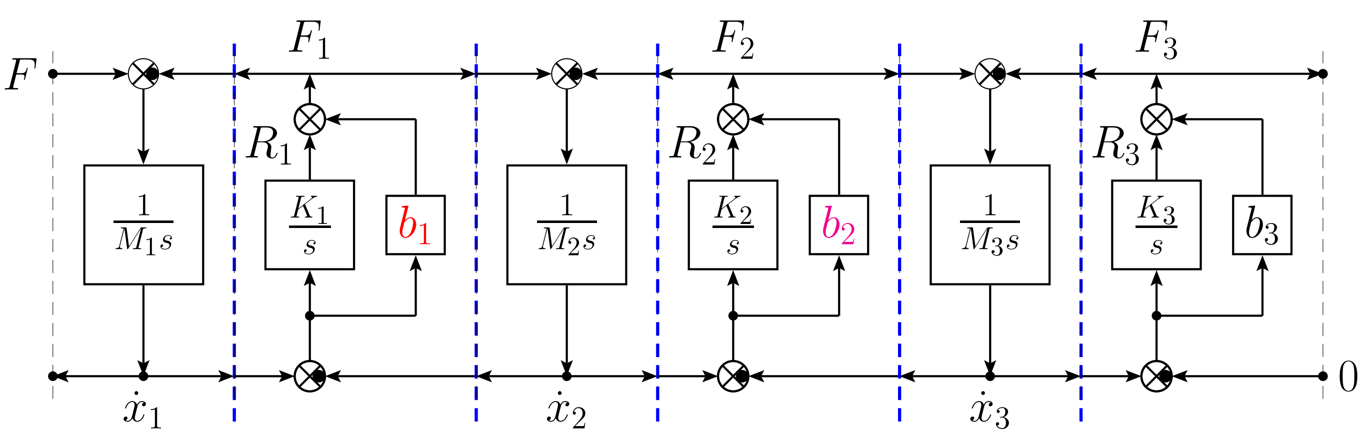


(a)

The symbolic scheme:



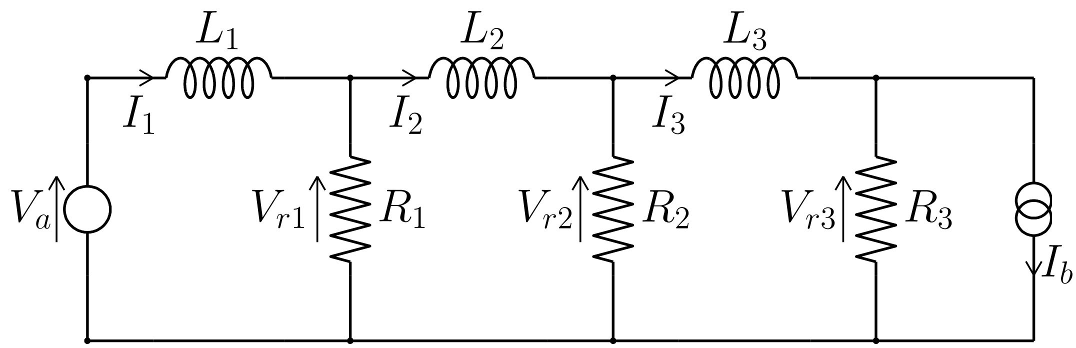
The corresponding POG block scheme:



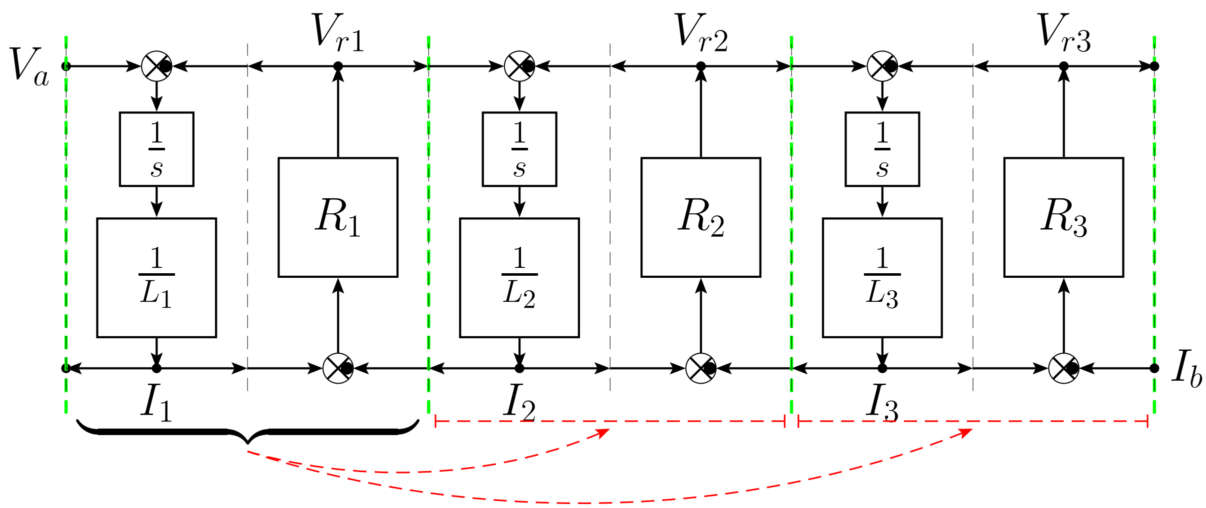
The POG state space dynamic model is:

$$
\begin{bmatrix} M_1 & & & & & \\ & \frac{1}{K_1} & & & & \\ & & M_2 & & & \\ & & & \frac{1}{K_2} & & \\ & & & & M_3 & \\ & & & & & \frac{1}{K_3} \end{bmatrix}
\begin{bmatrix} \ddot{x}_1 \\ \dot{R}_1 \\ \ddot{x}_2 \\ \dot{R}_2 \\ \ddot{x}_3 \\ \dot{R}_3 \end{bmatrix}
=
\begin{bmatrix} -b_1 & -1 & b_1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ b_1 & 1 & -b_1-b_2 & -1 & b_2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & b_2 & 1 & -b_2-b_3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} \dot{x}_1 \\ R_1 \\ \dot{x}_2 \\ R_2 \\ \dot{x}_3 \\ R_3 \end{bmatrix}
+
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F
$$

- **Example**. Let us consider the following electric circuit:



The corresponding POG block scheme:



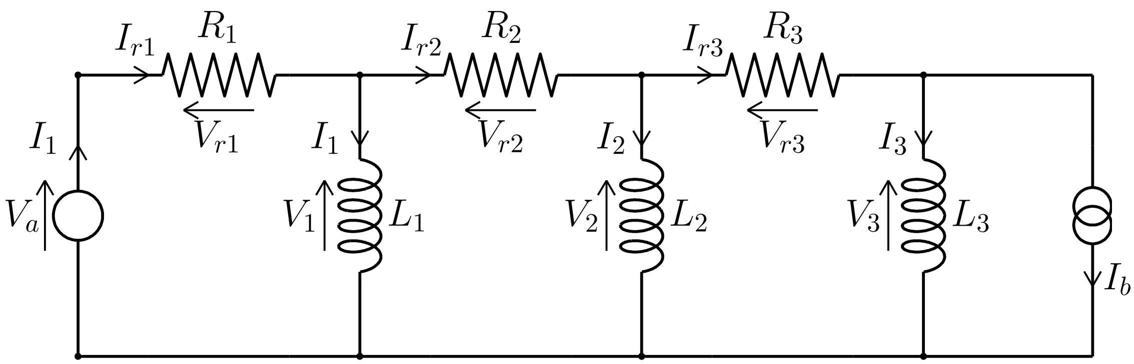
The POG state space dynamic model is:

$$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{bmatrix} = \begin{bmatrix} -R_1 & R_1 & 0 \\ R_1 & -R_1-R_2 & R_2 \\ 0 & R_2 & -R_2-R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & R_3 \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}$$

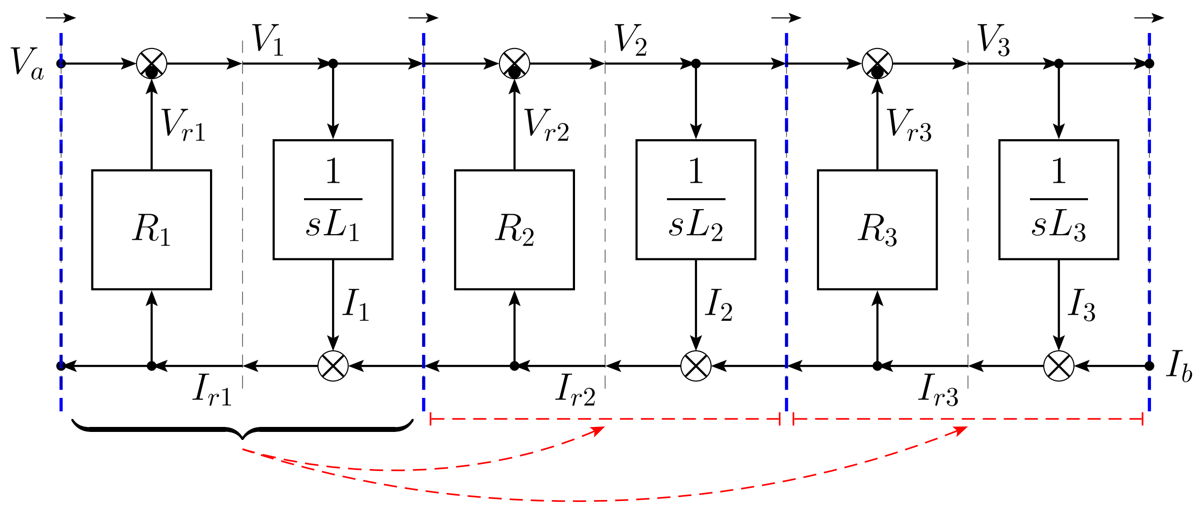
$$\begin{bmatrix} I_1 \\ V_{r3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & R_3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & -R_3 \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}$$

Note that, in this case, the power matrix $\mathbf{A}$ is symmetric because the system is characterized by only one type of stored energy. In this case the magnetic energy is stored in the three inductances L_1, L_2 and L_3.

- **Example**. Let us consider the following electric circuit:



The corresponding POG block scheme:



The POG state space dynamic model is:

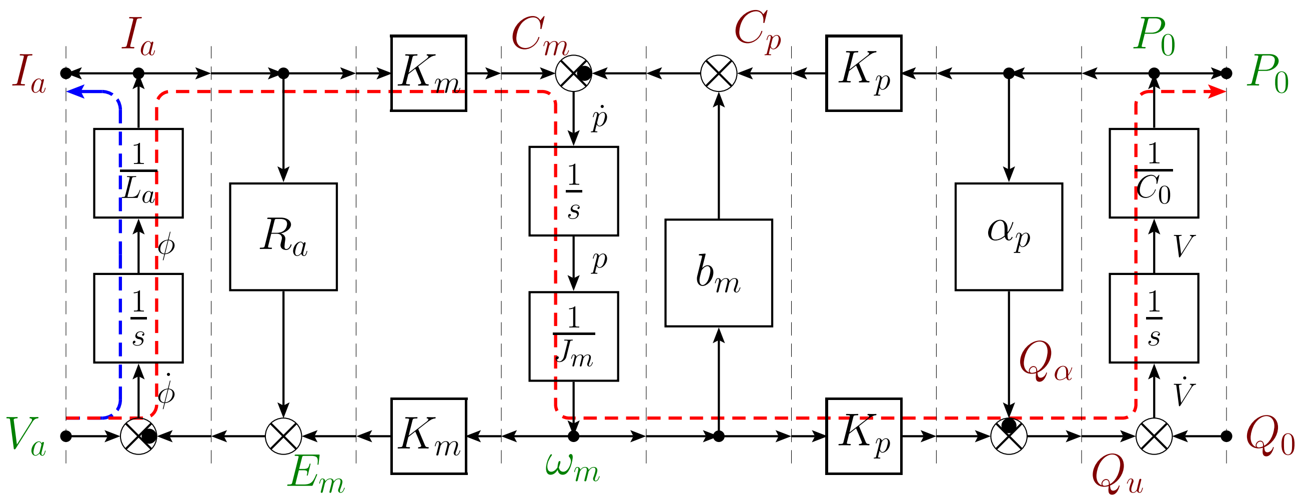
$$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{bmatrix} = \begin{bmatrix} -R_1 & -R_1 & -R_1 \\ -R_1 & -R_1-R_2 & -R_1-R_2 \\ -R_1 & -R_1-R_2 & -R_1-R_2-R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 1 & -R_1 \\ 1 & -R_1-R_2 \\ 1 & -R_1-R_2-R_3 \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}$$

$$\begin{bmatrix} I_{r1} \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -R_1 & -R_1-R_2 & -R_1-R_2-R_3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 1 \\ 1 & -R_1-R_2-R_3 \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}$$

The power matrix $\mathbf{A}$ is symmetric because the system is characterized only by one type of stored energy.

Relative degree of a transfer function $G(s)$

- Let us consider a generic block scheme:



- For each transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links an input $u(t)$ to the output $y(t)$, the following properties hold:

  1) the order of function $G(s)$ is equal to the number n of independent dynamic elements which store energy within the system;
  2) the poles of function $G(s)$ are equal to the solutions of equation $\Delta(s) = 0$ where $\Delta(s)$ is the determinant of the block scheme;
  3) the **relative degree** of function $G(s)$ is equal to the minimum number r of integrators present in all the paths that link the input $u(t)$ to the output $y(t)$;
  4) if there is only one path $\mathcal{P}_1$ that links the input $u(t)$ to the output $y(t)$, then the zeros of function $G(s)$ are equal to the solutions of equation $\Delta_1(s) = 0$ where $\Delta_1(s)$ is the determinant of the reduced block scheme obtained from the original one eliminating all the blocks touched by path $\mathcal{P}_1$;

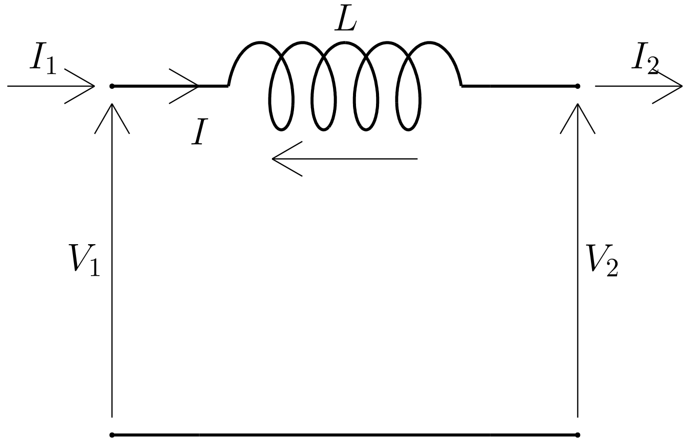
  $G(s) = \frac{P_0}{V_a}$ has 3 poles and 0 zeros because the relative degree is $r = 3$;

  $G(s) = \frac{I_a}{V_a}$ has 3 poles and 2 zeros because the relative degree is $r = 1$;

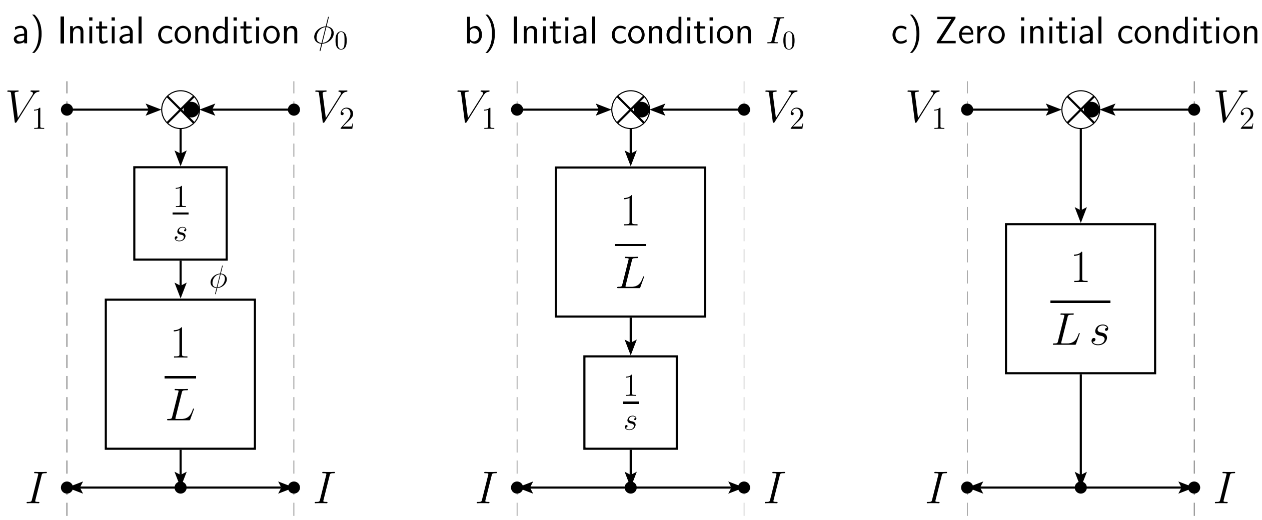
- Note: the higher is the relative degree the more difficult is the control.

Linear systems: equivalent block schemes

- Let us consider the following inductor connected in series:



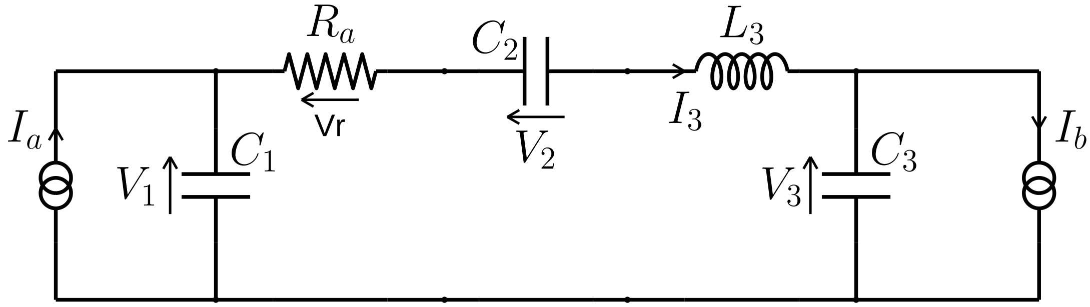
- For the considered linear system, three mathematically equivalent POG block schemes can be used:



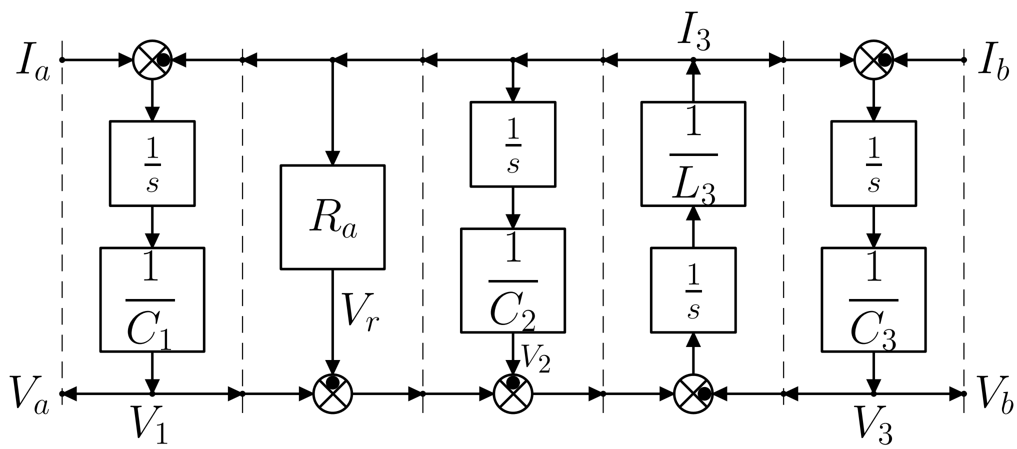
- For linear systems it is always possible to switch the position of two cascade connected blocks without modifying the input-output dynamic behavior of the considered system.
- If the three block schemes were implemented in Simulink, their initial conditions would be $\phi = \phi_0$, $I = I_0$ and $I = 0$, respectively.

POG state space equations: examples

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

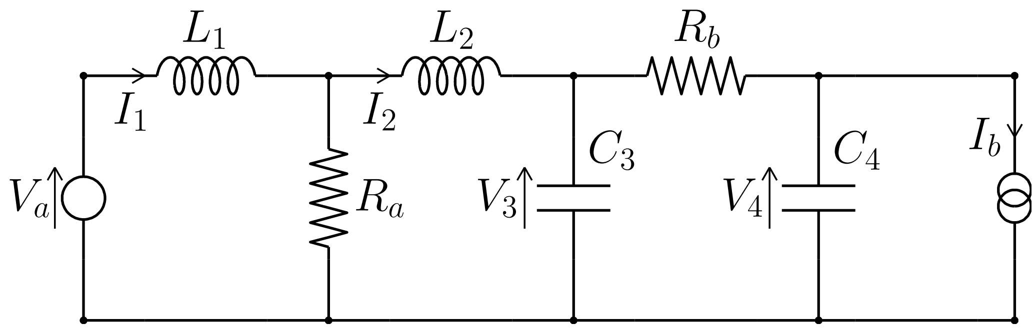
$$
\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_3 \end{bmatrix}}_{\mathbf{L}}
\underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_3 \end{bmatrix}}_{\dot{\mathbf{x}}}
=
\underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & -R_a & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ V_3 \end{bmatrix}}_{\mathbf{x}}
+
\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}}
\underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{u}}
$$

$$
\underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{y}}
=
\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}}
\mathbf{x} +
\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}}
\underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{u}}
$$

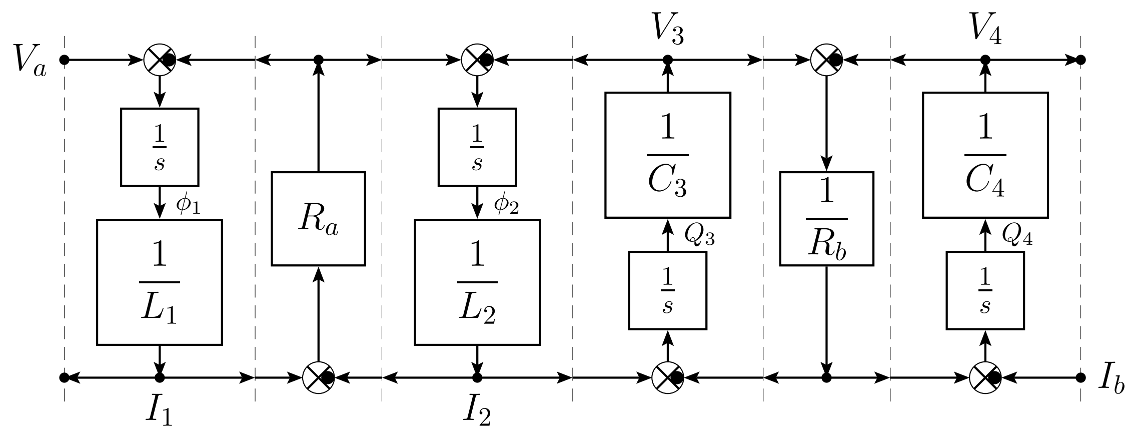
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$
\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R_a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad
\mathbf{A}_w = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

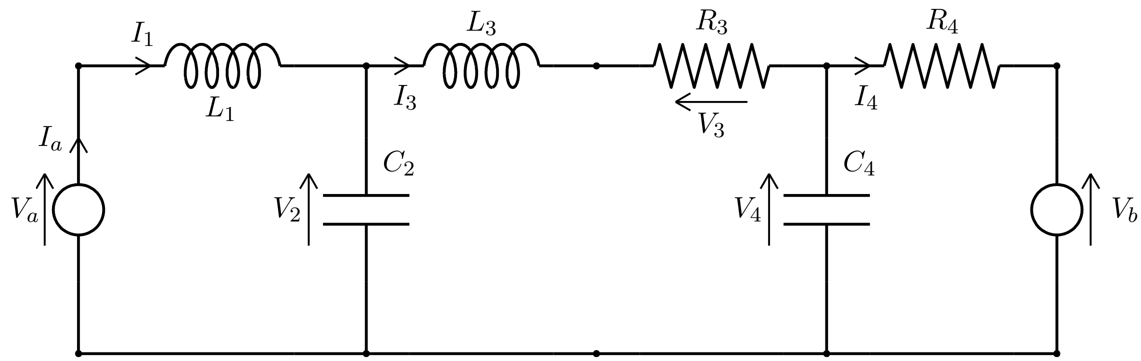
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & R_a & 0 & 0 \\ R_a & -R_a & -1 & 0 \\ 0 & 1 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_1 \\ V_4 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

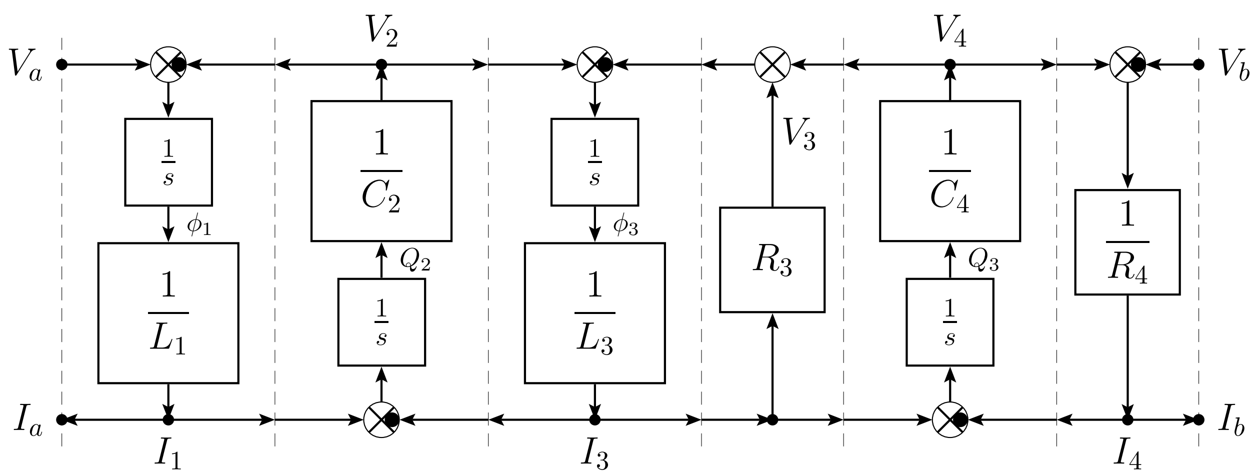
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_a & R_a & 0 & 0 \\ R_a & -R_a & 0 & 0 \\ 0 & 0 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

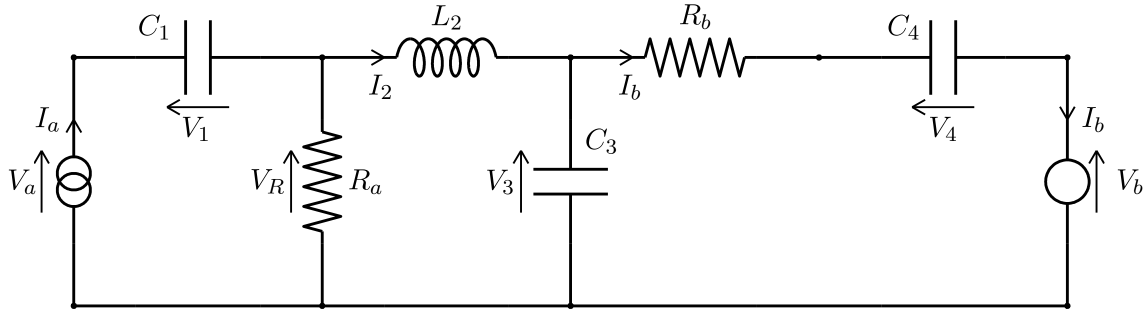
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -R_3 & -1 \\ 0 & 0 & 1 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{R_4} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_4} \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

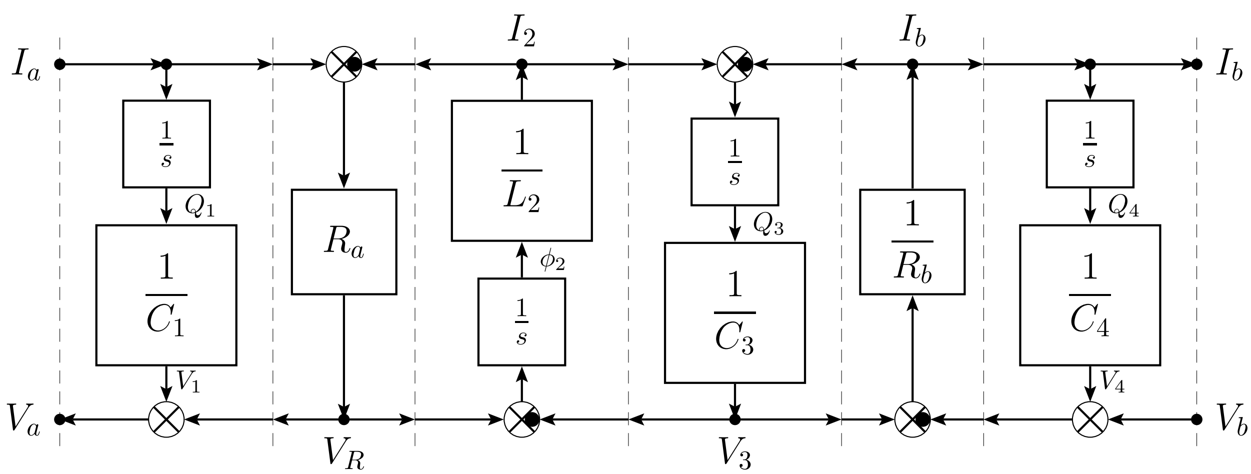
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_4} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

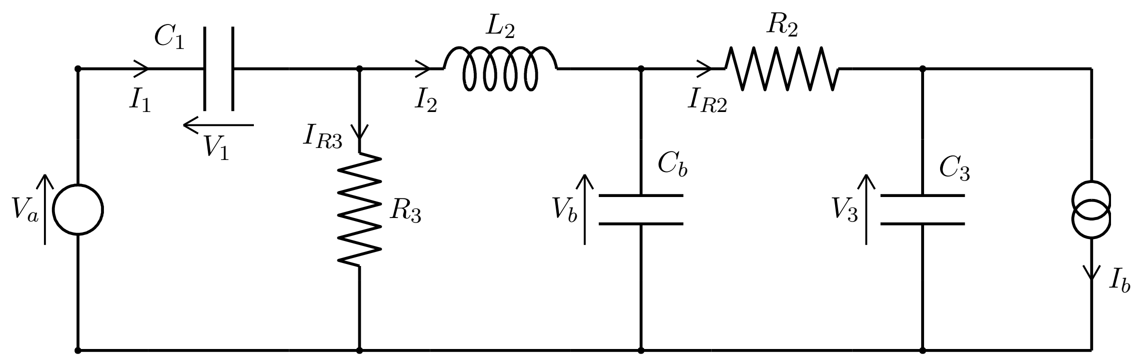
$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -R_a & -1 & 0 \\ 0 & 1 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ R_a & 0 \\ 0 & \frac{1}{R_b} \\ 0 & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -R_a & 0 & 0 \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} R_a & 0 \\ 0 & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

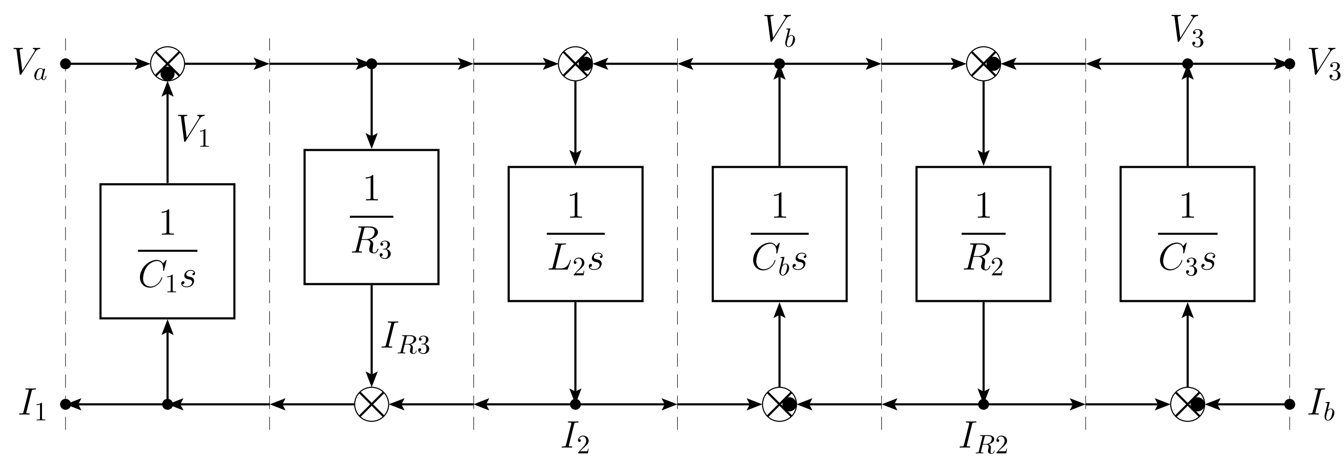
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -R_a & 0 & 0 \\ 0 & 0 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

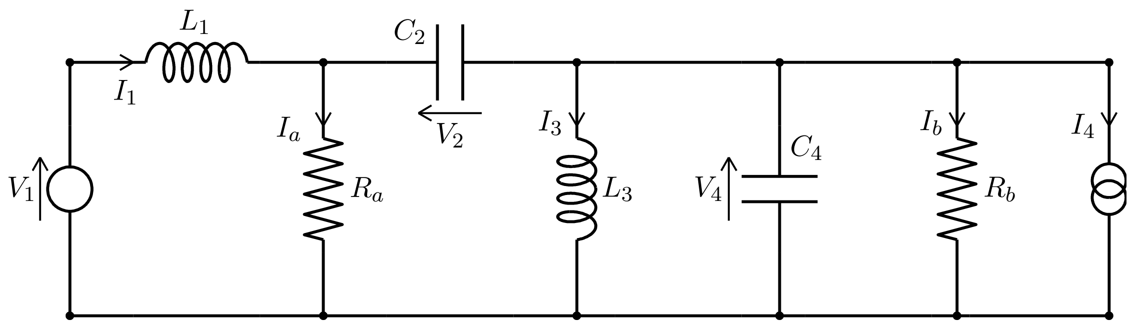
$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_b & 0 \\ 0 & 0 & 0 & C_3 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{I}_2 \\ \dot{V}_b \\ \dot{V}_3 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -\frac{1}{R_3} & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{R_2} & \frac{1}{R_2} \\ 0 & 0 & \frac{1}{R_2} & -\frac{1}{R_2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ V_b \\ V_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{1}{R_3} & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_1 \\ V_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -\frac{1}{R_3} & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_3} & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

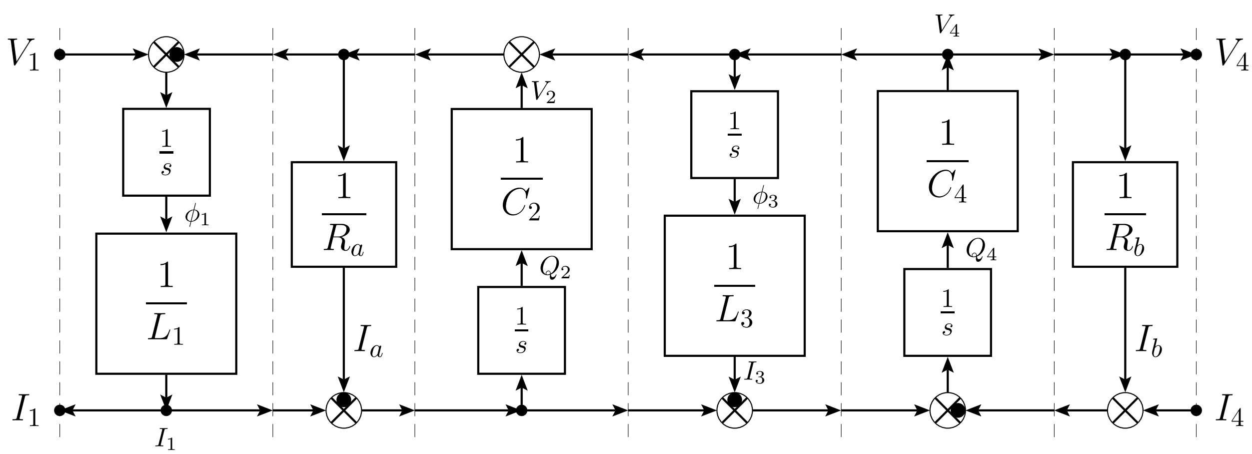
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_2} & \frac{1}{R_2} \\ 0 & 0 & \frac{1}{R_2} & -\frac{1}{R_2} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

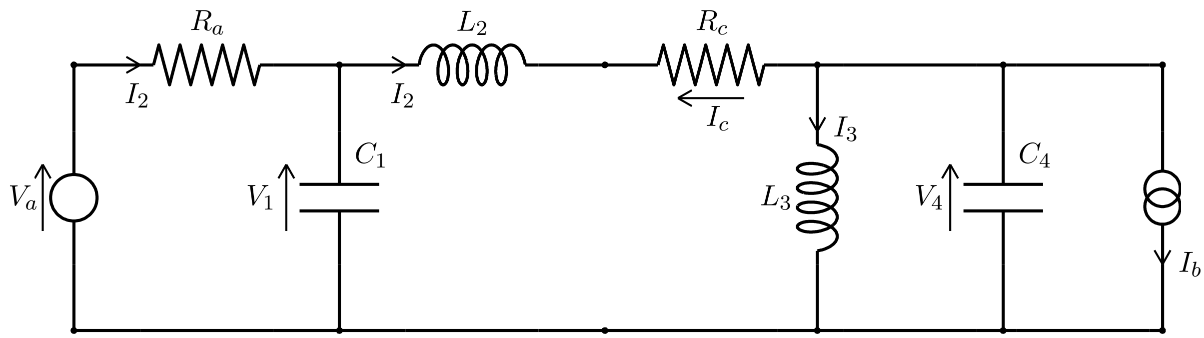
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & -\frac{1}{R_a} & 0 & -\frac{1}{R_a} \\ 0 & 0 & 0 & 1 \\ 1 & -\frac{1}{R_a} & -1 & -\frac{1}{R_a} - \frac{1}{R_b} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_1 \\ I_4 \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_1 \\ V_4 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_1 \\ I_4 \end{bmatrix}}_{\mathbf{u}}$$

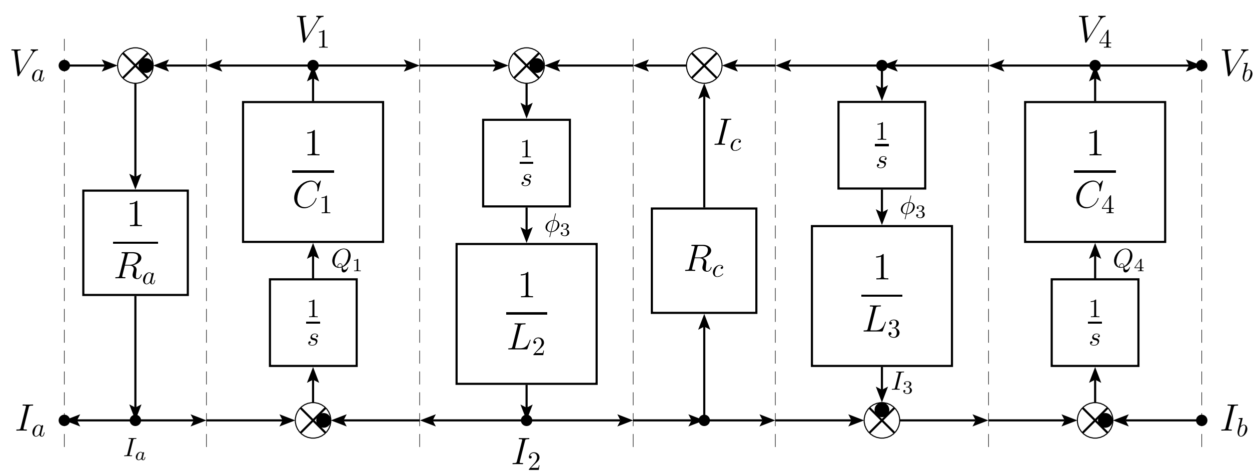
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_a} & 0 & -\frac{1}{R_a} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_a} & 0 & -\frac{1}{R_a} - \frac{1}{R_b} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

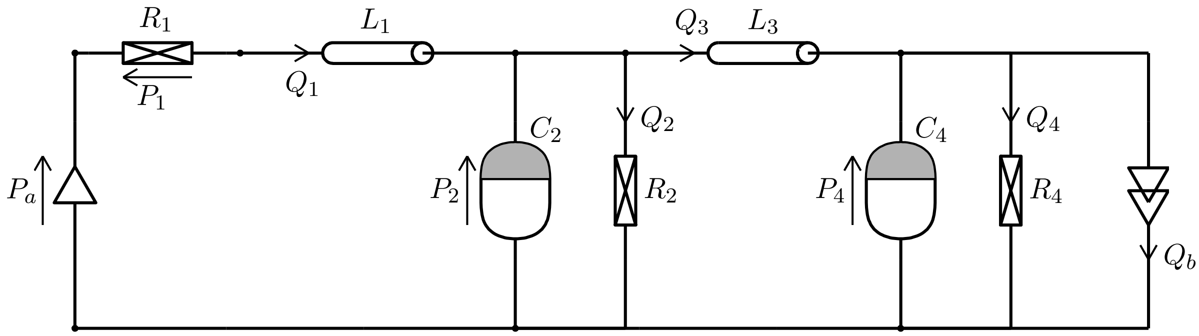
$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{I}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -\frac{1}{R_a} & -1 & 0 & 0 \\ 1 & -R_c & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -\frac{1}{R_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

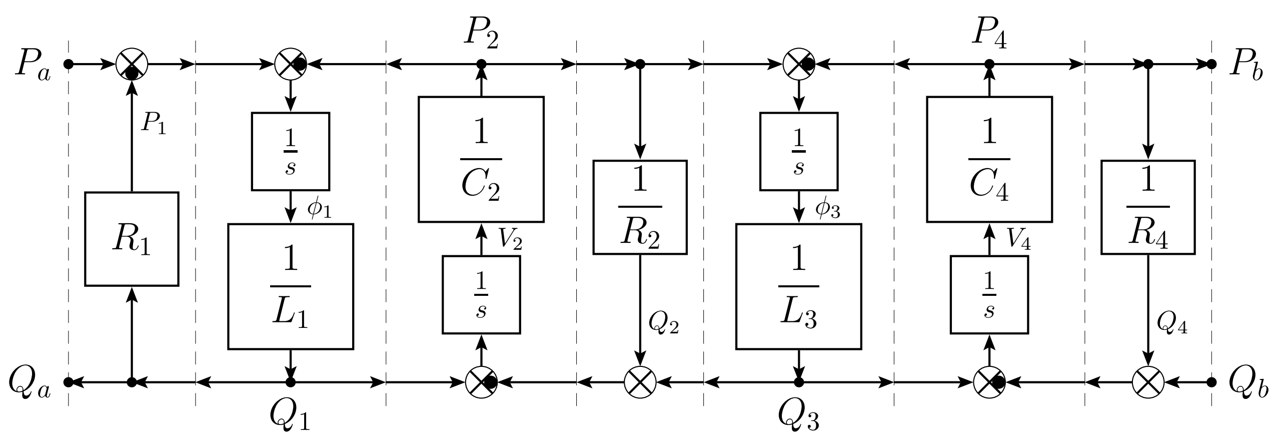
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -\frac{1}{R_a} & 0 & 0 & 0 \\ 0 & -R_c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

- The following hydraulic circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

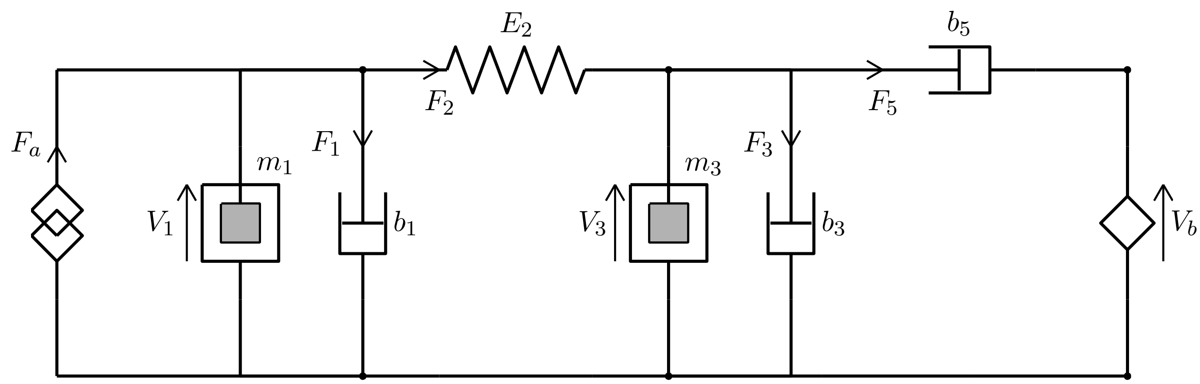
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{Q}_1 \\ \dot{P}_2 \\ \dot{Q}_3 \\ \dot{P}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_1 & -1 & 0 & 0 \\ 1 & -\frac{1}{R_2} & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} Q_1 \\ P_2 \\ Q_3 \\ P_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} Q_a \\ P_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

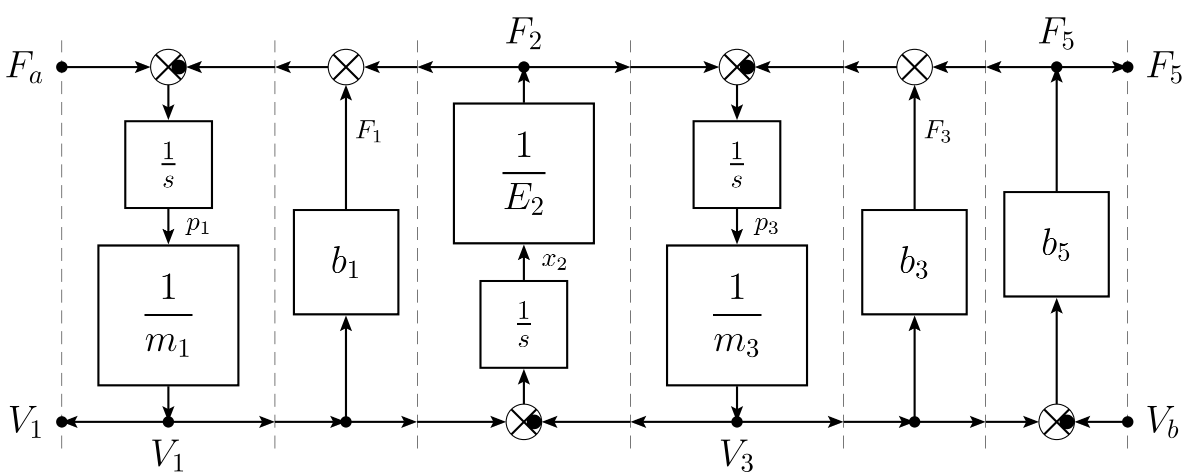
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_1 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_4} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The following mechanical system:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

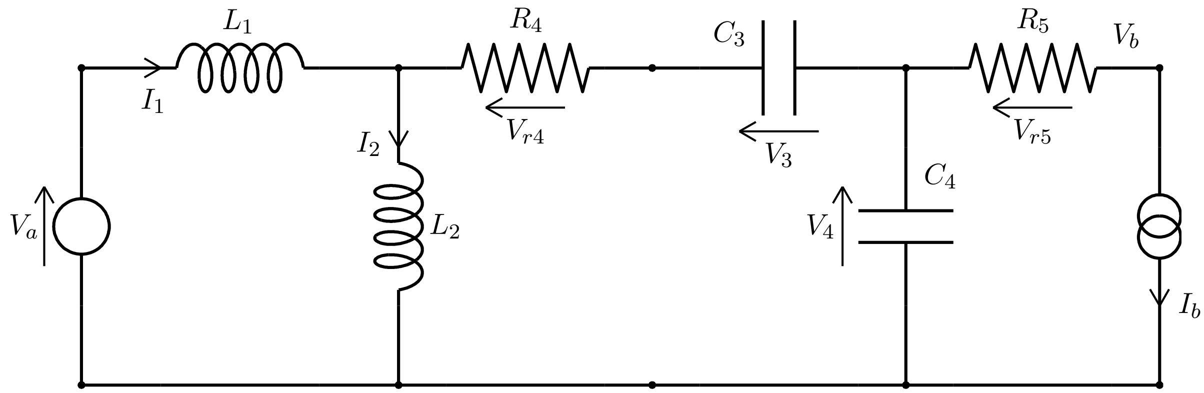
$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{F}_2 \\ \dot{V}_3 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -b_1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -b_3-b_5 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ F_2 \\ V_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b_5 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} F_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_1 \\ F_5 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & b_5 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -b_5 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} F_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

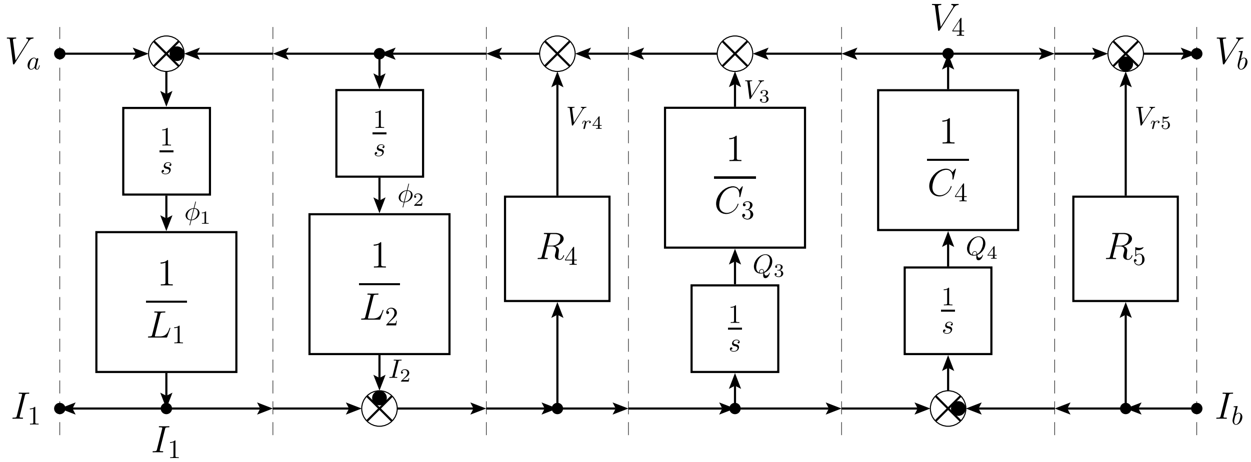
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -b_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b_3-b_5 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

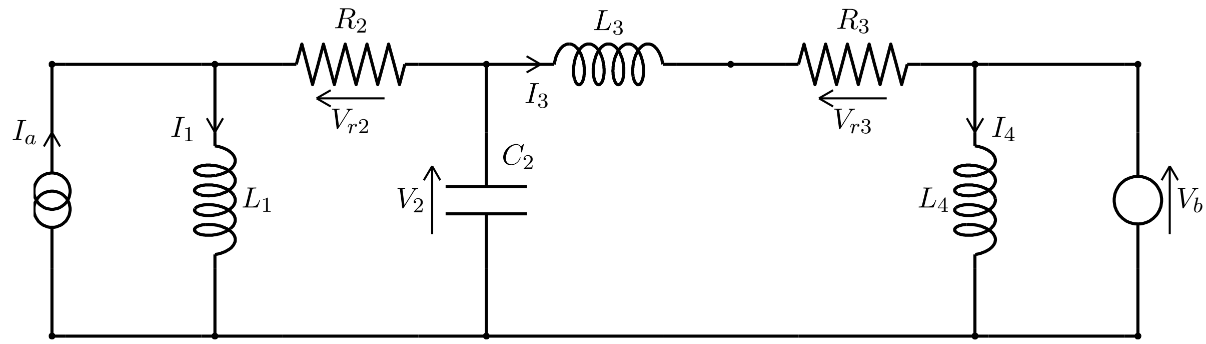
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_4 & R_4 & -1 & -1 \\ R_4 & -R_4 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_1 \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -R_5 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

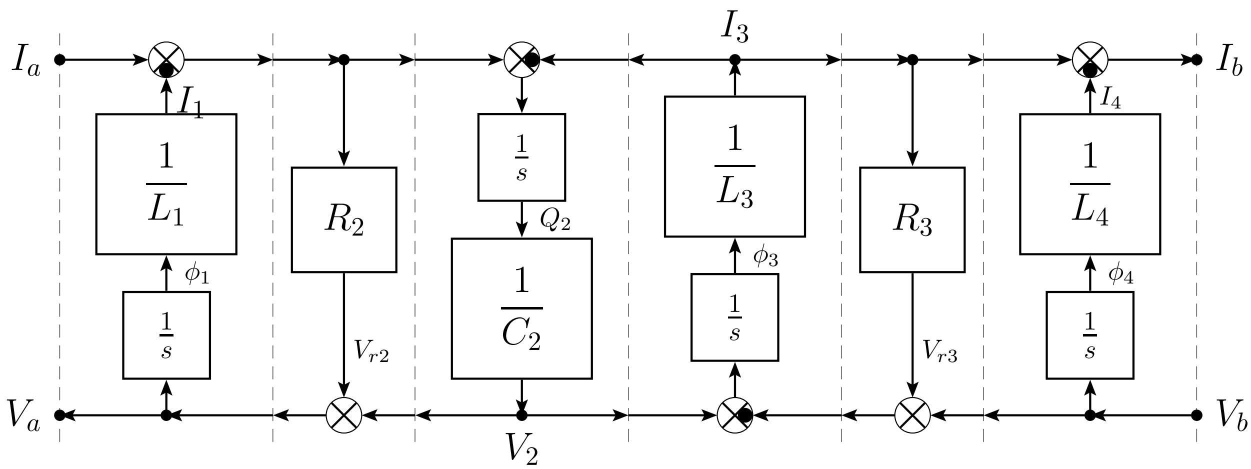
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_4 & R_4 & 0 & 0 \\ R_4 & -R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

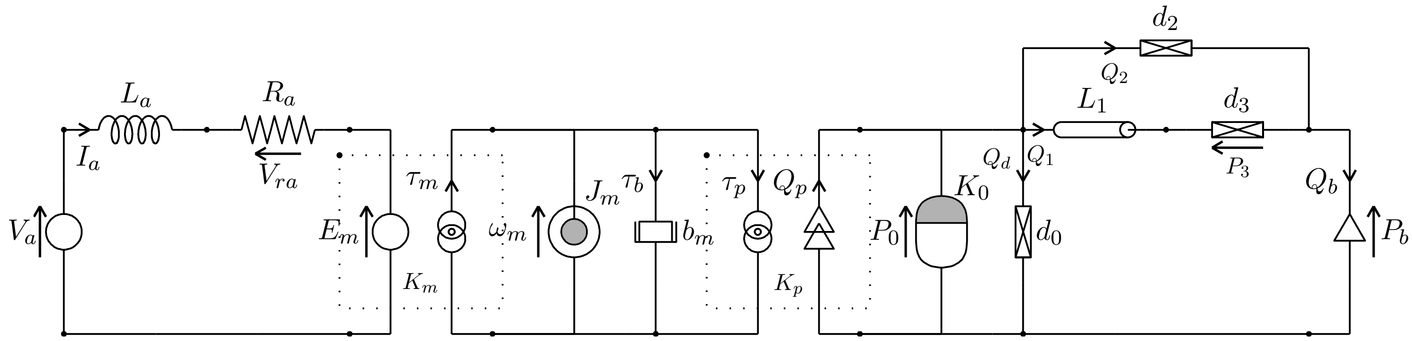
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{I}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -R_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} R_2 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -R_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} R_1 + R_2 & 0 \\ 0 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

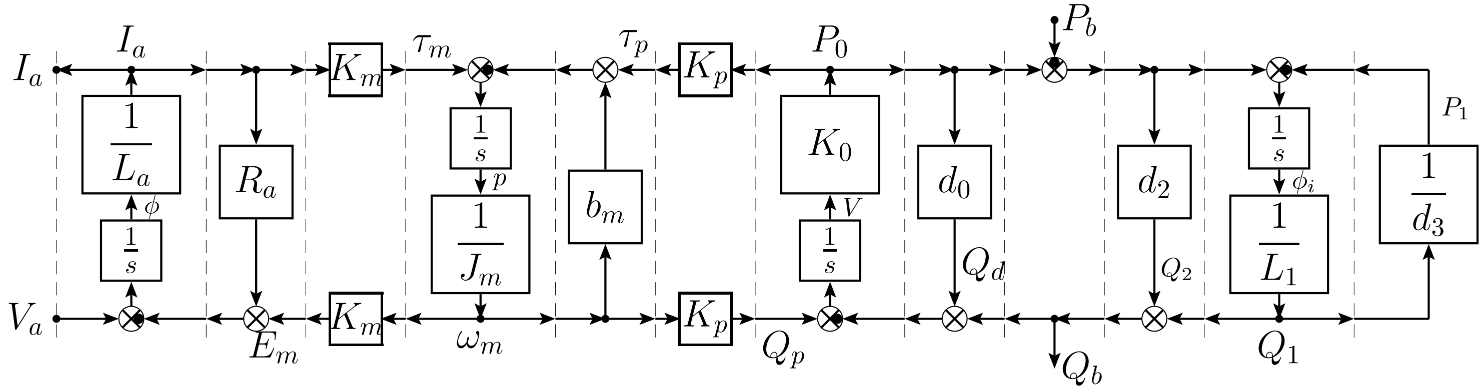
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The following dynamic system (DC motor with an hydraulic pump):



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

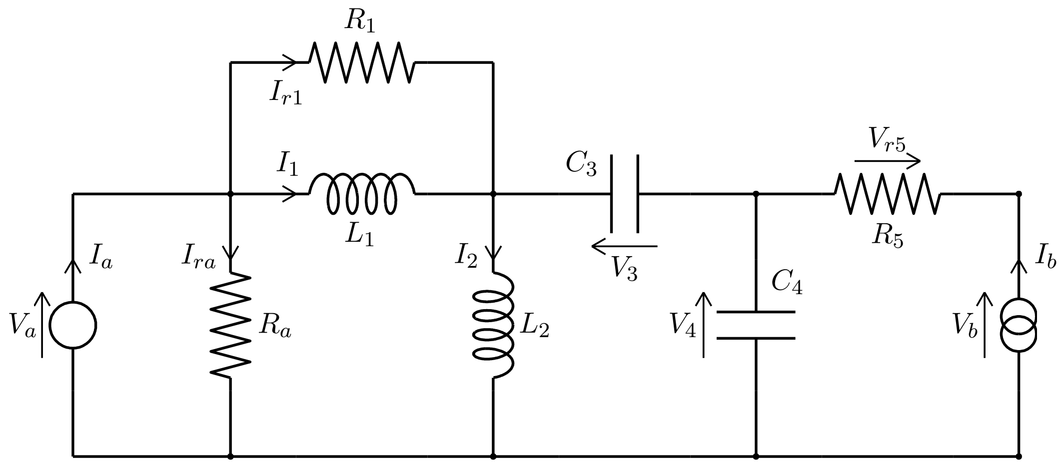
$$\underbrace{\begin{bmatrix} L_a & 0 & 0 & 0 \\ 0 & J_m & 0 & 0 \\ 0 & 0 & \frac{1}{K_0} & 0 \\ 0 & 0 & 0 & L_1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \\ \dot{Q}_1 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & -K_m & 0 & 0 \\ K_m & -b_m & -K_p & 0 \\ 0 & K_p & -d_0 - d_2 & -1 \\ 0 & 0 & 1 & -\frac{1}{d_3} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \\ Q_1 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & d_2 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ P_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ Q_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & d_2 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -d_2 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ P_b \end{bmatrix}}_{\mathbf{u}}$$

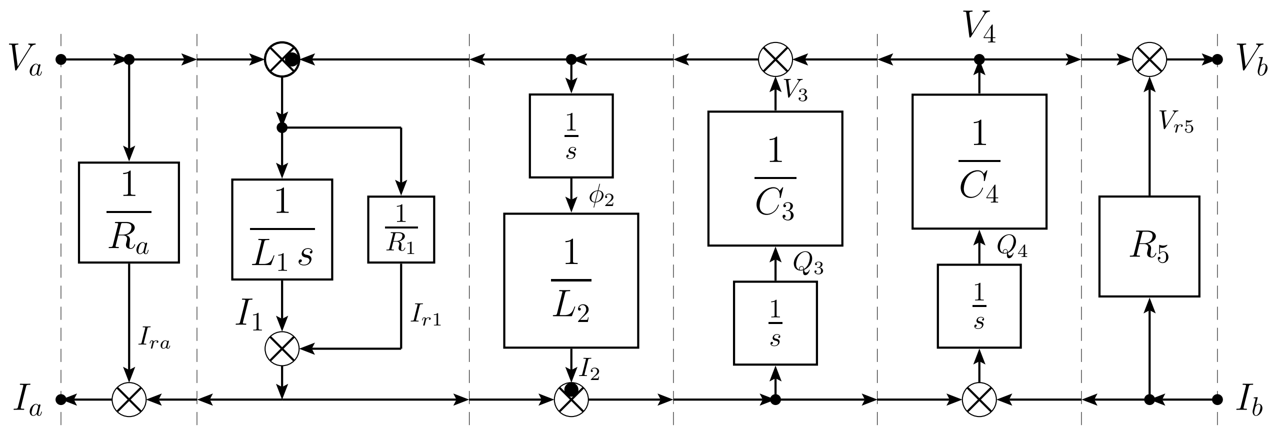
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_a & 0 & 0 & 0 \\ 0 & -b_m & 0 & 0 \\ 0 & 0 & -d_0 - d_2 & 0 \\ 0 & 0 & 0 & -\frac{1}{d_3} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -K_m & 0 & 0 \\ K_m & 0 & -K_p & 0 \\ 0 & K_p & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The following electric circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

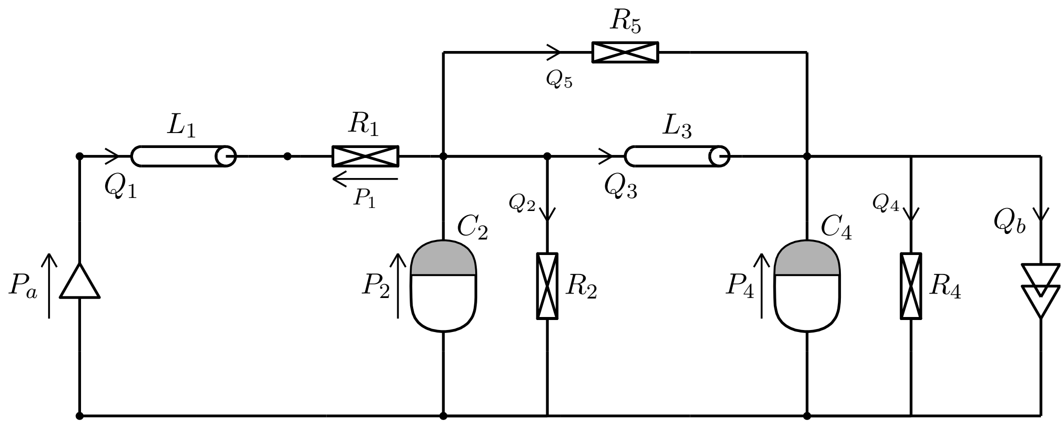
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 1 & -1 & -\frac{1}{R_1} & -\frac{1}{R_1} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \frac{1}{R_1} & 0 \\ \frac{1}{R_1} & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_a} + \frac{1}{R_1} & 0 \\ 0 & R_5 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

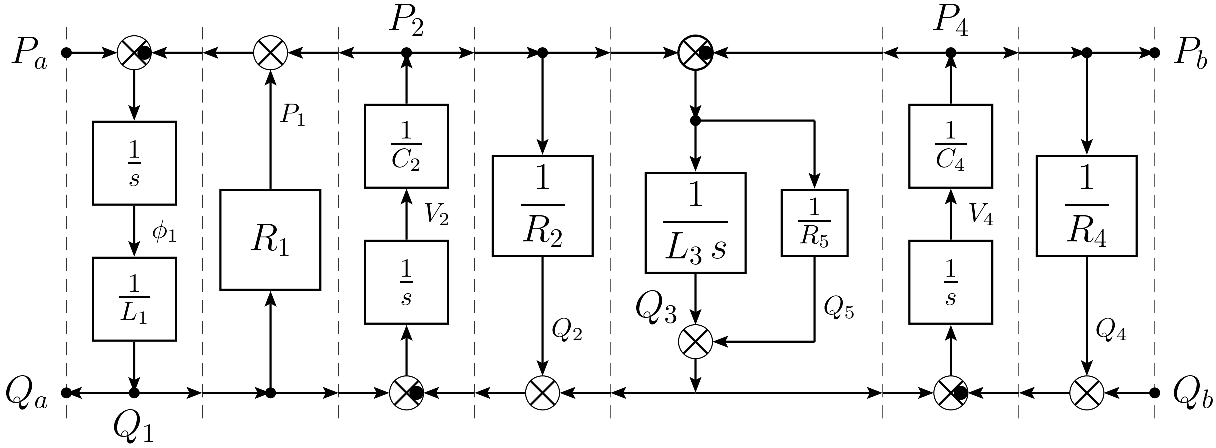
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 0 & 0 & -\frac{1}{R_1} & -\frac{1}{R_1} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

- The following hydraulic circuit:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

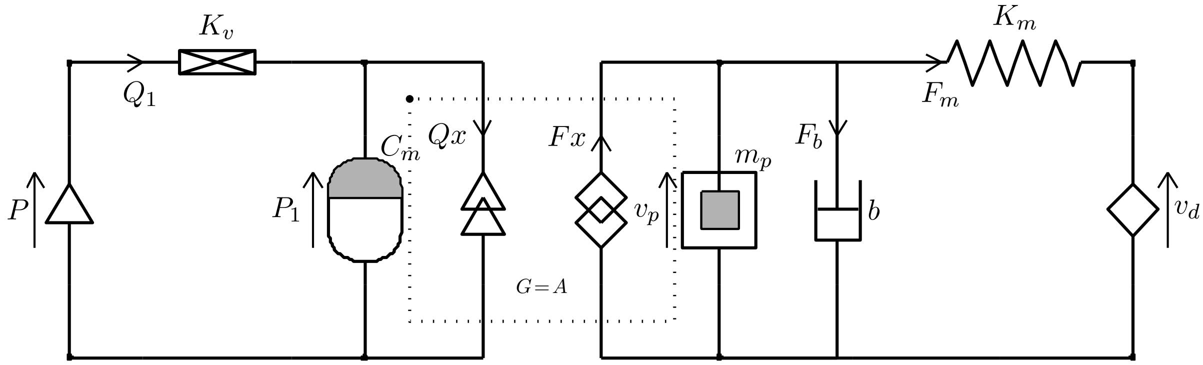
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{Q}_1 \\ \dot{P}_2 \\ \dot{Q}_3 \\ \dot{P}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_1 & -1 & 0 & 0 \\ 1 & -\frac{1}{R_2}-\frac{1}{R_5} & -1 & \frac{1}{R_5} \\ 0 & 1 & 0 & -1 \\ 0 & \frac{1}{R_5} & 1 & -\frac{1}{R_4}-\frac{1}{R_5} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} Q_1 \\ P_2 \\ Q_3 \\ P_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} Q_a \\ P_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

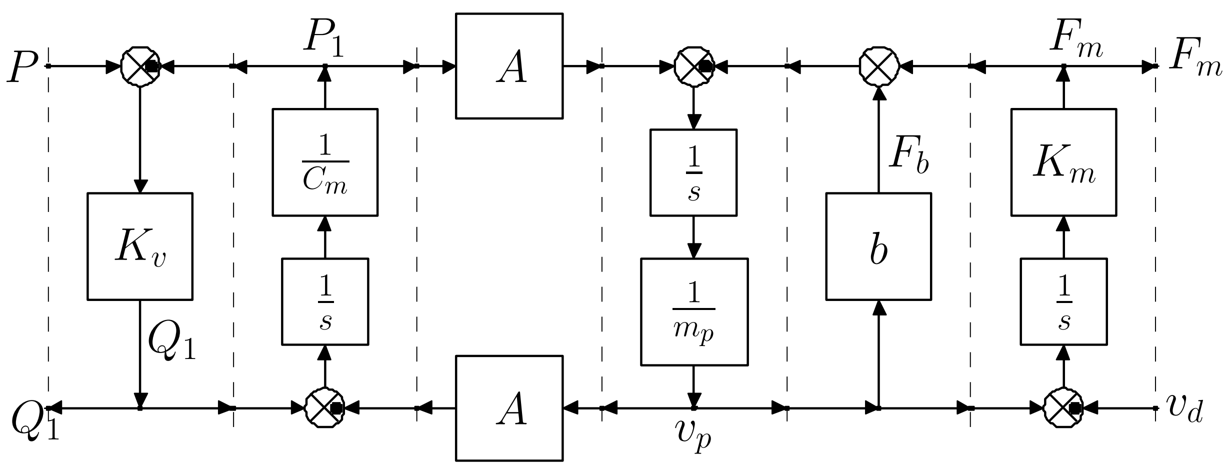
- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_1 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_2}-\frac{1}{R_5} & 0 & \frac{1}{R_5} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_5} & 0 & -\frac{1}{R_4}-\frac{1}{R_5} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The following friction system:



is described by the following POG scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

$$\underbrace{\begin{bmatrix} C_m & 0 & 0 \\ 0 & m_p & 0 \\ 0 & 0 & \frac{1}{K_m} \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{P}_1 \\ \dot{v}_p \\ \dot{F}_m \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -K_v & -A & 0 \\ A & -b & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} P_1 \\ v_p \\ F_m \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} P \\ v_d \end{bmatrix}}_{\mathbf{u}}$$

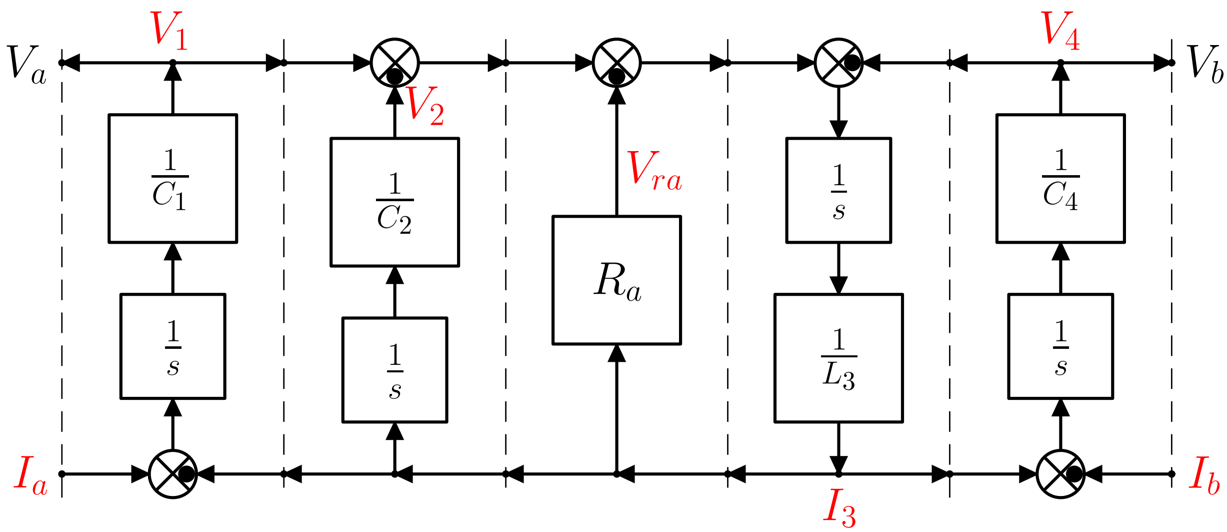
$$\underbrace{\begin{bmatrix} V_1 \\ F_5 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} P \\ v_d \end{bmatrix}}_{\mathbf{u}}$$

- Matrices $\mathbf{A}_s$ and $\mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -K_v & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -A & 0 \\ A & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

POG State Space model: examples

1.a) POG block scheme:



1.b) State space equations:

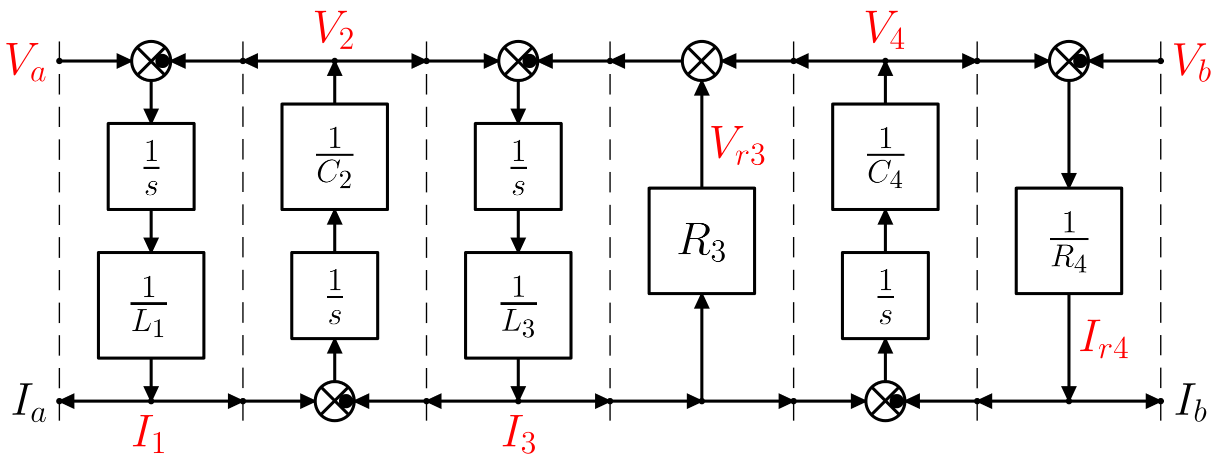
$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & -R_a & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R_a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

2.a) POG block scheme:



2.b) State space equations:

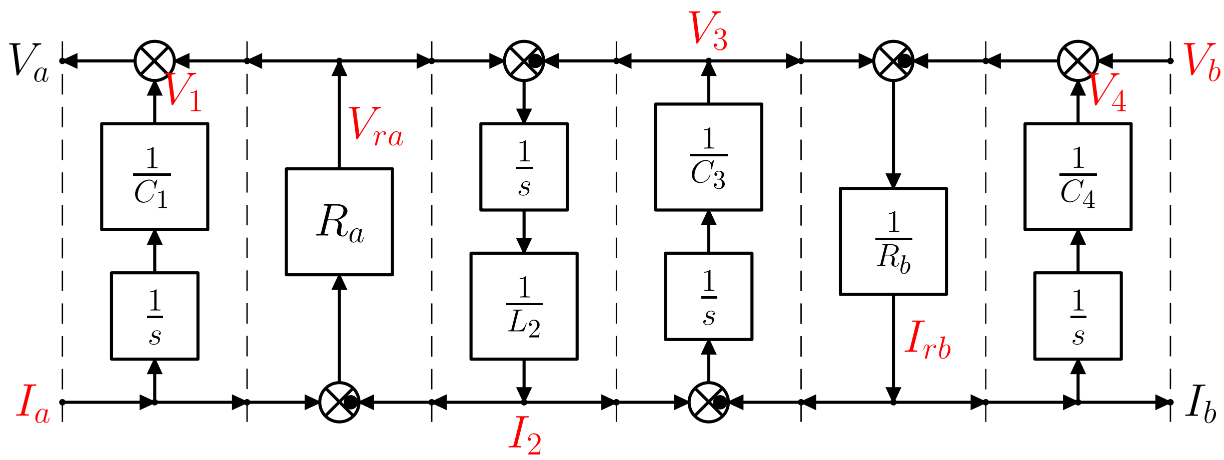
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -R_3 & -1 \\ 0 & 0 & 1 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{R_4} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_4} \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_4} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

3.a) POG block scheme:



3.b) State space equations:

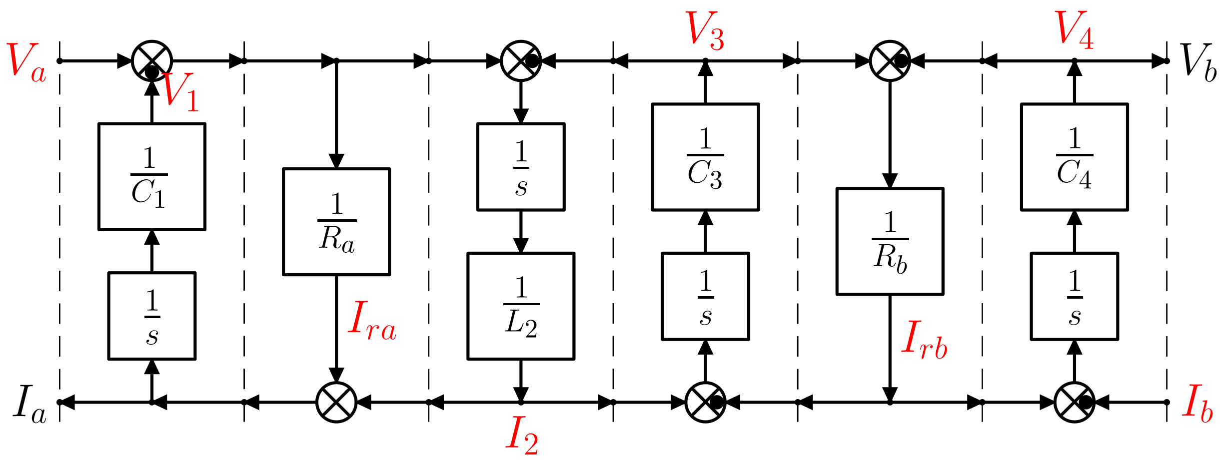
$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -R_a & -1 & 0 \\ 0 & 1 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ R_a & 0 \\ 0 & \frac{1}{R_b} \\ 0 & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -R_a & 0 & 0 \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} R_a & 0 \\ 0 & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -R_a & 0 & 0 \\ 0 & 0 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.a) POG block scheme:



4.b) State space equations:

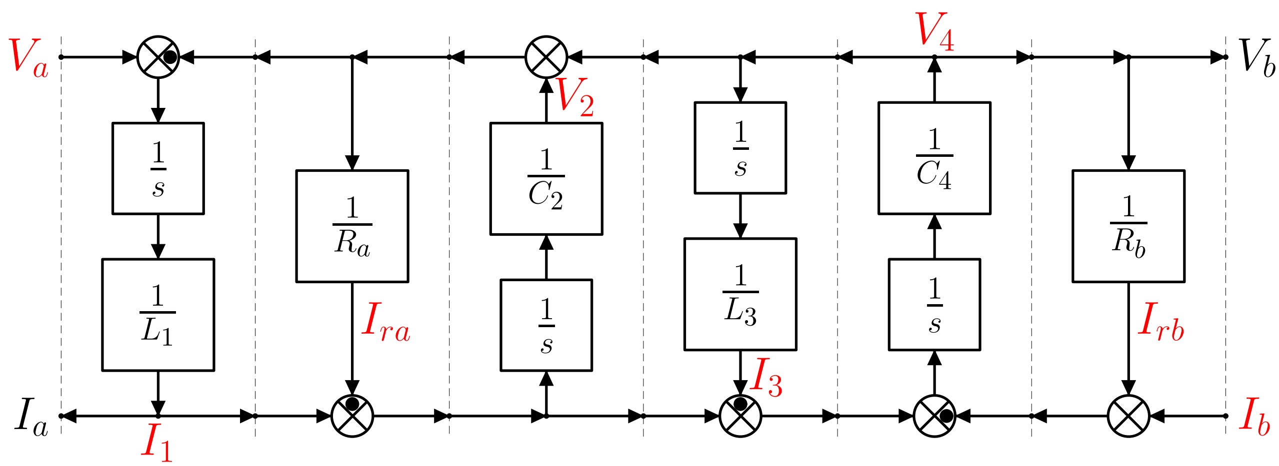
$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -\frac{1}{R_a} & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -\frac{1}{R_a} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -\frac{1}{R_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.a) POG block scheme:



5.b) State space equations:

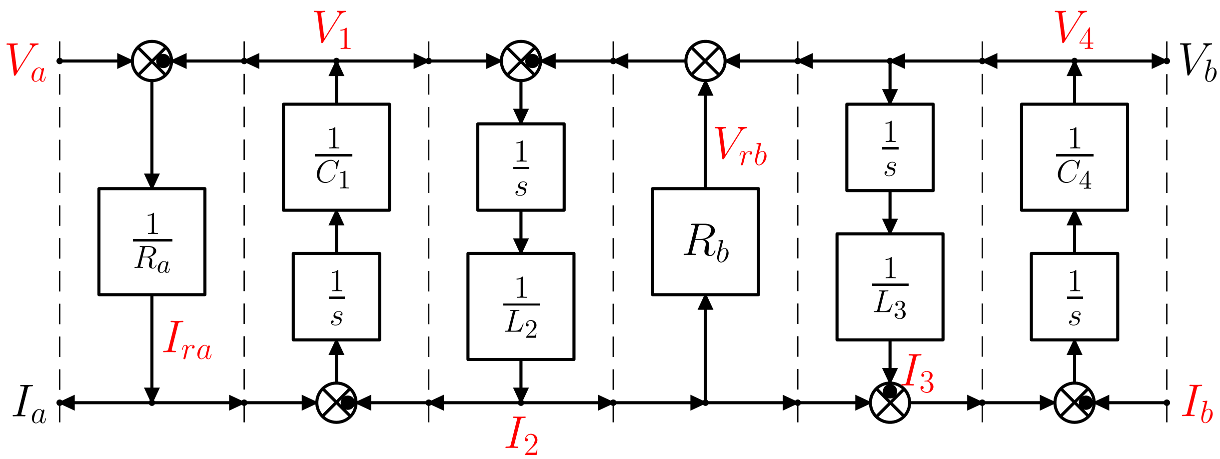
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & -\frac{1}{R_a} & 0 & -\frac{1}{R_a} \\ 0 & 0 & 0 & 1 \\ 1 & -\frac{1}{R_a} & -1 & -\frac{1}{R_a} - \frac{1}{R_b} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_a} & 0 & -\frac{1}{R_a} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_a} & 0 & -\frac{1}{R_a} - \frac{1}{R_b} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

6.a) POG block scheme:



6.b) State space equations:

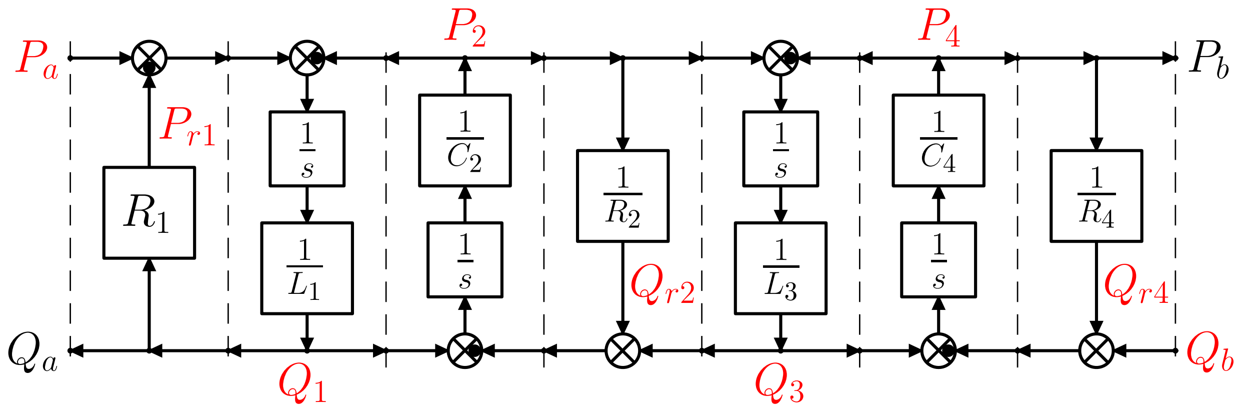
$$
\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -\frac{1}{R_a} & -1 & 0 & 0 \\ 1 & -R_b & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}
$$

$$
\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -\frac{1}{R_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}
$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$
\mathbf{A}_s = \begin{bmatrix} -\frac{1}{R_a} & 0 & 0 & 0 \\ 0 & -R_b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.
$$

7.a) POG block scheme:



7.b) State space equations:

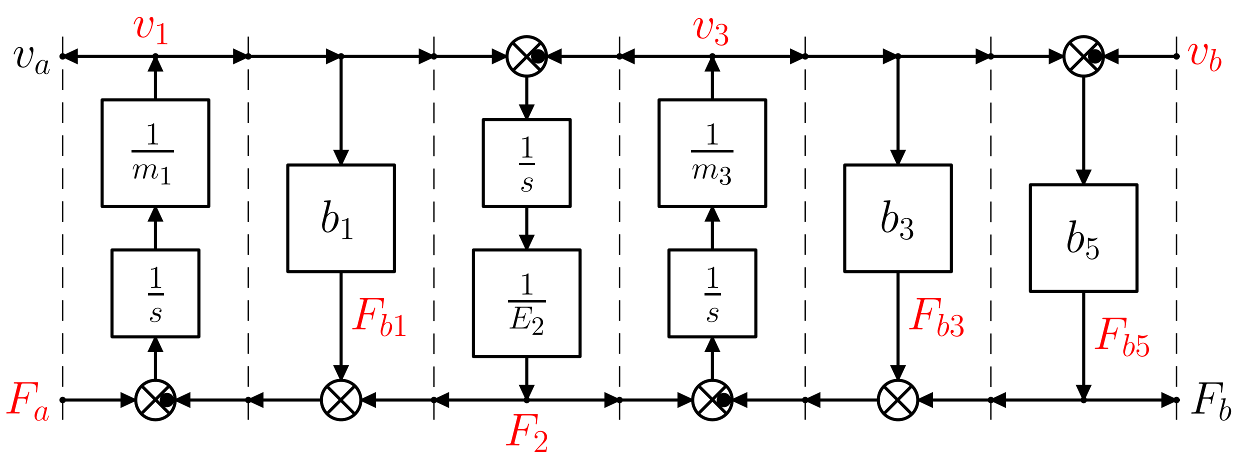
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -R_1 & -1 & 0 & 0 \\ 1 & -\frac{1}{R_2} & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} Q_1 \\ P_2 \\ Q_3 \\ P_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} Q_a \\ P_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_1 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_4} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

8.a) POG block scheme:



8.b) State space equations:

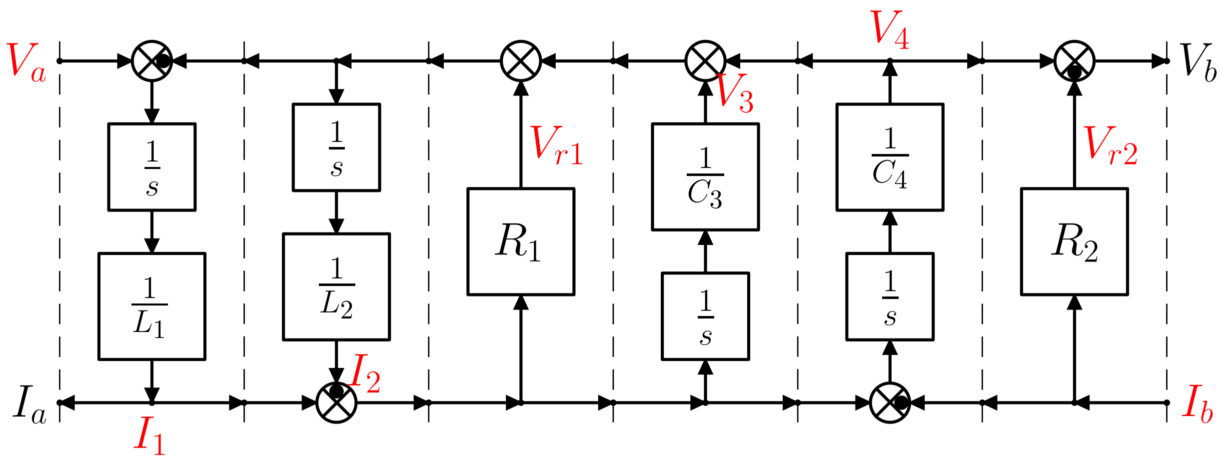
$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \dot{\mathbf{x}} = \begin{bmatrix} -b_1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -b_3 - b_5 \end{bmatrix} \begin{bmatrix} v_1 \\ F_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b_5 \end{bmatrix} \begin{bmatrix} F_a \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} v_a \\ F_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & b_5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & -b_5 \end{bmatrix} \mathbf{u}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -b_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b_3 - b_5 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

9.a) POG block scheme:



9.b) State space equations:

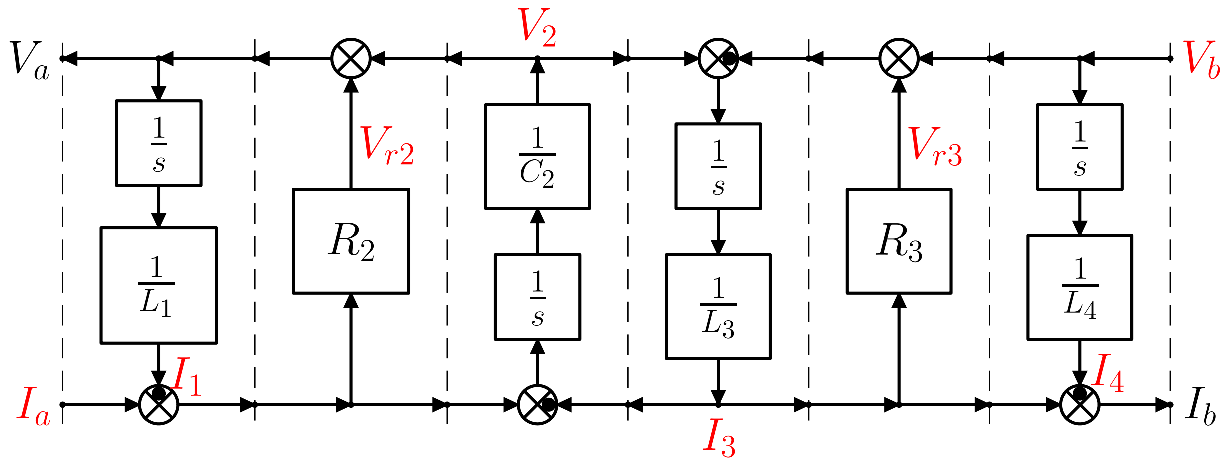
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -R_1 & R_1 & -1 & -1 \\ R_1 & -R_1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -R_2 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_1 & R_1 & 0 & 0 \\ R_1 & -R_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

10.a) POG block scheme:



10.b) State space equations:

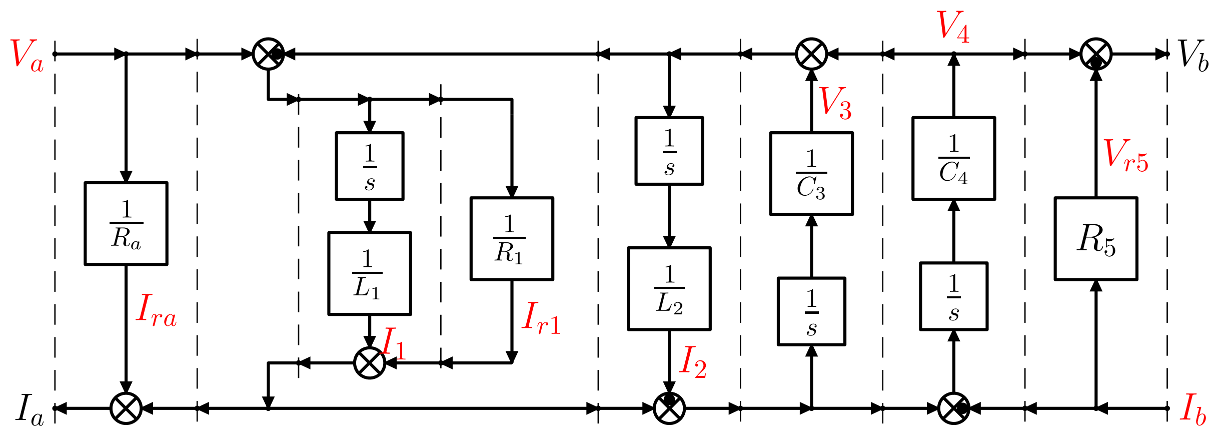
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -R_2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -R_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} R_2 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -R_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} R_2 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

11.a) POG block scheme:



11.b) State space equations:

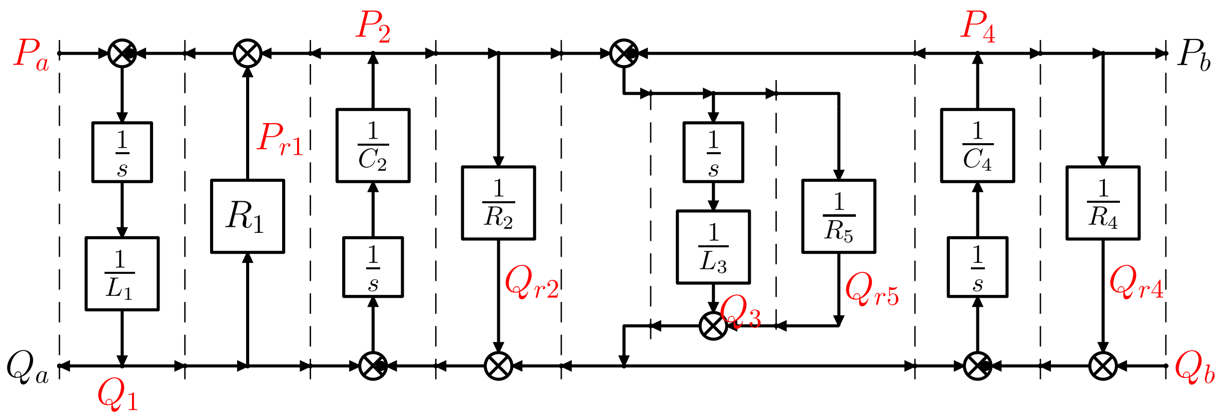
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 1 & -1 & -\frac{1}{R_1} & -\frac{1}{R_1} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \frac{1}{R_1} & 0 \\ \frac{1}{R_1} & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_a} & 0 \\ 0 & -R_5 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 0 & 0 & -\frac{1}{R_1} & -\frac{1}{R_1} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

12.a) POG block scheme:



12.b) State space equations:

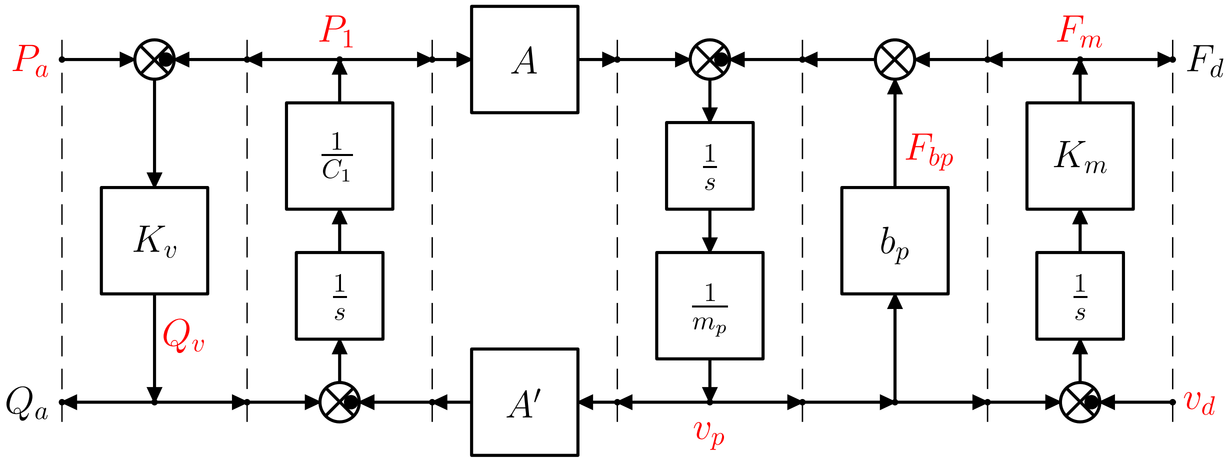
$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -R_1 & -1 & 0 & 0 \\ 1 & -\frac{1}{R_2}-\frac{1}{R_5} & -1 & \frac{1}{R_5} \\ 0 & 1 & 0 & -1 \\ 0 & \frac{1}{R_5} & 1 & -\frac{1}{R_4}-\frac{1}{R_5} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} Q_1 \\ P_2 \\ Q_3 \\ P_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}}$$

$$\underbrace{\begin{bmatrix} Q_a \\ P_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_1 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_2}-\frac{1}{R_5} & 0 & \frac{1}{R_5} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_5} & 0 & -\frac{1}{R_4}-\frac{1}{R_5} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

13.a) POG block scheme:



13.b) State space equations:

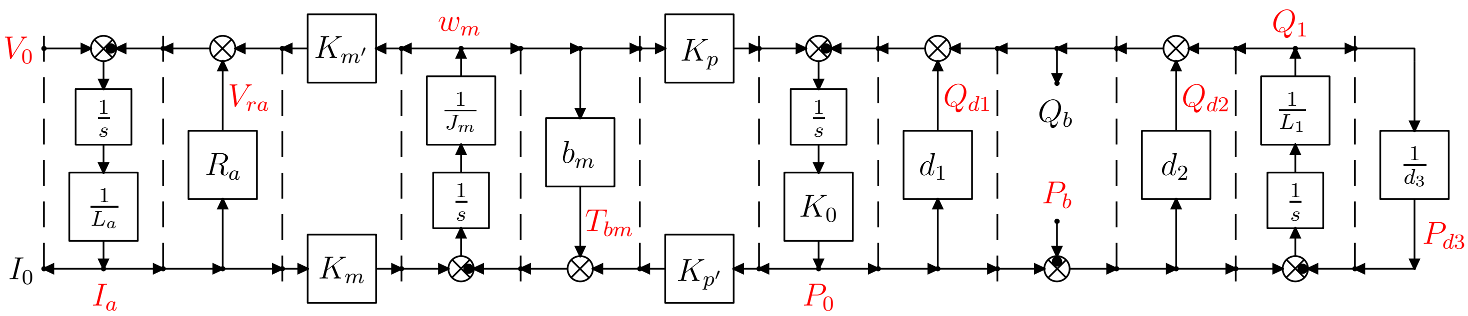
$$\begin{bmatrix} C_1 & 0 & 0 \\ 0 & m_p & 0 \\ 0 & 0 & \frac{1}{K_m} \end{bmatrix} \dot{\mathbf{x}} = \begin{bmatrix} -K_v & -A & 0 \\ A & -b_p & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ v_p \\ F_m \end{bmatrix} + \begin{bmatrix} K_v & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} P_a \\ v_d \end{bmatrix}$$

$$\begin{bmatrix} Q_a \\ F_d \end{bmatrix} = \begin{bmatrix} -K_v & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} K_v & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -K_v & 0 & 0 \\ 0 & -b_p & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -A & 0 \\ A & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

14.a) POG block scheme:



14.b) State space equations:

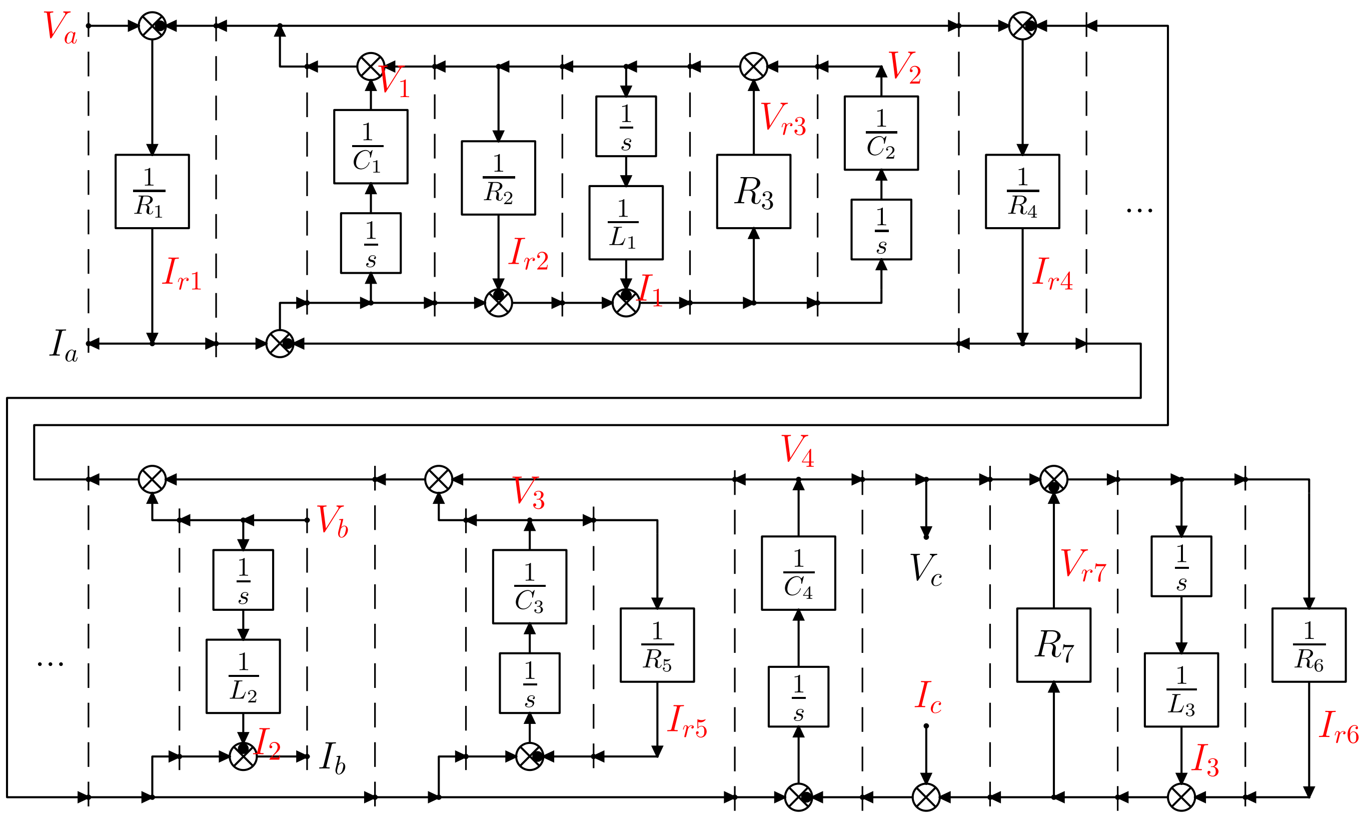
$$\underbrace{\begin{bmatrix} L_a & 0 & 0 & 0 \\ 0 & J_m & 0 & 0 \\ 0 & 0 & \frac{1}{K_0} & 0 \\ 0 & 0 & 0 & L_1 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -R_a & -K_m & 0 & 0 \\ K_m & -b_m & -K_p & 0 \\ 0 & K_p & -d_1-d_2 & -1 \\ 0 & 0 & 1 & -\frac{1}{d_3} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_a \\ w_m \\ P_0 \\ Q_1 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & d_2 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_0 \\ Q_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & d_2 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -d_2 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_0 \\ P_b \end{bmatrix}}_{\mathbf{u}}$$

Symmetric and skew-symmetric parts of power matrix $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_w$:

$$\mathbf{A}_s = \begin{bmatrix} -R_a & 0 & 0 & 0 \\ 0 & -b_m & 0 & 0 \\ 0 & 0 & -d_1-d_2 & 0 \\ 0 & 0 & 0 & -\frac{1}{d_3} \end{bmatrix}, \qquad \mathbf{A}_w = \begin{bmatrix} 0 & -K_m & 0 & 0 \\ K_m & 0 & -K_p & 0 \\ 0 & K_p & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

15.a) POG block scheme:



15.b) State space equations $\mathbf{S}$ and Energy matrix $\mathbf{L}$:

$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \;\;\mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases}, \qquad \mathbf{L} = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_4 \end{bmatrix}.$$

Power matrix $\mathbf{A}$ and Input matrix $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{R} & \frac{1}{2} & -\frac{1}{2R} & 0 & \frac{1}{2R} & 0 & \frac{1}{2R} \\ -\frac{1}{2} & -\frac{R}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{2R} & -\frac{1}{4} & -\frac{3}{4R} & 0 & \frac{1}{4R} & 0 & \frac{1}{4R} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2R} & -\frac{1}{4} & \frac{1}{4R} & 0 & -\frac{7}{4R} & 0 & -\frac{3}{4R} \\ 0 & 0 & 0 & 0 & 0 & -\frac{R}{2} & \frac{1}{2} \\ \frac{1}{2R} & -\frac{1}{4} & \frac{1}{4R} & 0 & -\frac{3}{4R} & -\frac{1}{2} & -\frac{5}{4R} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{2R} & \frac{1}{2R} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4R} & \frac{1}{4R} & 0 \\ 0 & 1 & 0 \\ \frac{1}{4R} & -\frac{3}{4R} & 0 \\ 0 & 0 & 0 \\ \frac{1}{4R} & -\frac{3}{4R} & -1 \end{bmatrix}.$$

Output matrix $\mathbf{C}$ and Input-output matrix $\mathbf{D}$:

$$\mathbf{C} = \begin{bmatrix} -\frac{1}{2R} & \frac{1}{4} & -\frac{1}{4R} & 0 & -\frac{1}{4R} & 0 & -\frac{1}{4R} \\ \frac{1}{2R} & -\frac{1}{4} & \frac{1}{4R} & -1 & -\frac{3}{4R} & 0 & -\frac{3}{4R} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \frac{3}{4R} & -\frac{1}{4R} & 0 \\ \frac{1}{4R} & -\frac{3}{4R} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

State vector $\mathbf{x}$, Input vector $\mathbf{u}$ and Output vector $\mathbf{y}$:

$$\mathbf{x} = \begin{bmatrix} V_1 \\ I_1 \\ V_2 \\ I_2 \\ V_3 \\ I_3 \\ V_4 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} V_a \\ V_b \\ I_c \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} I_a \\ I_b \\ V_c \end{bmatrix}.$$

Symmetric part of matrix $\mathbf{A}$:

$$\mathbf{A}_s = \frac{\mathbf{A}+\mathbf{A}^T}{2} = \begin{bmatrix} -\frac{1}{R} & 0 & -\frac{1}{2R} & 0 & \frac{1}{2R} & 0 & \frac{1}{2R} \\ 0 & -\frac{R}{4} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2R} & 0 & -\frac{3}{4R} & 0 & \frac{1}{4R} & 0 & \frac{1}{4R} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2R} & 0 & \frac{1}{4R} & 0 & -\frac{7}{4R} & 0 & -\frac{3}{4R} \\ 0 & 0 & 0 & 0 & 0 & -\frac{R}{2} & 0 \\ \frac{1}{2R} & 0 & \frac{1}{4R} & 0 & -\frac{3}{4R} & 0 & -\frac{5}{4R} \end{bmatrix}.$$

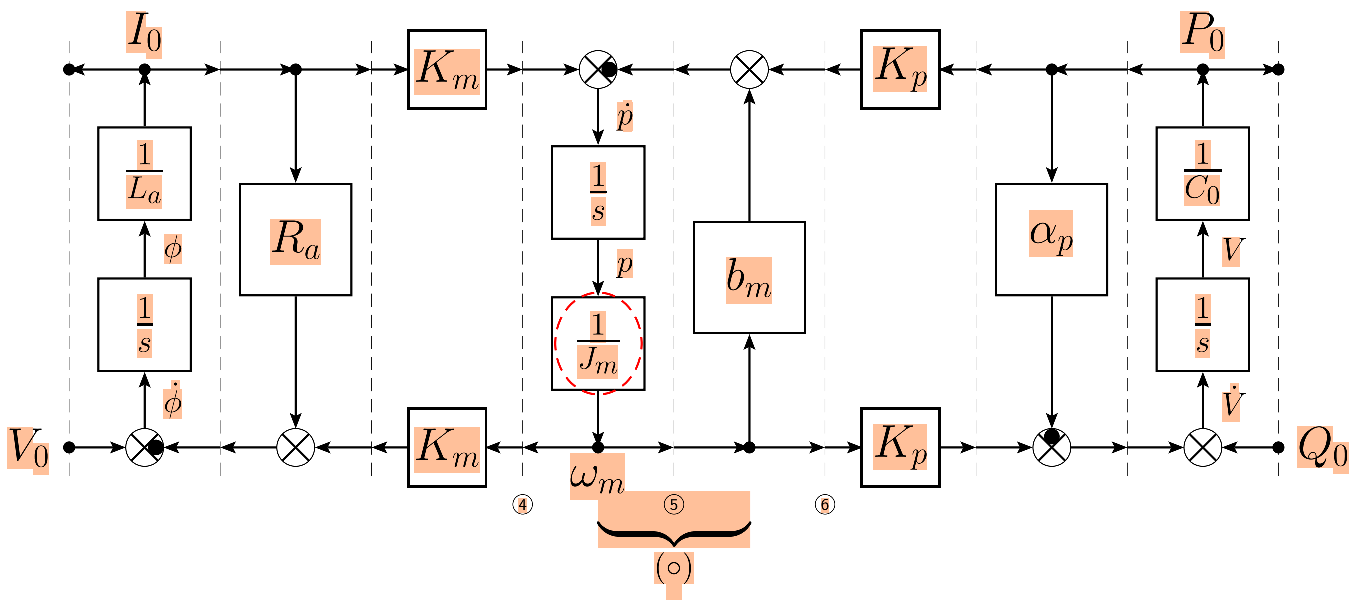
Skew-symmetric part of matrix $\mathbf{A}$:

$$\mathbf{A}_w = \frac{\mathbf{A}-\mathbf{A}^T}{2} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}.$$

Model reduction of a POG block scheme

- When a physical parameter of the system tends to zero (or to infinity) the POG system degenerates toward a lower dimension dynamic system. The POG dynamic model can be reduced "**graphically**" or "**analytically**"

- **Graphical model reduction**. Consider the following POG scheme:



- The corresponding POG state space model is:

$$\begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix} \dot{\mathbf{x}} = \begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix} \begin{bmatrix} I_0 \\ \omega_m \\ P_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_0 \\ Q_0 \end{bmatrix}$$

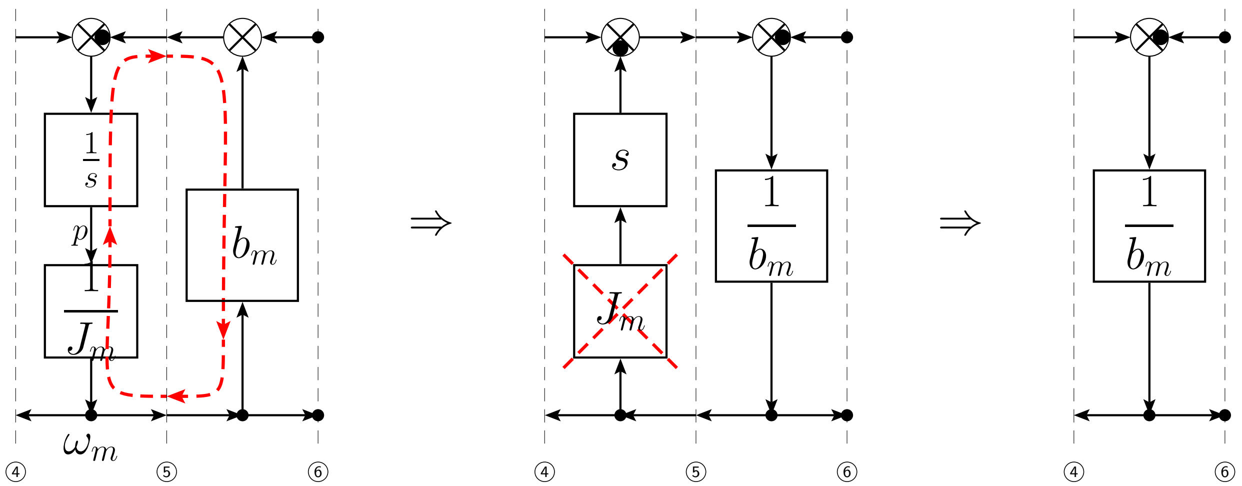
$$\begin{bmatrix} I_0 \\ P_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

- Let us show how it is possible to obtain the reduced POG state space model when one of the parameter of the system id neglected. Let us consider, for example, the case $J_m = 0$.

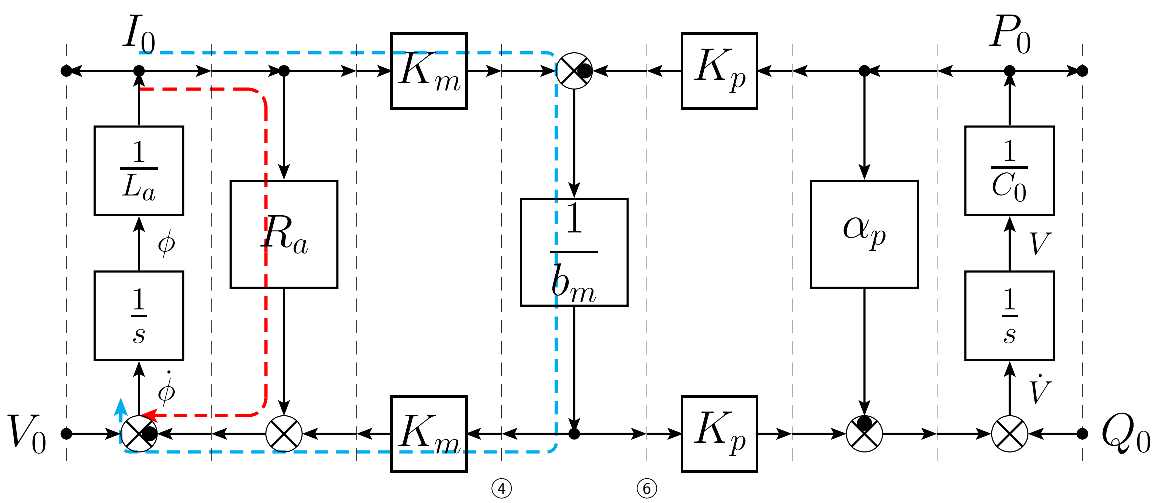
- When $J_m = 0$ the above POG block scheme cannot be used because the term $\frac{1}{J_m}$ present in the block scheme is infinite.

- In this case the central part (○) of the block scheme can be graphically transformed as follows:

a) Loop top be inverted b) Inverted loop c) Simplified scheme



- The simplified and transformed system has now the following structure:



- The corresponding POG state space model is:

$$\begin{bmatrix} L_a & 0 \\ 0 & C_0 \end{bmatrix} \begin{bmatrix} \dot{I}_0 \\ \dot{P}_0 \end{bmatrix} = \begin{bmatrix} -R_a - \frac{K_m^2}{b_m} & \frac{K_m K_p}{b_m} \\ \frac{K_m K_p}{b_m} & -\alpha_p - \frac{K_p^2}{b_m} \end{bmatrix} \begin{bmatrix} I_0 \\ P_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_0 \\ Q_0 \end{bmatrix} \qquad (*)$$

- **Analytical model reduction.** When $J_m = 0$ the state space model of the considered system can be rewritten as follows:

$$\begin{bmatrix} L_a & 0 & 0 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & C_0 \end{bmatrix} \begin{bmatrix} \dot{I}_0 \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix} = \begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix} \begin{bmatrix} I_0 \\ \omega_m \\ P_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_0 \\ Q_0 \end{bmatrix}$$

The second equation is an algebraic constraint between the state variables:

$$K_m I_0 - b_m \omega_m - K_p P_0 = 0$$

The angular velocity ω_m can be expressed as follows:

$$\omega_m = \frac{K_m}{b_m} I_0 - \frac{K_p}{b_m} P_0$$

- Applying the following "congruent" state space transformation:

$$\underbrace{\begin{bmatrix} I_0 \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{K_m}{b_m} & -\frac{K_p}{b_m} \\ 0 & 1 \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} I_0 \\ P_0 \end{bmatrix}}_{\mathbf{z}} \qquad \leftrightarrow \qquad \mathbf{x} = \mathbf{T}\,\mathbf{z}$$

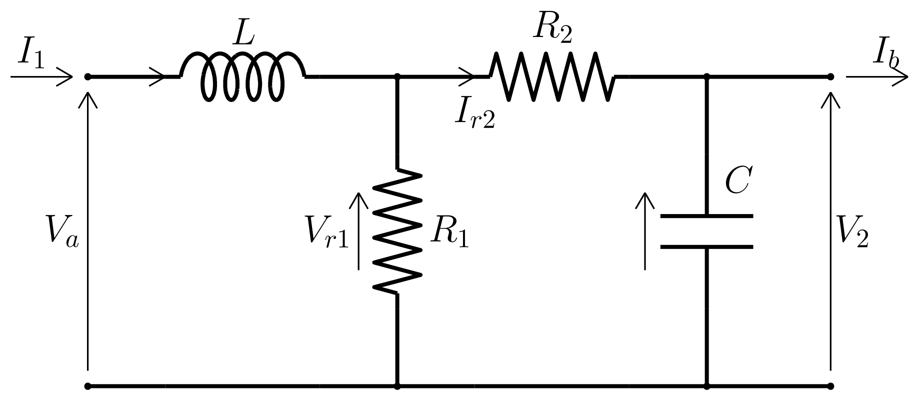
to the given system one directly obtains the reduced system (*).

- The calculations can be checked in Matlab using the Symbolic Toolbox:

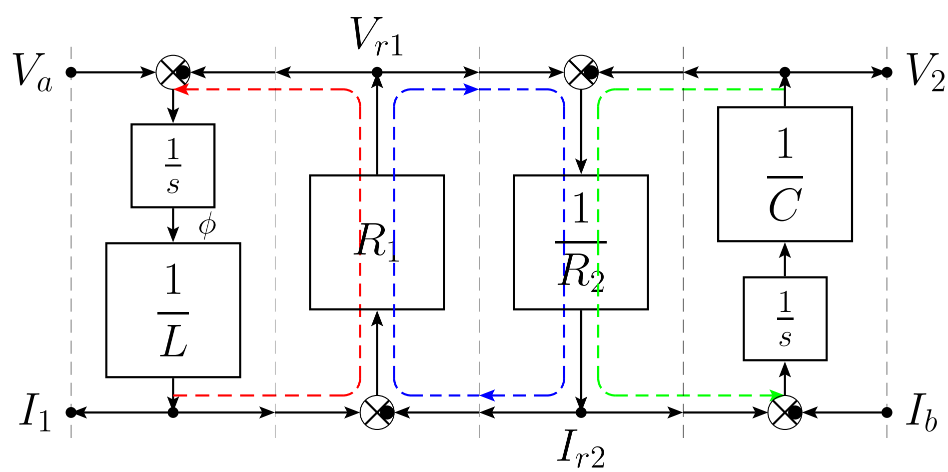
```
-- Matlab commands ------------------------------------------------------------
syms La Ra Km Jm bm Kp C0 alpha_p
Jm=0;
LM=diag([La Jm C0]);
AM=[ -Ra     -Km           0;
      Km     -bm         -Kp;
       0      Kp    -alpha_p];
BM=[1  0; ...
    0  0; ...
    0  1];
T=[     1           0;
    Km/bm      -Kp/bm;
        0           1];
LT=T.'*LM*T
AT=simplify(T.'*AM*T)
BT=T.'*BM

-- Matlab output --------------------------------------------------------------
LT =            AT =                                          BT =
[La,   0]       [- Km^2/bm - Ra,           (Km*Kp)/bm]      [1, 0]
[ 0,  C0]       [     (Km*Kp)/bm, - Kp^2/bm - alpha_p]      [0, 1]
-------------------------------------------------------------------------------
```

Algebraic loops in the POG schemes

- Let us consider the following electric circuit:



In the corresponding POG scheme is present an "algebraic loop":



The corresponding POG state space model can be determined as follows:

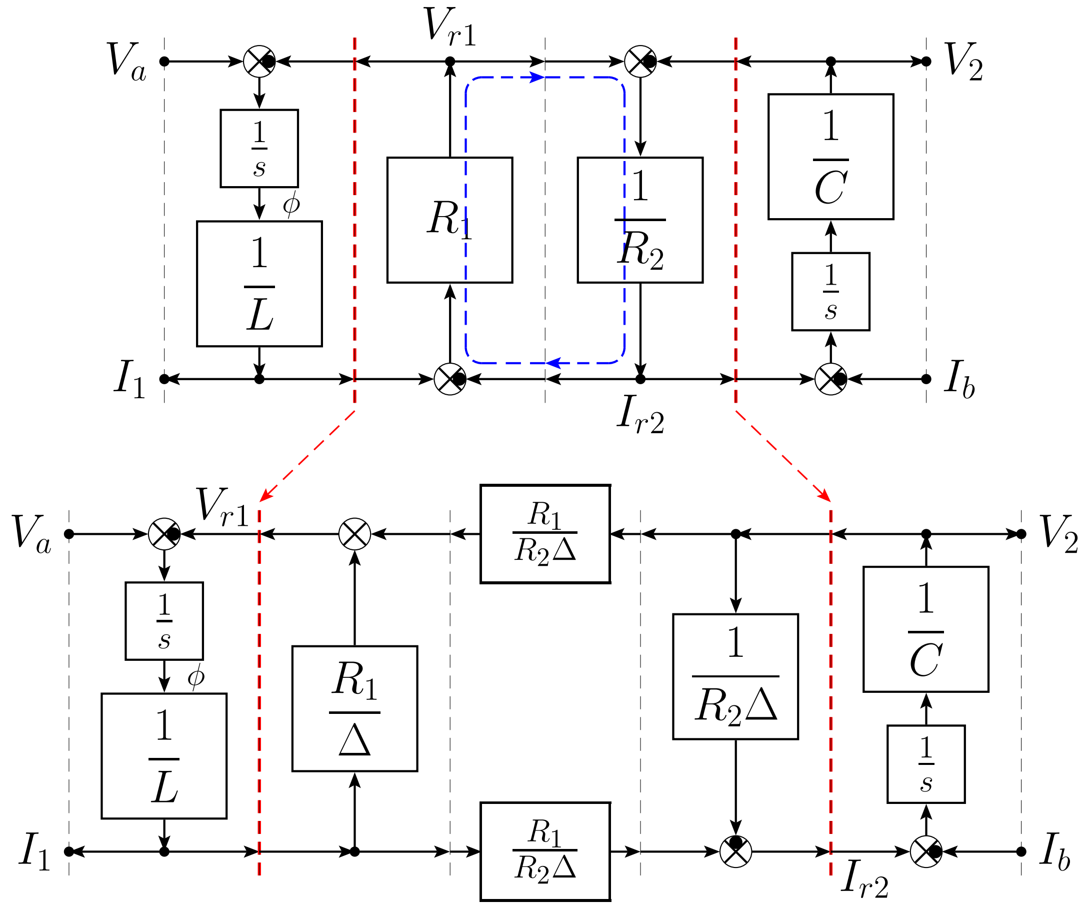
$$\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R_1}{\Delta} & -\frac{R_1}{R_2\Delta} \\ \frac{R_1}{R_2\Delta} & -\frac{1}{R_2\Delta} \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix} \qquad (\diamond)$$

where Δ is the determinant of the POG scheme without the dynamic blocks:

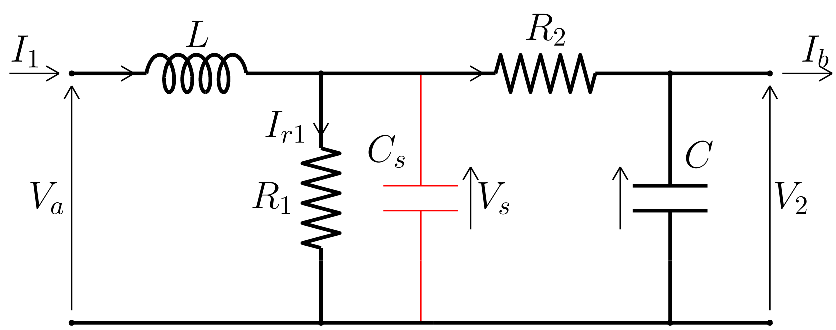
$$\Delta = 1 + \frac{R_1}{R_2}.$$

The coefficient A_{ij} of matric $\mathbf{A}$ is the "transfer gain" (computed using the "Mason formula") of the path that links the j-th state variable x_j with the input of the i-th integrator.

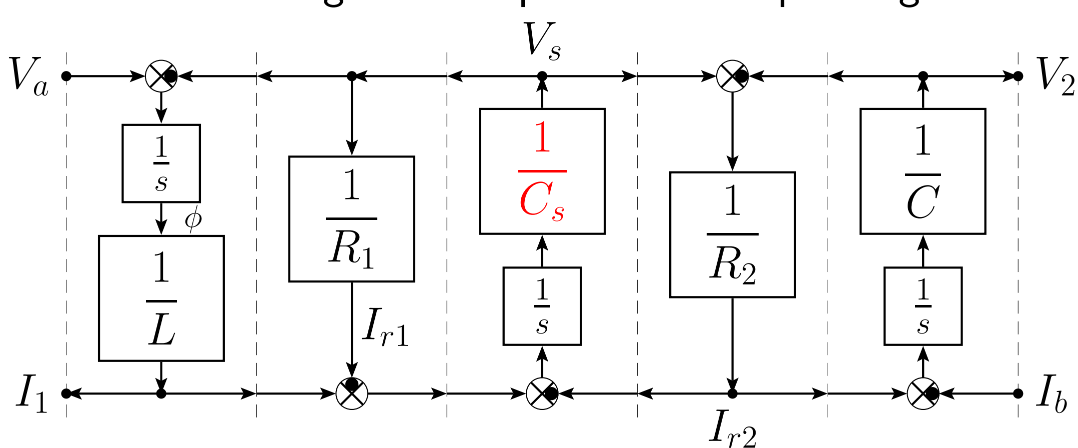
- The "algebraic loop" can also be eliminated "*graphically*" as follows:



- The same result can also be obtained using the following "mathematical" approach. Add a "spurious" physical element C_s to the original system



such to eliminate the algebraic loop in the corresponding POG scheme:



The corresponding POG state space model can now be written easily:

$$\begin{bmatrix} L & 0 & 0 \\ 0 & C_s & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{V}_s \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -\frac{1}{R_1} - \frac{1}{R_2} & \frac{1}{R_2} \\ 0 & \frac{1}{R_2} & -\frac{1}{R_2} \end{bmatrix} \begin{bmatrix} I_1 \\ V_s \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}$$

When $C_s = 0$ one obtains the following constraint:

$$I_1 + \left(-\frac{1}{R_1} - \frac{1}{R_2}\right) V_s + \frac{1}{R_2} V_2 = 0$$

which can be rewritten as follows:

$$V_s = \frac{R_1 R_2}{R_1 + R_2} I_1 + \frac{R_1}{R_1 + R_2} V_2$$

Let us consider the following "congruent" state space transformation:

$$\mathbf{x} = \mathbf{T}\overline{\mathbf{x}} \qquad \text{where} \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 \\ \frac{R_1 R_2}{R_1+R_2} & \frac{R_1}{R_1+R_2} \\ 0 & 1 \end{bmatrix}$$

The transformed and reduced matrices have the following structure:

$$\overline{\mathbf{x}} = \mathbf{T}\mathbf{x} = \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}, \qquad \overline{\mathbf{L}} = \mathbf{T}^T \mathbf{L} \mathbf{T} = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}$$

$$\overline{\mathbf{A}} = \mathbf{T}^T \mathbf{A} \mathbf{T} = \begin{bmatrix} -\frac{R_1 R_2}{R_1+R_2} & -\frac{R_1}{R_1+R_2} \\ \frac{R_1}{R_1+R_2} & -\frac{1}{R_1+R_2} \end{bmatrix}, \qquad \overline{\mathbf{B}} = \mathbf{T}^T \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

One can easily verify that the obtained reduced system is equal to the system ($\diamond$) obtained applying the Mason formula to solve the algebraic loop. One can easily verify that the two matrices $\overline{\mathbf{A}}$ and $\tilde{\mathbf{A}}$ are equal:

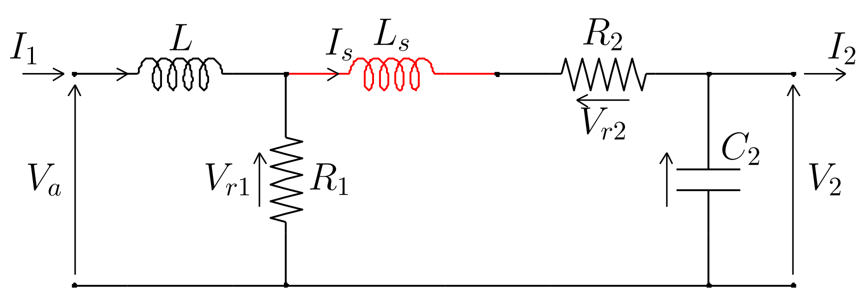
$$\overline{\mathbf{A}} = \begin{bmatrix} -\frac{R_1 R_2}{R_1+R_2} & -\frac{R_1}{R_1+R_2} \\ \frac{R_1}{R_1+R_2} & -\frac{1}{R_1+R_2} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{1+\frac{R_1}{R2}} & -\frac{R_1}{R_2(1+\frac{R_1}{R2})} \\ \frac{R_1}{R_2(1+\frac{R_1}{R2})} & -\frac{1}{R_2(1+\frac{R_1}{R2})} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{\Delta} & -\frac{R_1}{R_2 \Delta} \\ \frac{R_1}{R_2 \Delta} & -\frac{1}{R_2 \Delta} \end{bmatrix} = \tilde{\mathbf{A}}$$

- Solution using the spurious element C_s. Calculations with Matlab:

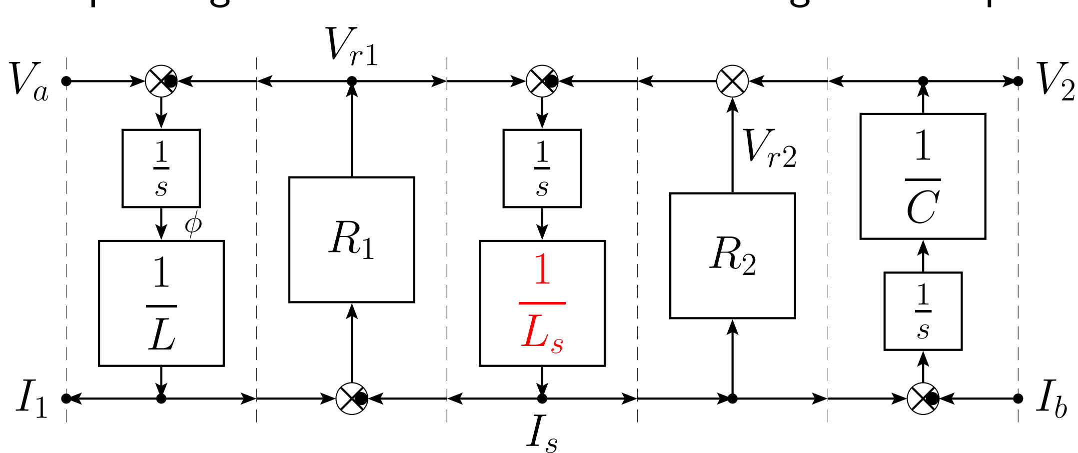
```
-- Matlab commands -------------------------------------------------------
syms L C Cs R1 R2
LM=diag([L 0 C]);
AM=[ 0          -1      0;
     1   -1/R1-1/R2  1/R2;
     0         1/R2 -1/R2];
BM=[1 0; 0 0; 0 -1];
T=[              1          0;
    R1*R2/(R1+R2) R1/(R1+R2);
                 0         1];
LT=T.'*LM*T
AT=simplify(T.'*AM*T)
BT=T.'*BM
Delta=1+R1/R2;
V=simplify(AT-[-R1/Delta -R1/Delta/R2; ...
                R1/Delta/R2 -1/Delta/R2 ])
-- Matlab output ---------------------------------------------------------
LT =          AT =                                      BT =          V =

[ L, 0]       [ -(R1*R2)/(R1 + R2), -R1/(R1 + R2)]      [ 1,  0]      [ 0, 0]
[ 0, C]       [       R1/(R1 + R2),  -1/(R1 + R2)]      [ 0, -1]      [ 0, 0]

--------------------------------------------------------------------------
```

- Different types of spurious elements can be used. Nevertheless the final reduced system will be always the same.

- The same result can also be obtained using a different "spurious" element, for example an additional inductance L_s.



The corresponding POG scheme does not have algebraic loops:



The POG state space model is:

$$\begin{bmatrix} L & 0 & 0 \\ 0 & L_s & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_s \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} -R_1 & R_1 & 0 \\ R_1 & -R_1 - R_2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_s \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}$$

When $L_s = 0$, one obtains the following constraint:

$$R_1 I_1 + (-R_1 - R_2) I_s - V_2 = 0 \qquad \Rightarrow \qquad I_s = \frac{R_1 I_1}{R_1 + R_2} - \frac{V_2}{R_1 + R_2}$$

Using the following "congruent" state space transformation:

$$\mathbf{x} = \mathbf{T}\overline{\mathbf{x}} \qquad \Leftrightarrow \qquad \begin{bmatrix} I_1 \\ I_s \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{R_1}{R_1+R_2} & \frac{-1}{R_1+R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

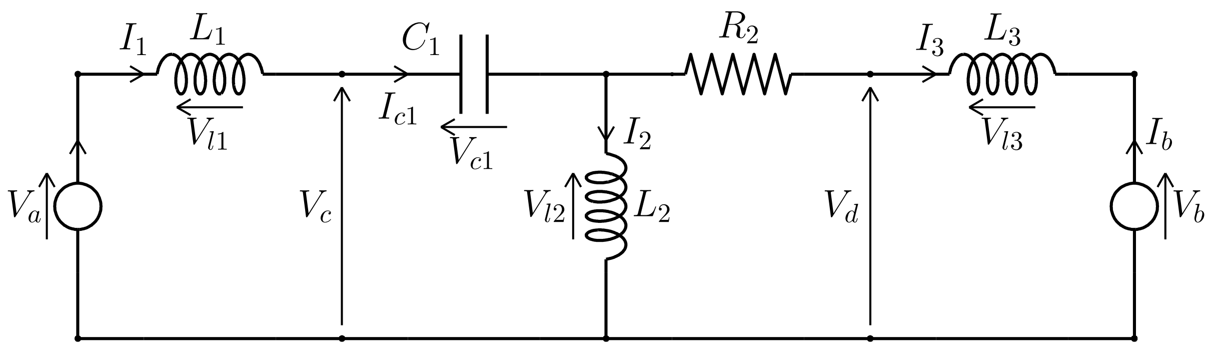
The transformed and reduced matrices have the following structure:

$$\overline{\mathbf{L}} = \mathbf{T}^T \mathbf{L} \mathbf{T} = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}, \qquad \overline{\mathbf{A}} = \mathbf{T}^T \mathbf{A} \mathbf{T} = \begin{bmatrix} -\frac{R_1 R_2}{R_1+R_2} & -\frac{R_1}{R_1+R_2} \\ \frac{R_1}{R_1+R_2} & -\frac{1}{R_1+R_2} \end{bmatrix}$$

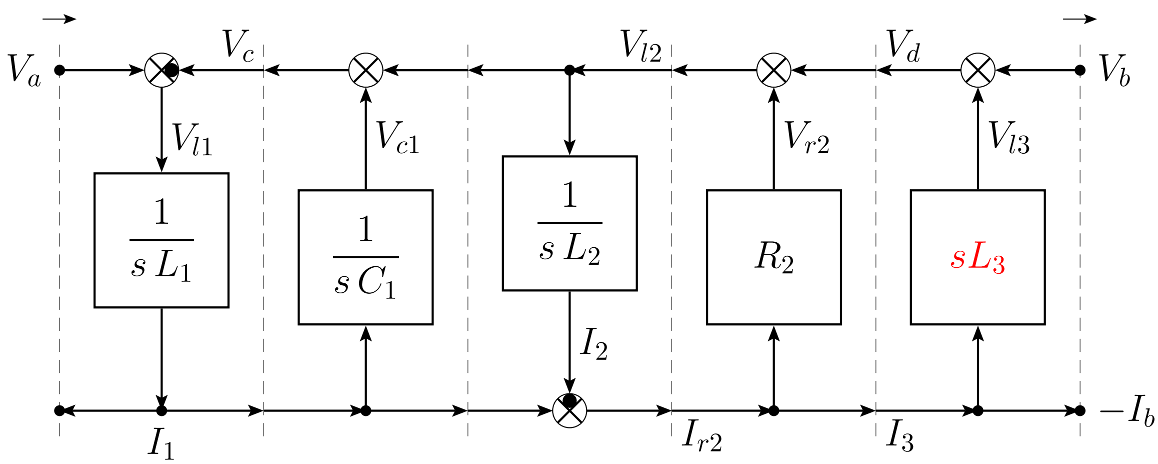
- Solution using the spurious element L_s. Calculations with Matlab:

```
-- Matlab commands ----------------------------------------------------------
syms L C Ls R1 R2
LM=diag([L 0 C]);
AM=[ -R1      R1    0;
      R1   -R1-R2  -1;
       0       1    0];
BM=[1 0; 0 0; 0 -1];
T=[         1          0;
   R1/(R1+R2) -1/(R1+R2);
            0         1];
LT=T.'*LM*T
AT=simplify(T.'*AM*T)
BT=T.'*BM
Delta=1+R1/R2;
V=simplify(AT-[-R1/Delta -R1/Delta/R2; ...
                R1/Delta/R2 -1/Delta/R2 ])
-- Matlab output ------------------------------------------------------------
LT =          AT =                                     BT =          V =

[ L, 0]       [ -(R1*R2)/(R1 + R2), -R1/(R1 + R2)]     [ 1,  0]      [ 0, 0]
[ 0, C]       [       R1/(R1 + R2),  -1/(R1 + R2)]     [ 0, -1]      [ 0, 0]

-----------------------------------------------------------------------------
```

- **Example**. Let us consider the following electric circuit:



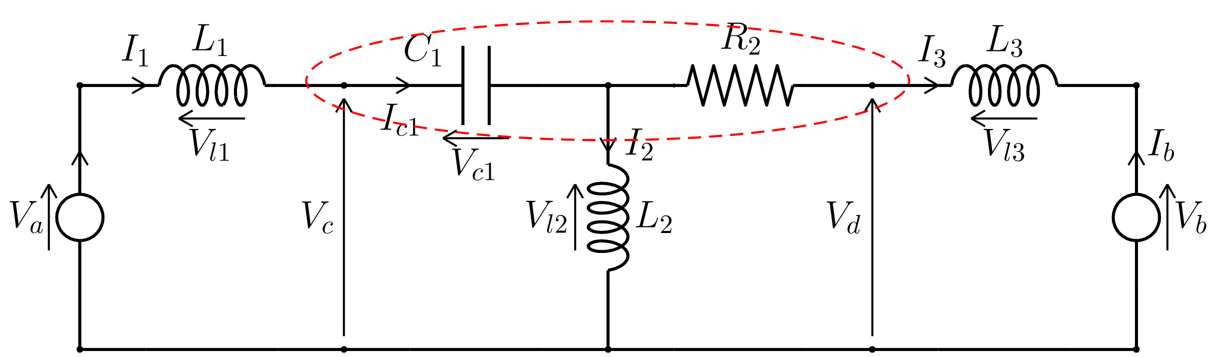
The corresponding POG block scheme:



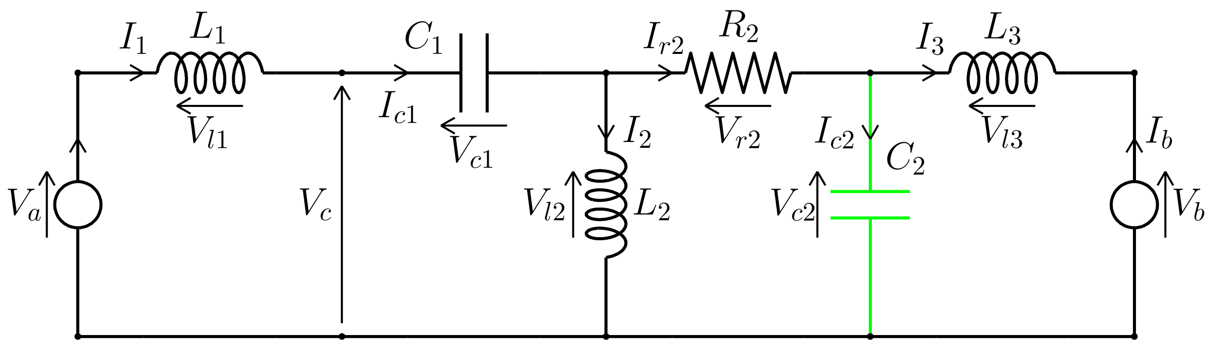
- The system CANNOT be modeled using the "integral causality": the inductance L_3 is graphically represented using "derivative causality".
- From the physical scheme it is evident that the state variables I_1, I_2 and I_3 are constrained as following:

$$I_1 = I_2 + I_3$$

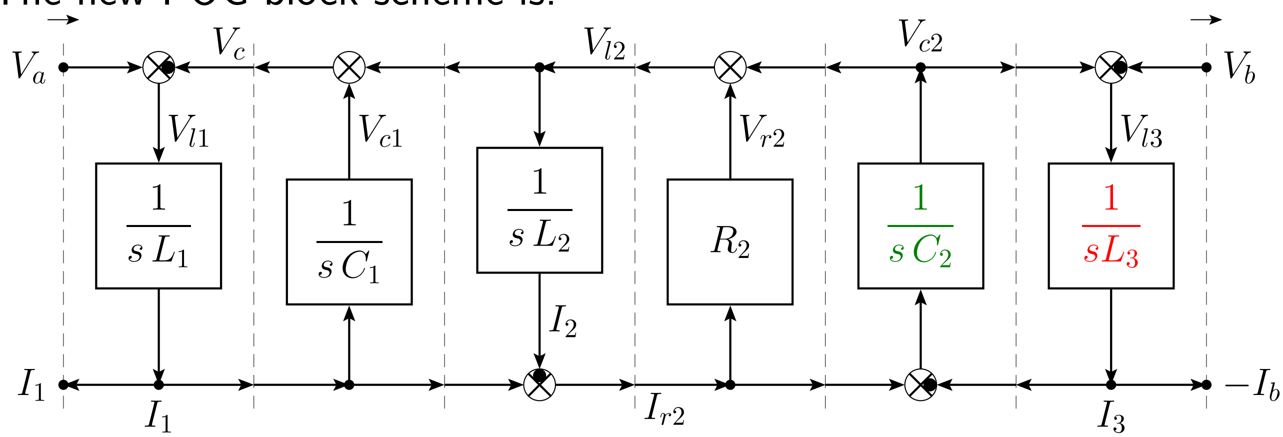
- The constraint is due to the node evidenced in red in the following scheme:



- The system can be modeled adding a spurious dynamic element C_2 in parallel connection between elements R_2 and L_3:



The new POG block scheme is:



The corresponding POG state space dynamic model is:

$$\begin{bmatrix} L_1 & & & & \\ & C_1 & & & \\ & & L_2 & & \\ & & & C_2 & \\ & & & & L_3 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{V}_{c1} \\ \dot{I}_2 \\ \dot{V}_{c2} \\ \dot{I}_3 \end{bmatrix} = \begin{bmatrix} -R_2 & -1 & R_2 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ R_2 & 0 & -R_2 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \\ V_{c2} \\ I_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ -I_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

When $C_2 = 0$ from the forth equation one obtains the following constraint:

$$I_1 - I_2 - I_3 = 0 \qquad \Rightarrow \qquad I_3 = I_1 - I_2$$

Let us consider the following state space "congruent" transformation:

$$\mathbf{x} = \mathbf{T}\overline{\mathbf{x}} \qquad \Leftrightarrow \qquad \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \\ V_{c2} \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \end{bmatrix}$$

Using the congruent transformation $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$ one obtains the following transformed and reduced POG dynamic system:

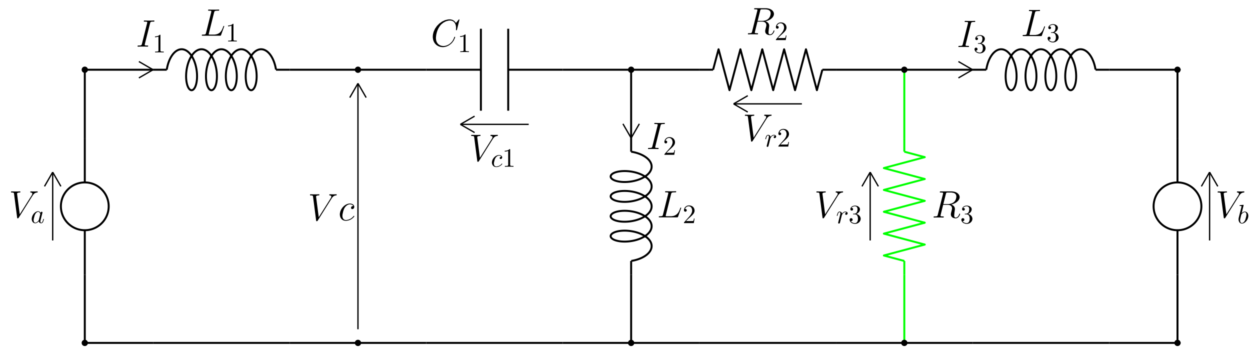
$$\begin{bmatrix} L_1+L_3 & & -L_3 \\ 0 & C_1 & 0 \\ -L_3 & & L_2+L_3 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{V}_{c1} \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} -R_2 & -1 & R_2 \\ 1 & 0 & 0 \\ R_2 & 0 & -R_2 \end{bmatrix} \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ -I_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x}$$

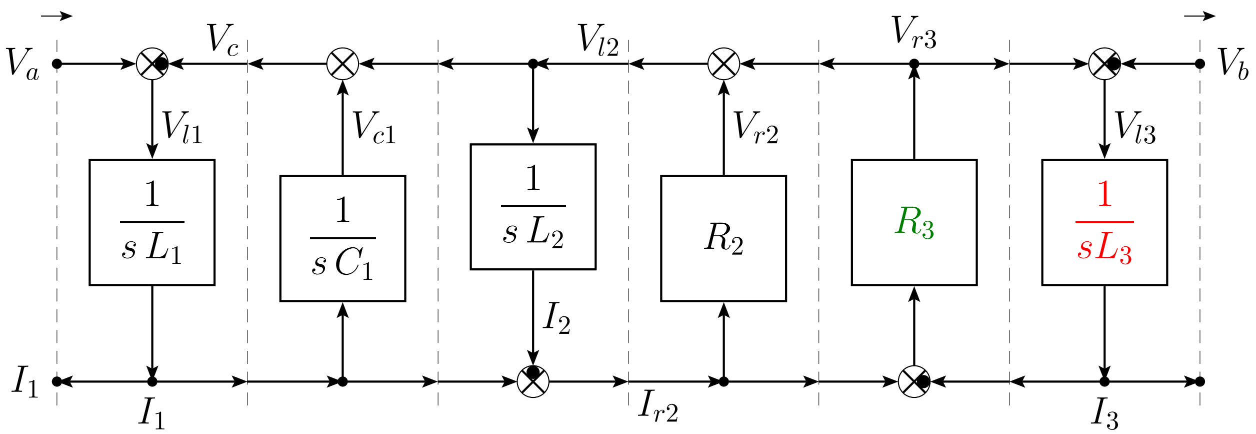
- The state variables V_{c2} and I_3 are no more present because $C_2 = 0$ and the constraint $I_3 = I_1 - I_2$.
- The transformed system is NO more decoupled because matrix $\overline{\mathbf{L}} = \mathbf{T}^T\mathbf{L}\mathbf{T}$ is not diagonal. The reduced system has still the structure of a POG dynamic model. Matrix $\overline{\mathbf{L}}$, for example, is still symmetric and definite positive.
- Calculations using Matlab:

```
-- Matlab commands -------------------------------------------------------
syms L1 L2 L3 C1 C2 R2
 C2=0;
 LM=diag([L1 C1 L2 C2 L3]);
 AM=[-R2 -1 R2  -1  0;
      1  0   0   0  0;
     R2  0 -R2   1  0;
      1  0  -1   0 -1;
      0  0   0   1  0];
BM=[1 0; 0 0; 0 0; 0 0; 0 -1];
CM=[1 0 0 0 0;
    0 0 0 0 1];
T1=sym([ 1  0  0;
         0  1  0;
         0  0  1;
         0  0  0;
         1  0 -1]);
LT=T1.'*LM*T1
AT=simplify(T1.'*AM*T1)
BT=T1.'*BM
CT=CM*T1

-- Matlab output ---------------------------------------------------------

    LT =                        AT =                BT =           CT =

   [ L1 + L3,  0,     -L3]     [ -R2, -1,  R2]     [ 1, -1]      [ 1, 0,  0]
   [       0, C1,       0]     [   1,  0,   0]     [ 0,  0]      [ 1, 0, -1]
   [     -L3,  0, L2 + L3]     [  R2,  0, -R2]     [ 0,  1]

--------------------------------------------------------------------------
```

- The same final result can also be obtained adding a dissipative spurious element R_3 in parallel connection between elements R_2 and $L3$:



The new POG block scheme is:



The corresponding POG state space dynamic model is:

$$\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{bmatrix} \dot{\mathbf{x}} = \begin{bmatrix} -R_2 - R_3 & -1 & R_2 + R_3 & R_3 \\ 1 & 0 & 0 & 0 \\ R_2 + R_3 & 0 & -R_2 - R_3 & -R_3 \\ R_3 & 0 & -R_3 & -R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

When $R_3 = \infty$, from the forth equation one obtains:

$$\frac{L_3 \dot{I}_3}{R_3} = (I_1 - I_2 - I_3) + \frac{V_b}{R_3} \qquad \Rightarrow \qquad I_3 = I_1 - I_2$$

Let us consider the following state space "congruent" transformation:

$$\mathbf{x} = \mathbf{T}\overline{\mathbf{x}} \qquad \Leftrightarrow \qquad \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \end{bmatrix}$$

Using the congruent transformation $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$ one obtains the following transformed and reduced POG dynamic system:

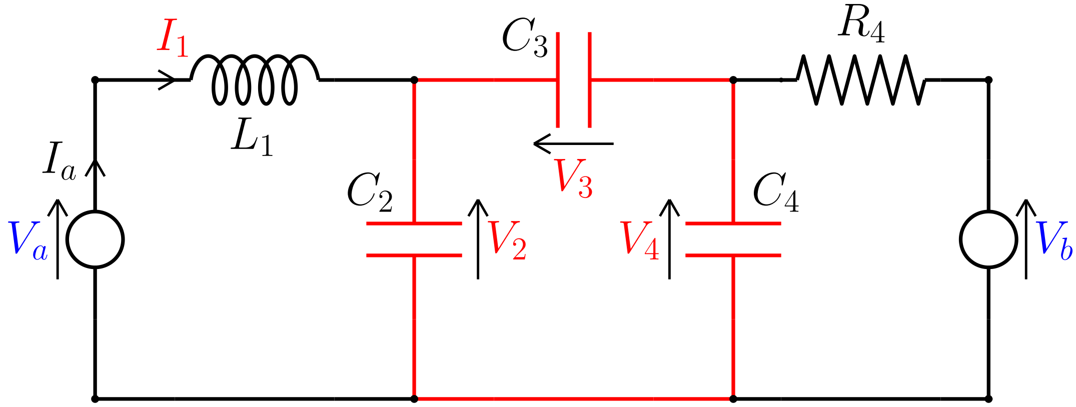
$$\begin{bmatrix} L_1+L_3 & & -L_3 \\ 0 & C_1 & 0 \\ -L_3 & & L_2+L_3 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{V}_{c1} \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} -R_2 & -1 & R_2 \\ 1 & 0 & 0 \\ R_2 & 0 & -R_2 \end{bmatrix} \begin{bmatrix} I_1 \\ V_{c1} \\ I_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

which is equal to the reduced system obtained in the previous case.

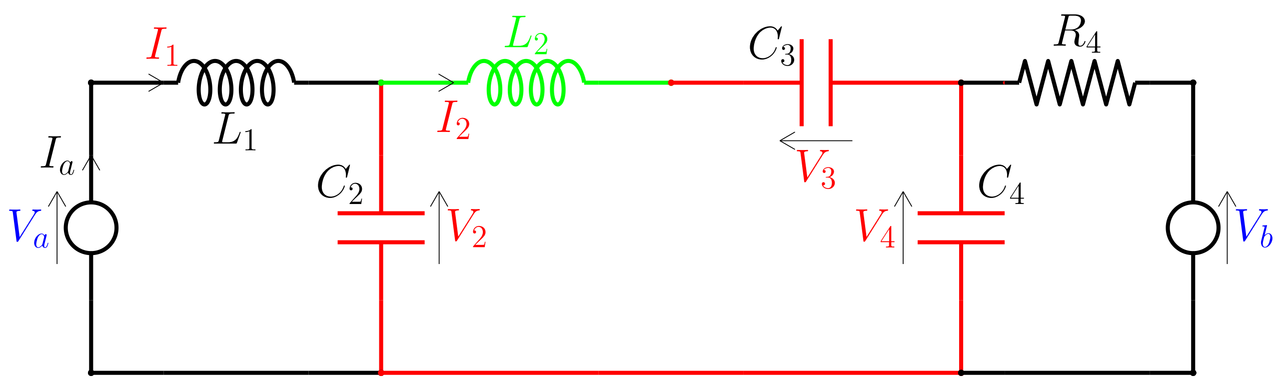
- Note that in the final obtained system the parameter R_3 is no more present.
- Calculations using Matlab:

```
-- Matlab commands ---------------------------------------------------------
 syms L1 L2 L3 C1 R3 R2
 C2=0;
 LM=diag([L1 C1 L2 L3]);
 AM=[-R2-R3  -1   R2+R3   R3;
          1   0       0    0;
      R2+R3   0  -R2-R3  -R3;
         R3   0     -R3  -R3];
 BM=[1 0; 0 0; 0 0; 0 -1];
 T1=sym([ 1  0  0;
          0  1  0;
          0  0  1;
          1  0 -1]);
 LT=T1.'*LM*T1
 AT=simplify(T1.'*AM*T1)
 BT=T1.'*BM

-- Matlab output -----------------------------------------------------------

   LT =                          AT =                BT =

  [ L1 + L3,  0,     -L3]      [ -R2, -1,  R2]      [ 1, -1]
  [       0, C1,       0]      [   1,  0,   0]      [ 0,  0]
  [     -L3,  0, L2 + L3]      [  R2,  0, -R2]      [ 0,  1]

---------------------------------------------------------------------------
```

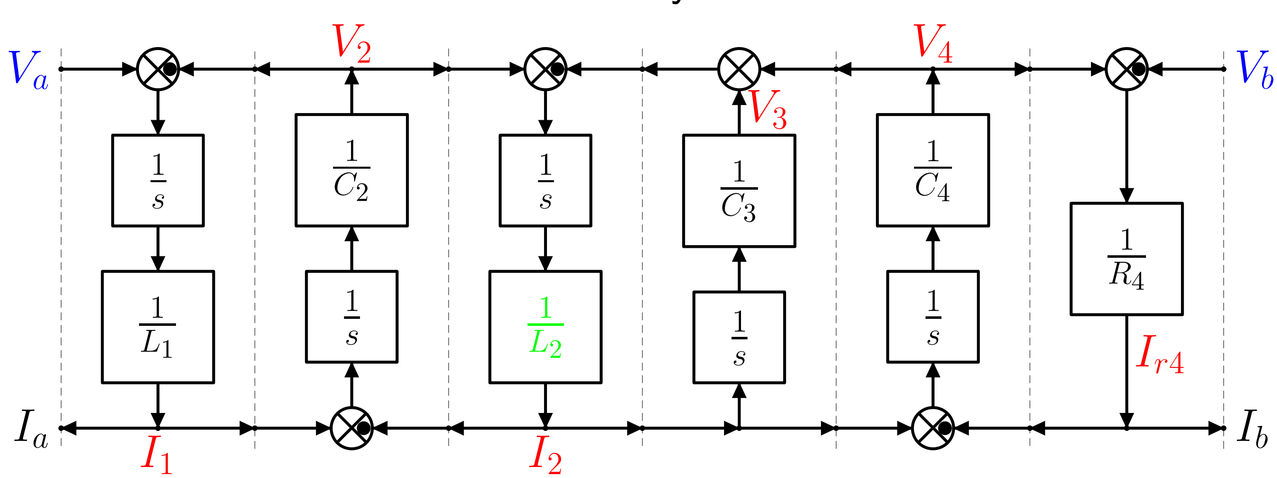
- Example. The following system cannot be easily modeled due to the presence of a closed path characterized only by effort dynamic blocks.



To solve this problem, an additional spurious element can be added to the system (for example an inductance L_2 in series with capacitance C_3):



POG block scheme of the considered system:



State space equations:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & L_2 & 0 & 0 \\ 0 & 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & 0 & C_4 \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{R_4} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{R_4} \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R_4} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

When $L_2 = 0$, from the state space equations one obtains the following static constraint:

$$V_2 - V_3 - V_4 = 0 \qquad \rightarrow \qquad V_2 = V_3 + V_4$$

The system can be transformed and reduced at the same time using the following congruent state-space transformation $\mathbf{x} = \mathbf{T}_x \overline{\mathbf{x}}$:

$$\underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}_x} \underbrace{\begin{bmatrix} I_1 \\ V_3 \\ V_4 \end{bmatrix}}_{\overline{\mathbf{x}}}$$

The transformed and reduced system has the following structure:

$$\underbrace{\left[\begin{array}{c|cc} L_1 & 0 & 0 \\ \hline 0 & C_2 + C_3 & C_2 \\ 0 & C_2 & C_2 + C_4 \end{array}\right]}_{\overline{\mathbf{L}}} \dot{\overline{\mathbf{x}}} = \underbrace{\left[\begin{array}{c|cc} 0 & -1 & -1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & -\frac{1}{R_4} \end{array}\right]}_{\overline{\mathbf{A}}} \underbrace{\left[\begin{array}{c} I_1 \\ \hline V_3 \\ V_4 \end{array}\right]}_{\overline{\mathbf{x}}} + \underbrace{\left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 0 \\ 0 & \frac{1}{R_4} \end{array}\right]}_{\overline{\mathbf{B}}} \mathbf{u}$$

$$\underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_4} \end{array}\right]}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R_4} \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

- Calculations using Matlab:

```
-- Matlab commands ----------------------------------------------------------
 syms L1 L2 C2 C3 C4 R4
 L2=0;
 LM=diag([L1 C2 L2 C3 C4]);
 AM=[     0    -1       0    0    0;
          1     0      -1    0    0;
          0     1       0   -1   -1;
          0     0       1    0    0;
          0     0       1    0 -1/R4];
 BM=[1 0; 0 0; 0 0; 0 0; 0 1/R4];
 Tr=    [ 1  0  0;
          0  1  1;
          0  0  0;
          0  1  0;
          0  0  1];
 LT=Tr.'*LM*Tr
 AT=simplify(Tr.'*AM*Tr)
 BT=Tr.'*BM

-- Matlab output ------------------------------------------------------------

LT =                          AT =                 BT =

[L1,        0,        0]     [0, -1,    -1]       [1,    0]
[ 0, C2 + C3,       C2]     [1,  0,     0]       [0,    0]
[ 0,      C2, C2 + C4]     [1,  0, -1/R4]       [0, 1/R4]

 ----------------------------------------------------------------------------
```

- The reduced system can also be expressed as follows:

$$\underbrace{\begin{bmatrix} L_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{bmatrix}}_{\overline{\mathbf{L}}} \dot{\overline{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & \mathbf{K}^T \\ \mathbf{K} & \mathbf{0} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ \mathbf{x}_2 \end{bmatrix}}_{\overline{\mathbf{x}}} + \underbrace{\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{B}_1^T & \mathbf{B}_2^T \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \overline{\mathbf{D}}\, \mathbf{u}$$