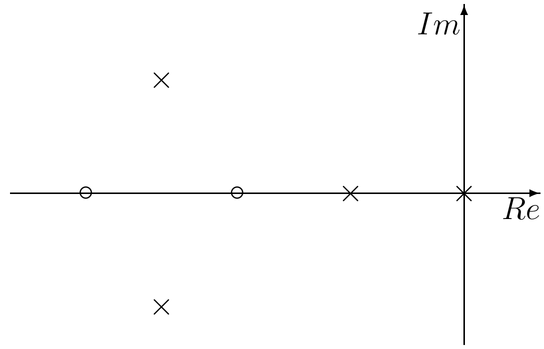
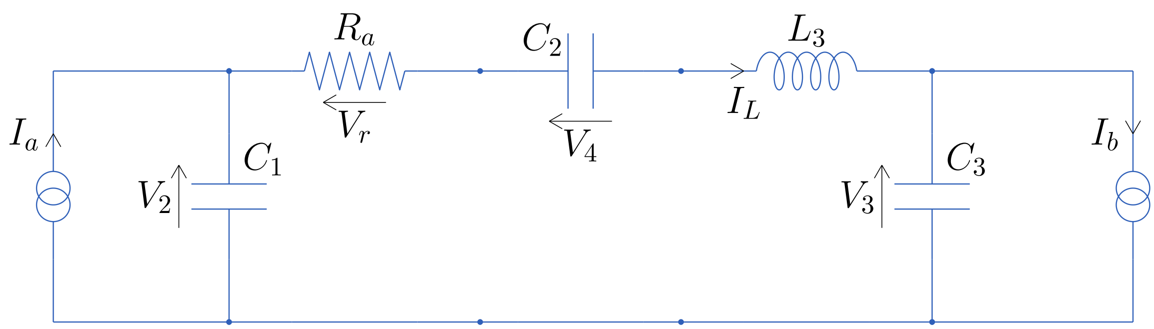
Minimum-phase systems

• Minimum-phase systems are those with all the poles and zeros having non-positive real parts. Example:

$$G(s) = \frac{(s+6)(s+10)}{s(s+3)[(s+8)^2+3^2])}$$



• All physical systems without an internal energy source are necessarily stable: all poles of the system have non-positive real parts.



The transfer function $H(s) = \frac{V_b(s)}{I_a(s)}$ which links $I_a(s)$ to $V_3(s)$ is the following:

$$H(s) = \frac{C_2}{s\left[C_1C_2C_3L_3s^2+C_1C_2C_3R_as+C_1C_2+C_1C_3+C_2C_3\right]}$$

Parameters: $C_1 = 0.003$ F; $R_a = 20$ Ohm; $C_2 = 30$ F; $L_3 = 0.002$ H; $C_3 = 50$ F. The numerical value of the transfer function $H(s)$ is:

$$H(s) = \frac{V_3(s)}{I_a(s)} = \frac{30}{s\left[0.009s^2+90.0s+1500.24\right]}.$$

The poles of the system are:

$$\text{Poles} = \left[\begin{array}{ccc} 0 & -16.697 & -9983.3 \end{array}\right].$$

The system is stable because it does not have energy sources: it can only accumulate energy (in the dynamic elements: C_1, C_2, C_3, and L_3) or dissipate energy (in the dissipative element: R_a).

• A physical system is only marginally stable only if part of the system is not influenced by the dissipative elements present within the system. For instance, the previous system has a pole at the origin because there are no resistances allowing the two capacitors C_1 and C_3 to discharge to ground.

• If a physical system contains no dissipative elements ($R_a = 0$), all its poles are located on the imaginary axis.

$$H(s) = \frac{N(s)}{D(s)} = \frac{C_2}{s\left[C_1C_2C_3L_3s^2+C_1C_2+C_1C_3+C_2C_3\right]}$$

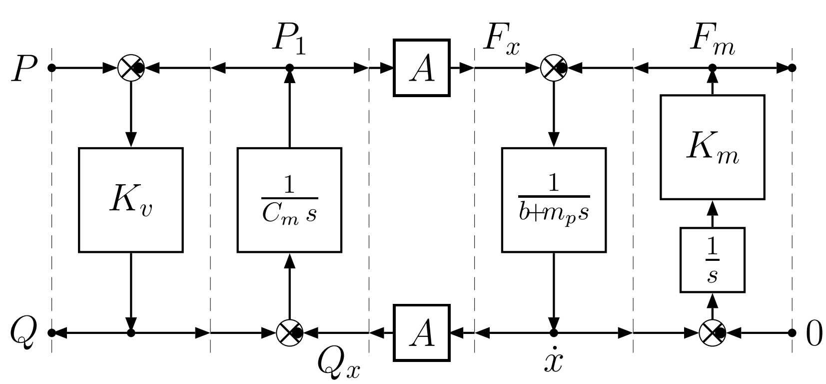
Systems of this type are characterized by a denominator polynomial $D(s)$ with only even (or odd) powers of the variable s:

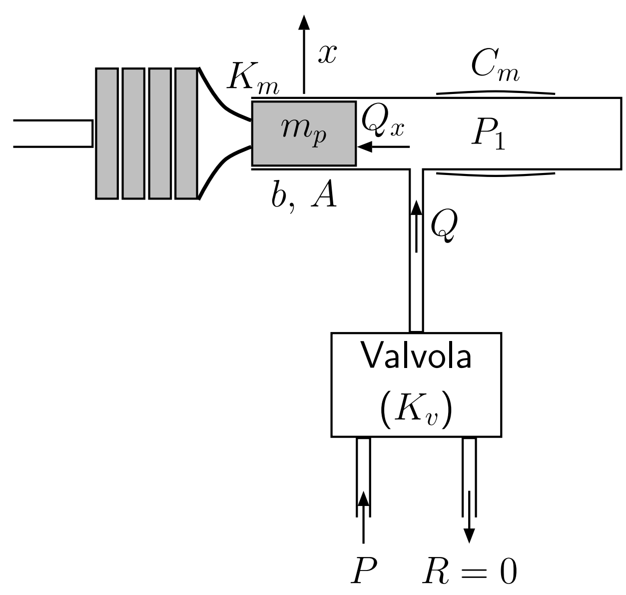
$$D(s) = s\left[C_1C_2C_3L_3s^2 + C_1C_2 + C_1C_3 + C_2C_3\right]$$

For systems of this type, the imaginary part of the poles corresponds to the resonance frequencies ω_r of the system:

$$\omega_r = \sqrt{\frac{C_1C_2 + C_1C_3 + C_2C_3}{C_1C_2C_3L_3}} = \frac{1500.24}{0.009} = 166690.$$

• A second example: an hydraulic clutch.





The transfer function $G(s)$ which links the input P to the output F_m is:

$$G(s) = \frac{F_m(s)}{P(s)} = \frac{AK_mK_v}{C_m m_p s^3 + (C_m\, b + K_v m_p)s^2 + (A^2 + C_m K_m + K_v\, b)s + K_m K_v}$$

When the system's dissipative elements are null ($K_v = 0$ and $b = 0$), the denominator $D(s)$ of the transfer function $G(s)$ contains only odd terms in s:

$$D(s) = C_m m_p s^3 + (A^2 + C_m K_m)s$$

Also in this case, without dissipative elements, all the poles of the system are located on the imaginary axis (ω_r is the resonance frequency):

$$p_1 = 0, \qquad p_{2,3} = \pm j\,\omega_r = \pm j\sqrt{\frac{(A^2 + C_m K_m)}{C_m m_p}}$$

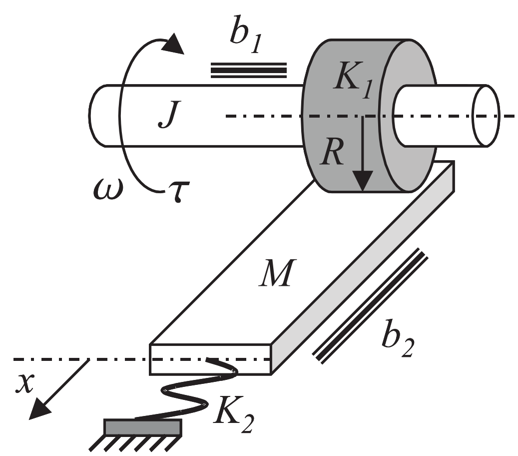
Third example: a mechanical system.

Transfer function $G(s)$ of the system:

$$G(s) = \frac{F_2}{\tau} = \frac{K_1 K_2 R}{a_4\, s^4 + a_3\, s^3 + a_2 s^2 + a_1 s + a_0}$$

where:

$$\begin{aligned} a_4 &= J\,M\,R^2 \\ a_3 &= (b_2 J + b_1 M) R^2 \\ a_2 &= J\,K_1 + b_1\,b_2\,R^2 + J\,K_2\,R^2 + K_1\,M\,R^2 \\ a_1 &= b_1\,K_1 + b_2\,K_1\,R^2 + b_1\,K_2\,R^2 \\ a_0 &= K_1\,K_2\,R^2 \end{aligned}$$



When the system's dissipative elements are null ($b_1 = 0$ and $b_2 = 0$), the odd coefficients a_1 and a_3 are null, and the denominator $D(s)$ of the transfer function $G(s)$ contains only even terms in s:

$$D(s) = \underbrace{J\,M\,R^2}_{\bar a_4} s^4 + \underbrace{(J\,K_1 + J\,K_2\,R^2 + K_1\,M\,R^2)}_{\bar a_2}\, s^2 + \underbrace{K_1\,K_2\,R^2}_{\bar a_0}$$

Also in this case, without dissipative elements, all the poles of the system are located on the imaginary axis. In this case, the system has two resonance frequencies $\omega_{1,2}$:

$$\omega_{1,2} = \sqrt{\frac{-\bar a_2 \pm \sqrt{\bar a_2^2 - 4\,\bar a_4\,\bar a_0}}{2\,\bar a_4}}$$

Nyquist Diagrams of minimum-phase systems

Most linear physical systems that accumulate and dissipate energy are minimum-phase, of type 0 or 1, and with a relative degree $r = n - m \geq 1$. Examples:

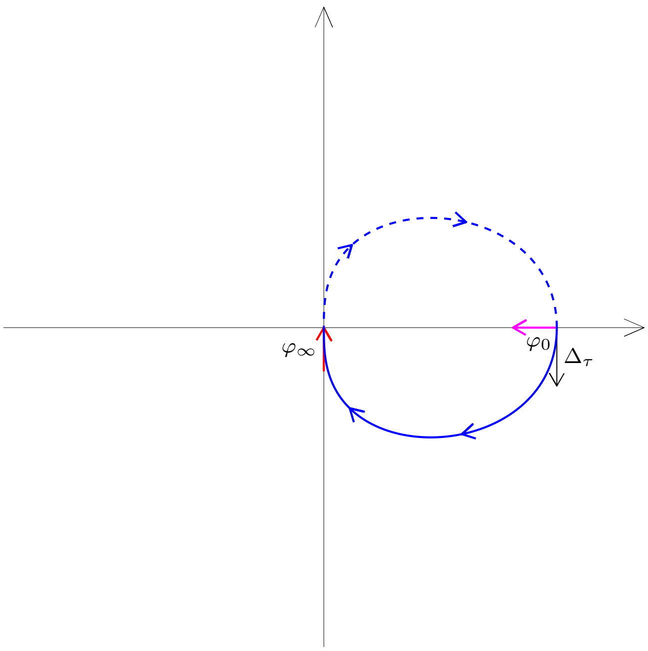
$$G_1(s) = \frac{20}{(s+10)^2}, \quad G_2(s) = \frac{6(s+1)}{s(s+3)^2}, \quad G_3(s) = \frac{16(s+2)}{s(s+1)(s+9)}, \quad G_4(s) = \frac{10}{s[(s+3)^2+5^2]}.$$

For systems of this type, the qualitative plotting of the Nyquist diagram of the function $G(s)$ is particularly simple. The plotting rules to follow are as follows:

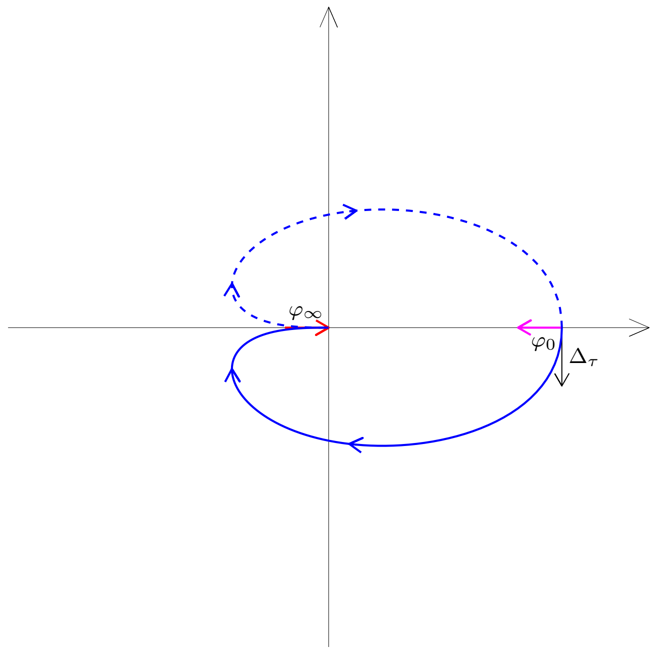
- If the system $G(s)$ is of type $t = 0$, the starting point of the diagram for $\omega = 0^+$ lies on the positive real axis and is equal to the static gain $G(0)$.
- If the system $G(s)$ is of type $t = 1$, the starting point of the Nyquist diagram for $\omega = 0^+$ lies at infinity with an initial phase $\varphi_0 = -\frac{\pi}{2}$.
- If the relative degree $r = n - m$ of the system $G(s)$ is greater than or equal to 1, the endpoint of the Nyquist diagram for $\omega = \infty$ is the origin with a final phase $\varphi_\infty = -\frac{r\,\pi}{2}$.
- For $\omega \in [0^+ \;\; \infty]$, the Nyquist diagram $G(j\omega)$ is plotted on the complex plane starting from the initial point, ending at the final point, and moving clockwise around the origin by an angle equal to $\Delta\varphi = \varphi_\infty - \varphi_0 = -\frac{(r-t)\,\pi}{2}$.
- After plotting the Nyquist diagram for $\omega > 0$, i.e., $\omega \in [0^+ \;\; \infty]$, the portion of the diagram for $\omega < 0$ is obtained by reflecting the diagram for $\omega > 0$ across the real axis.
- The closure at infinity of the Nyquist diagram for $\omega \in [0^- \;\; 0^+]$ is done as follows: 1) start from the point $\omega = 0^-$; 2) move clockwise; 3) arrive at the point $\omega = 0^+$ by adding a number of semicircles at infinity equal to the type t of the system $G(s)$.

• Qualitative plotting of the Nyquist diagrams of $G(s)$ systems of type $t = 0$ and relative degree $r \in [1,\ 2,\ 3,\ 4]$:

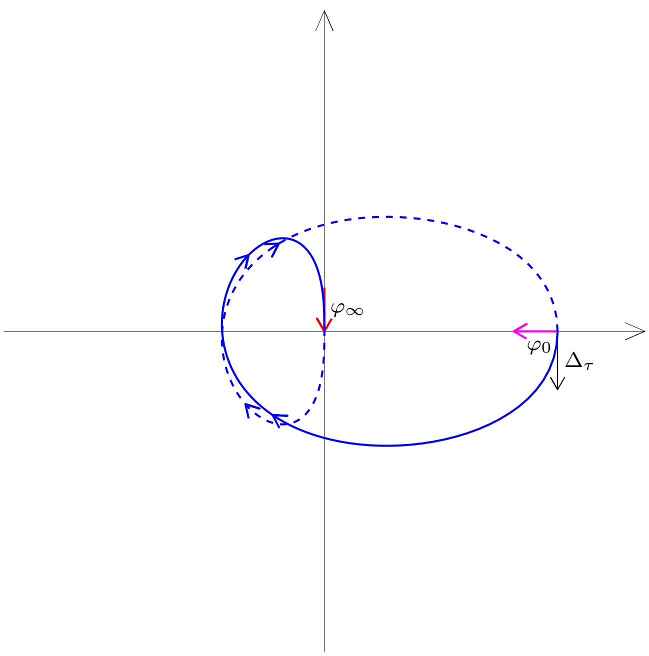
Nyquist Diagram: $G(s)$ with $t = 0$ and $r = 1$



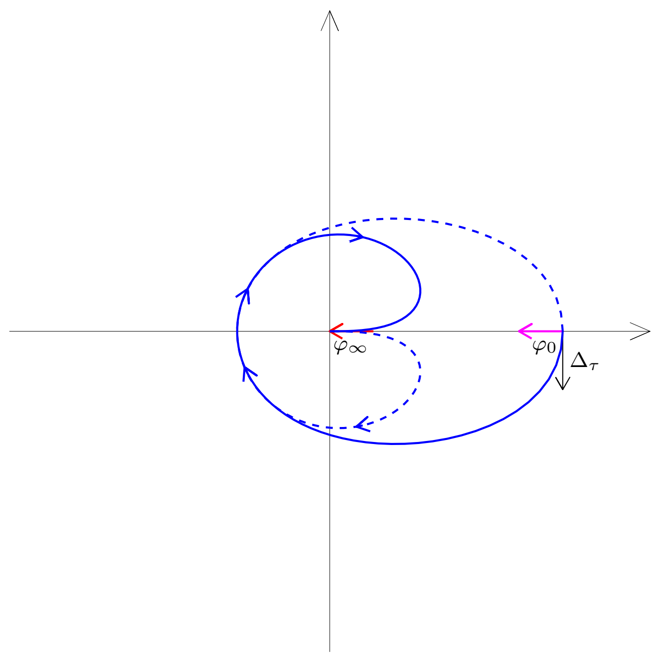
Nyquist Diagram: $G(s)$ with $t = 0$ and $r = 2$



Nyquist Diagram: $G(s)$ with $t = 0$ and $r = 3$

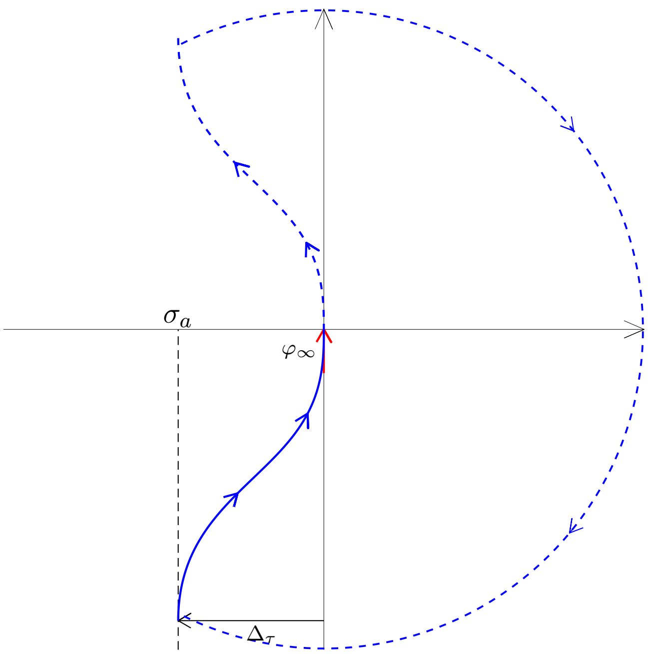


Nyquist Diagram: $G(s)$ with $t = 0$ and $r = 4$

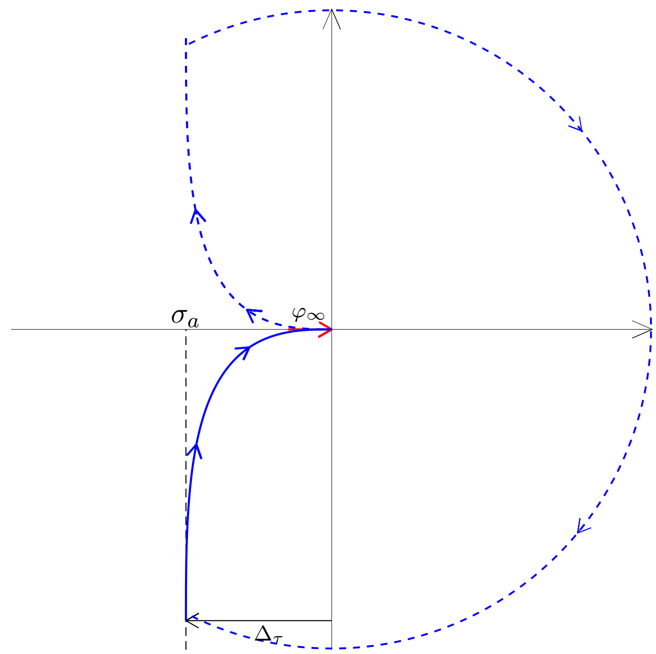


• Qualitative plotting of the Nyquist diagrams of $G(s)$ systems of type $t = 1$ and relative degree $r \in [1,\ 2,\ 3,\ 4]$:

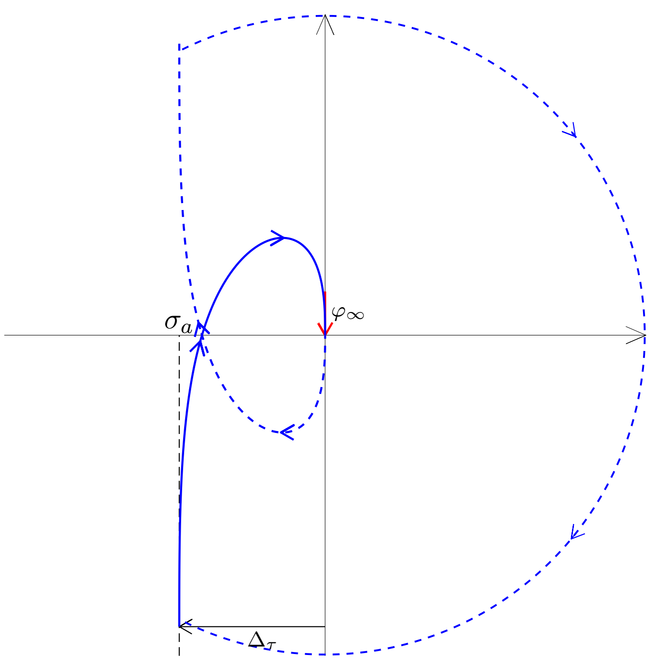
Nyquist Diagram: $G(s)$ with $t = 1$ and $r = 1$



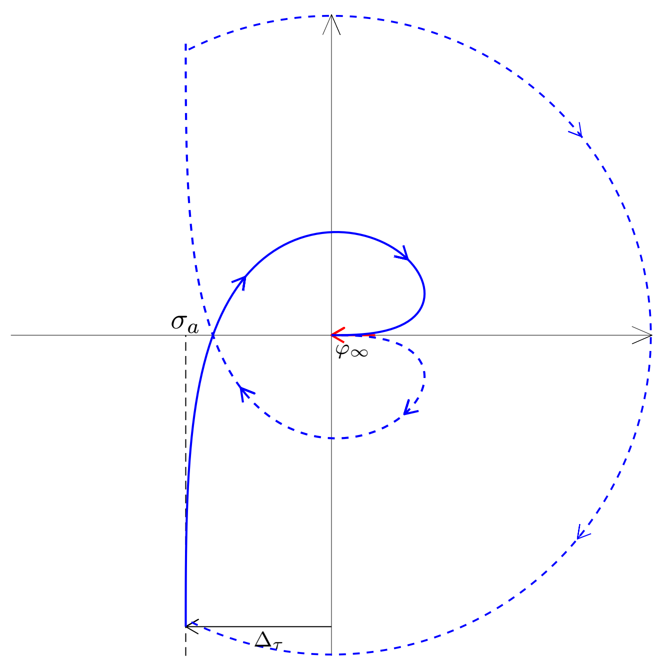
Nyquist Diagram: $G(s)$ with $t = 1$ and $r = 2$



Nyquist Diagram: $G(s)$ with $t = 1$ and $r = 3$

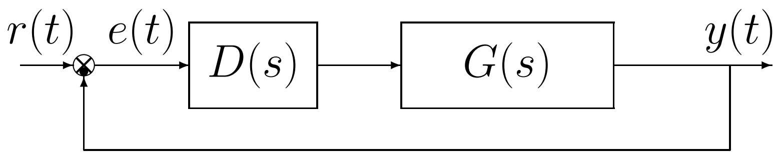


Nyquist Diagram: $G(s)$ with $t = 1$ and $r = 4$



Relative degree

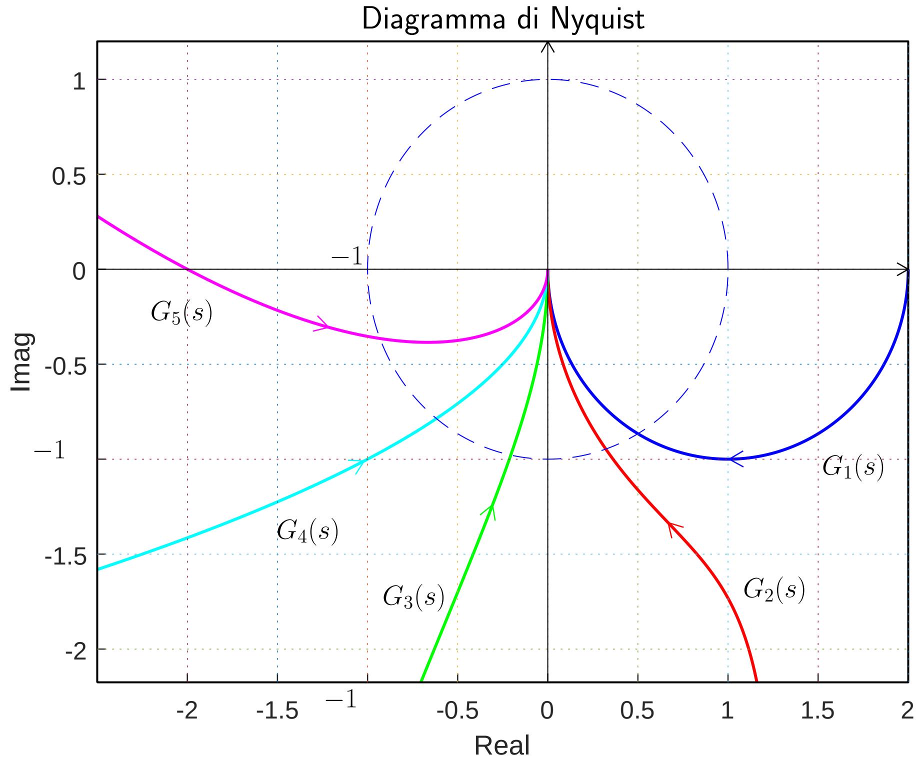
• Let us consider a minimum-phase system $G(s)$ controlled in feedback:



• The difficulty of the project is typically a function of the relative degree $r = n - m$: the higher the **relative degree** r, the more difficult it is to stabilize the feedback system and the longer is the settling time T_s.

Minimum-phase systems $G(s)$ with $r = 1$

• All minimum phase systems $G(s)$ with **relative degree** $r = 1$ can be easily stabilized in feedback using a sufficiently high gain K.



Functions $G(s)$ with relative degree $r = 1$ displayed on the Nyquist diagram:

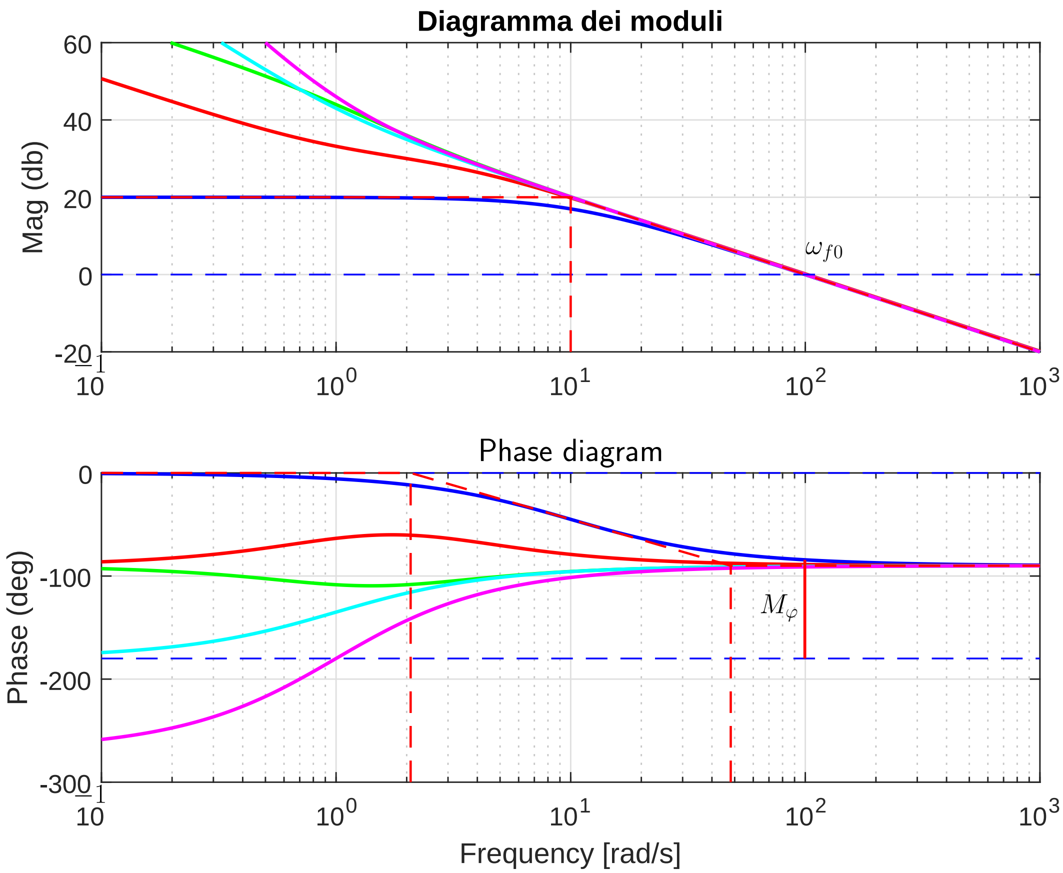
$$G_1(s) = \frac{20}{s+10}, \quad G_2(s) = \frac{6(s+1)}{s(s+3)}, \quad G_3(s) = \frac{16(s+2)}{s(s+1)}, \quad G_4(s) = \frac{(s+1)}{s^2}, \quad G_5(s) = \frac{(s+1)^2}{s^3}$$

• If the relative degree is $r = 1$, for $\omega \to \infty$ the Nyquist diagrams tends to zero with phase $\varphi_\infty = -\frac{\pi}{2}$. It follows that choosing K_i sufficiently high the feedback system will be stable with a high phase margin M_φ. Choosing:

$$K_1 = 5, \qquad K_2 = 17, \qquad K_3 = 25, \qquad K_4 = 100, \qquad K_5 = 100.$$

all the systems $K_i G_i(s)$ share the same crossing frequency $\omega_{f0} = 100$, that is the frequency where $|K_i\, G_i(j\,\omega_{f0})| = 1$.

• Bode Diagrams of functions $K_i G_i(s)$:



• All the system $K_i G_i(s)$ are characterized by a phase margin $M_\varphi \simeq \frac{\pi}{2}$. When phase margin M_φ approaches $\frac{\pi}{2}$ the step response of the feedback system is "aperiodic", characterized by a transient without oscillations.

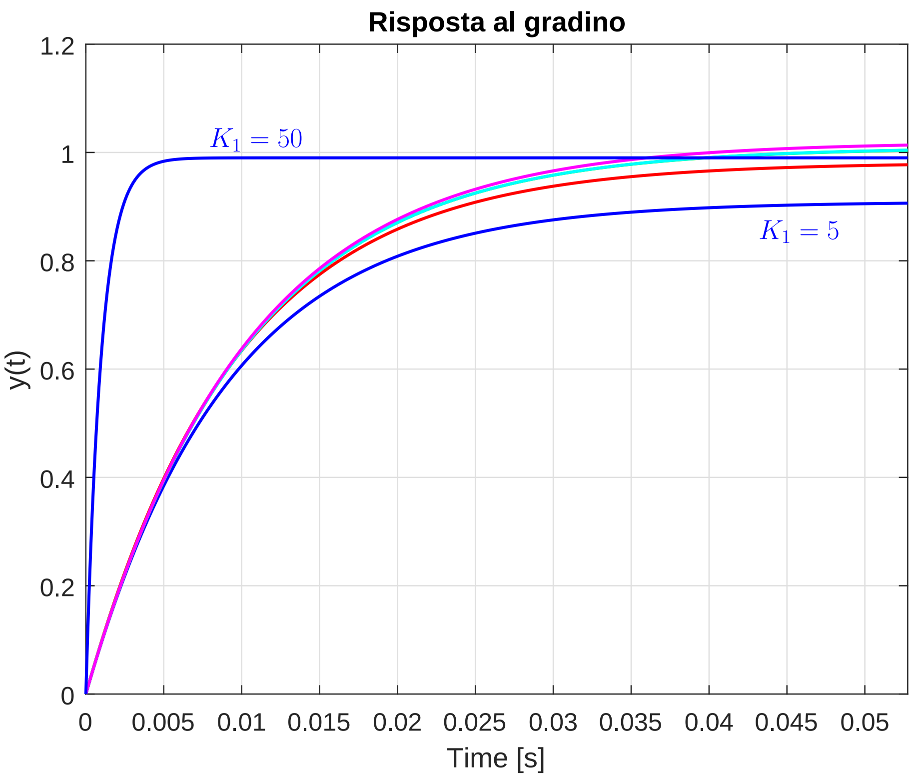
• The crossing frequency $\omega_{f0} = 100$ and the rising time T_r are relates as follows:

$$T_r \propto \frac{1}{\omega_{f0}} = 0.01$$

• Since the step response of the feedback system is "aperiodic", the corresponding settling time T_s is

$$T_s \simeq 3\,T_r \simeq 0.03.$$

• Step response of the feedback systems:



• Note: since all the functions share the same crossing frequency $\omega_{f0} = 100$, then they share the same settling time $T_s \simeq 0.03$.

• The step response of system $K_1 G_1(s)$ does not have a zero tracking error $e_\infty = 0$ because system $K_1 G_1(s)$ is of type $t = 0$. Using $K_1 = 5$, the steady-state error of function $K_1 G_1(s)$ is:

$$e_\infty = \frac{1}{1 + K_1 G_1(0)} = \frac{1}{11} = 0.09$$

• The value of parameter K_1 to be used in order to to obtain a steady-state error $e_\infty = 0.01$ is:

$$K_1 = \frac{1}{G_1(0)}\left(\frac{1}{e_\infty} - 1\right) = \frac{1}{2}\left(\frac{1}{0.01} - 1\right) = \frac{99}{2} \simeq 50$$

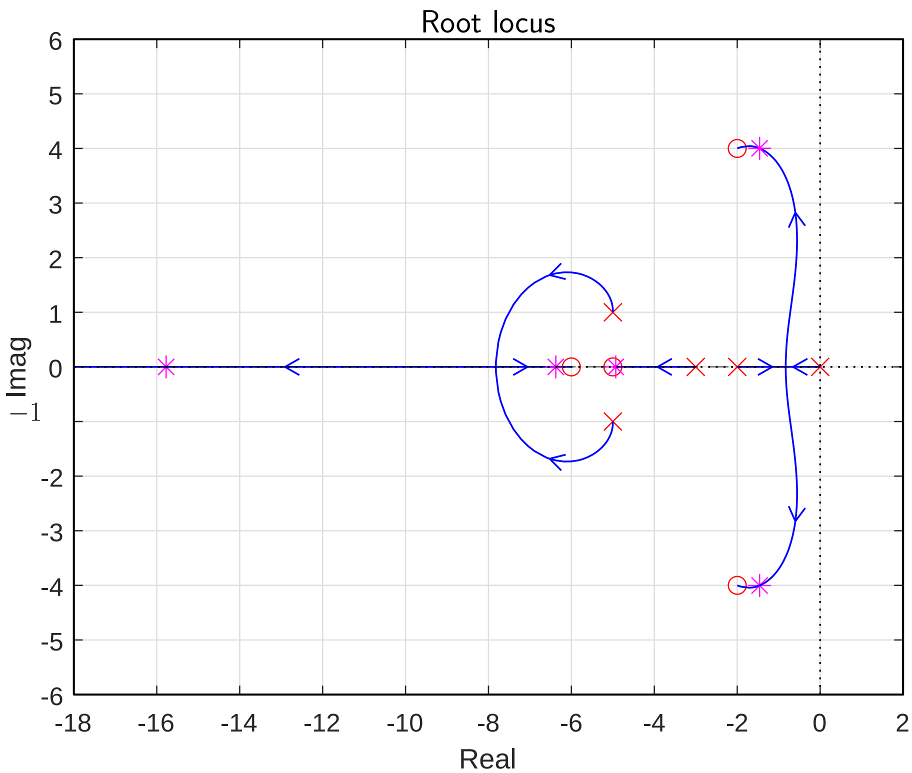
Using $K_1 = 50$, see the figure, the step response of system $K_1 G_1(s)$ has an higher value for the crossing frequency ω_{f0} and therefore a smaller values for the rising time T_r and for the settling time T_s.

• The fact that a minimum phase system with relative degree $r = 1$ is easily controllable in feedback using a gain K sufficiently high can also be highlighted using the root locus.

• Consider, for example, the following minimum phase system $G(s)$ with relative degree $r = 1$:

$$G(s) = \frac{(s^2 + 4s + 13)(s + 4)(s + 6)}{s(s + 2)(s + 3)(s^2 + 10s + 50)}$$

• Root locus of function $G(s)$ for $K > 0$:

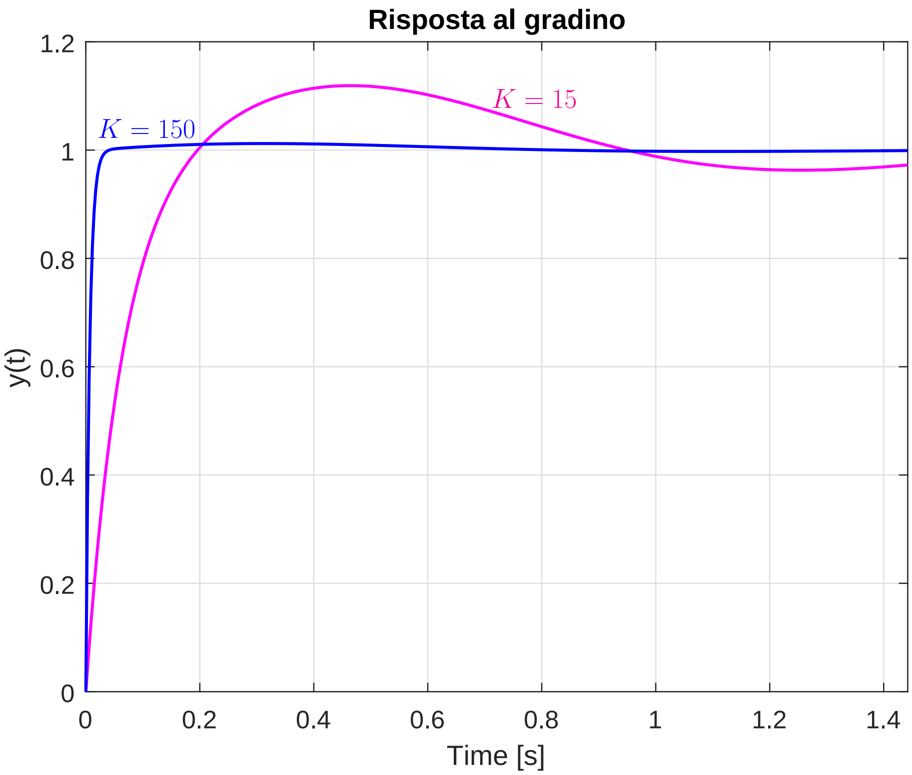


• The magenta asterisks "$*$" denote the position of the poles of the feedback system when $K = 15$.

• Four of the five poles of the feedback system are located "very close" to the four zeros of the system, so for these poles there is, in practice, a "pole-zero cancellation" which makes them practically irrelevant in the step response of the feedback system. These pole-zero cancellations are more precise the higher the value of K.

• The fifth pole "$*$" of the feedback system positioned in $p_5 \simeq -16$ is the only one not to undergo a pole-zero cancellation and therefore becomes the "dominant pole" of the feedback system.

• Step response of function $G(s)$ using the gains $K = 15$ and $K = 150$:



• Using the gain $K = 15$ the poles-zeros cancellations in the feedback system are not complete, and therefore a residual oscillation, due to the two complex pole in $-2 \pm 4\,j$, still remains.

• Using the gain $K = 150$ the poles-zeros cancellations in the feedback system are almost complete, and therefore the shape of the transient is mainly determined by the only pole on the real negative axis which is not cancelled by a zero.

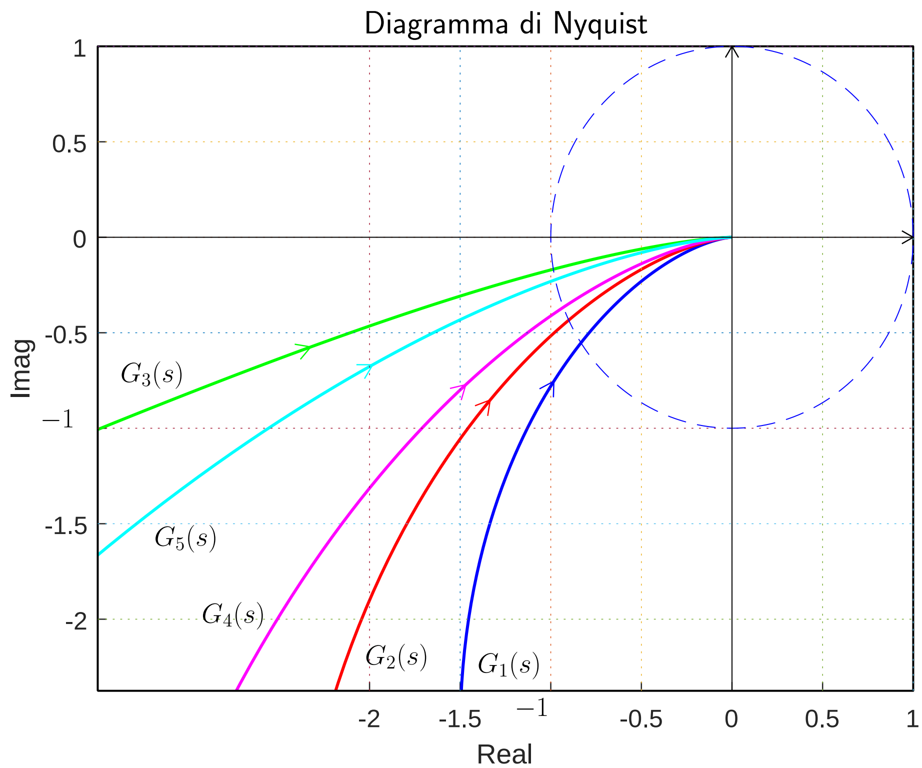
Minimum-phase systems $G(s)$ with $r = 2$

• All minimum phase systems $G(s)$ with relative degree $r = 2$ and with $\Delta_p > 0$ can typically be stabilized using a gain K and a lead network.

• Let us consider the following minimum-phase systems $G(s)$ with relative degree $r = 2$:

$$G_1(s) = \frac{1200}{(s+10)^2}, \quad G_2(s) = \frac{324(s+1)}{s(s+3)^2}, \quad G_3(s) = \frac{32(s+1)}{s(s+2)}, \quad G_4(s) = \frac{64(s+1)}{s(s+2)^2}, \quad G_5(s) = \frac{320(s+1)^2}{s(s+2)^3}$$

• Nyquist diagrams of the considered functions $G(s)$:



• The value of the following parameter Δ_p determines if the Nyquist diagram of function $G(s)$ intersects the negative real axis:

$$\Delta_p = \sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i \qquad \Rightarrow \qquad \begin{cases} \text{if } \Delta_p > 0 \rightarrow & \text{no intersection} \\ \text{if } \Delta_p < 0 \rightarrow & \text{yes intersection} \end{cases}$$

where z_i and p_i are the zeros and the poles of function $G(s)$.

• All the considered functions have positive parameter Δ_p:

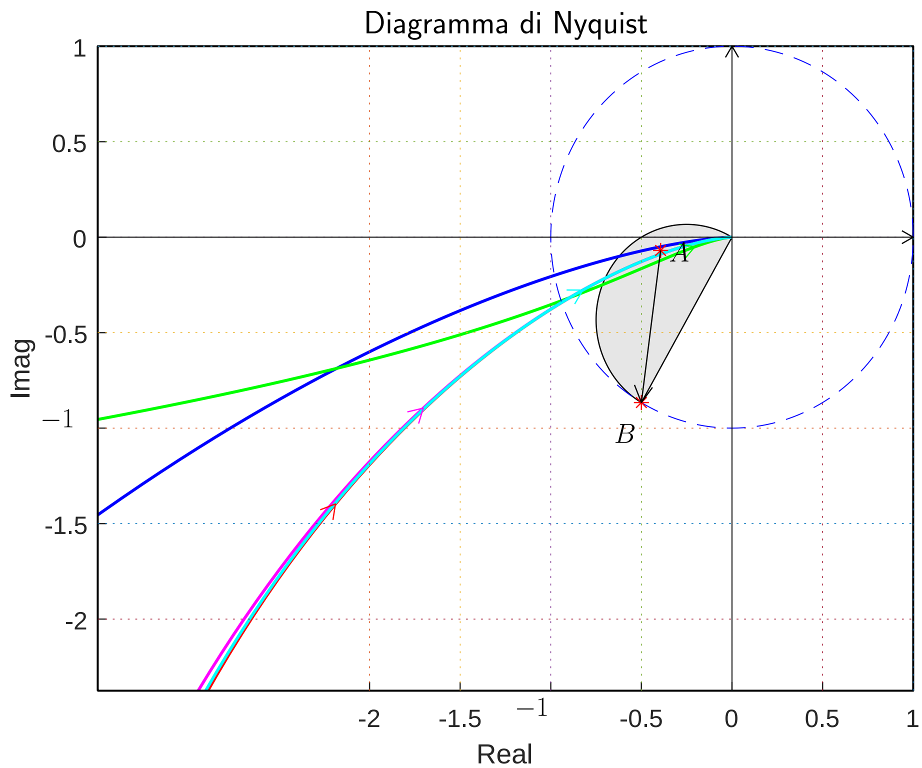
$$\Delta_{p1} = 20, \qquad \Delta_{p2} = 5, \qquad \Delta_{p3} = 1, \qquad \Delta_{p4} = 3, \qquad \Delta_{p5} = 4.$$

• For all the systems $G(s)$ of type 0, as for $G_1(s)$, the gain K_1 must be chosen sufficiently high to satisfy the desired tracking error, for example $e_\infty = 0.01$:

$$K_1 = \frac{1}{G_1(0)}\left(\frac{1}{e_\infty} - 1\right) = \frac{1}{12}\left(\frac{1}{0.01} - 1\right) = \frac{99}{12} = 8.25$$

• For all the systems $G(s)$ of type 1, as for $G_2(s)$, $G_3(s)$, $G_4(s)$ and $G_5(s)$, the gains K_i must increased such to obtain the following condition for which it is easy to design a lead network: the frequency response $G(j\omega)$ passes in the vicinity of the point A shown in the following figure.

• If the desired phase margin is $M_\varphi \simeq 60°$, then the admissible region for a lead network is the grey region reported in the following figure.



• A first possibile solution:

1) choose the gains K_i ($K_2 = 1.64$, $K_3 = 0.088$, $K_4 = 1.16$ and $K_5 = 0.415$) in order to have a phase margin $M_\varphi \simeq 20°$(see the figure above);

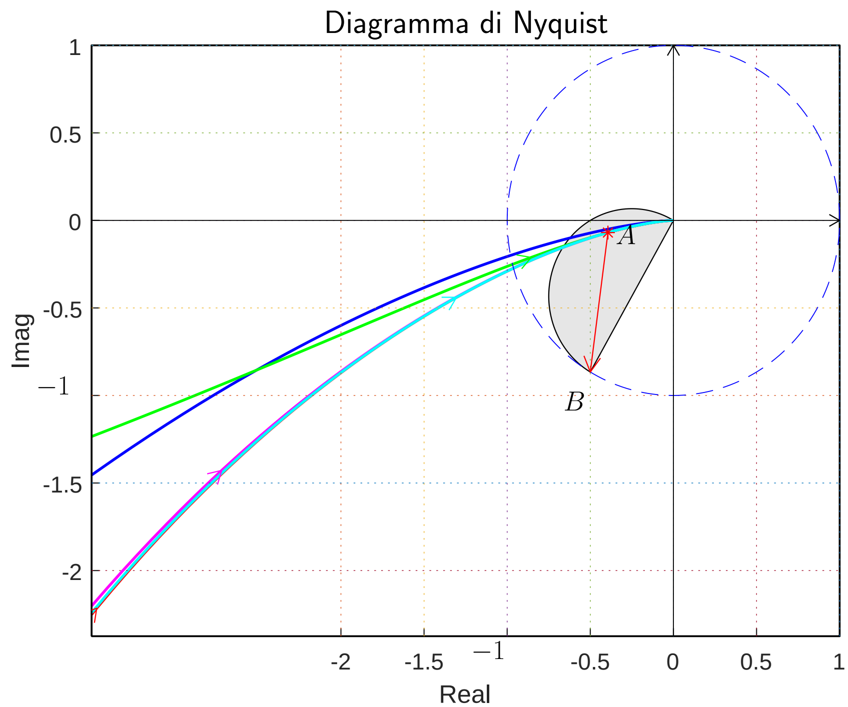
2) choose a point A belonging to the admissible region;

3) design a lead network $C(s)$ which moves point A to point B.

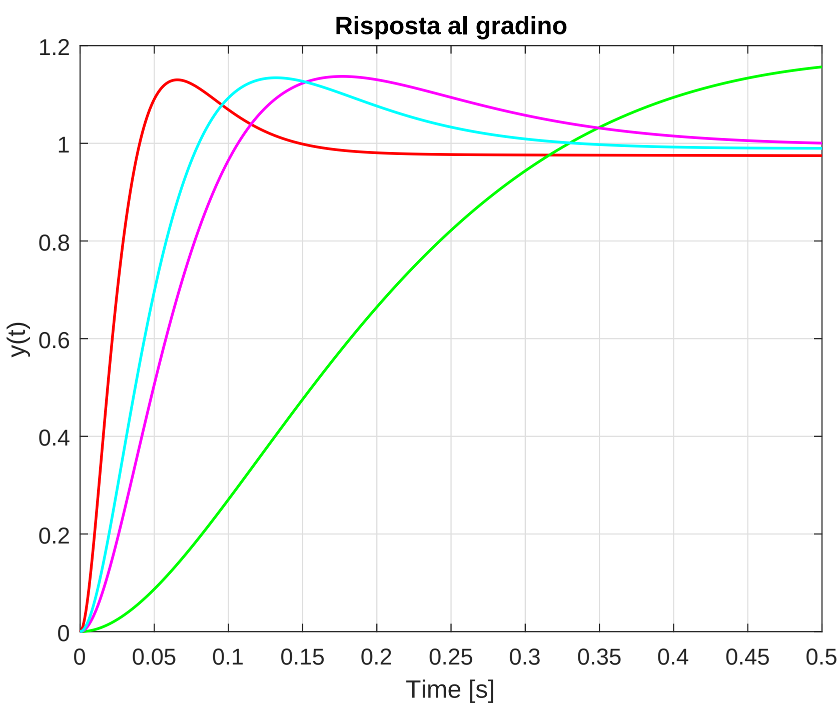
$$C(s) = \frac{1 + \tau_1\, s}{1 + \tau_2\, s}, \qquad \tau_1 = \frac{M - \cos(\varphi)}{\omega_A \sin(\varphi)}, \qquad \tau_2 = \frac{\cos(\varphi) - \frac{1}{M}}{\omega_A \sin(\varphi)}$$

where $M = \frac{M_B}{M_A}$ and $\varphi = \varphi_B - \varphi_A$.

• A second possibile solution:
1) choose the gains K_i ($K_2 = 1.64$, $K_3 = 0.88$, $K_4 = 1.16$ and $K_5 = 0.415$) and read the frequencies ω_i ($\omega_2 = 2.6$, $\omega_3 = 0.368$, $\omega_4 = 1.85$ and $\omega_5 = 0.658$) for which the functions $K_i\, G_i(s)$ pass through the point $A = 0.4\, e^{j(190^\circ)}$;
2) design a lead network $C(s)$ which moves point A to point B.



• Step response of the controlled systems $K_i\, C(s)\, G_i(s)$:



• The settling time T_s of the controlled systems is inversely proportional to the frequency ω_i of system $K_i\, G_i(s)$ in point A: $T_s \simeq \frac{3}{\omega_i}$.

Minimum-phase systems $G(s)$ of type 0 with $r \geq 3$

• All minimum phase systems $G(s)$ of type 0 with relative degree $r \geq 3$ can usually be stabilized by using a gain K and a lag network (or a lead-lag network).

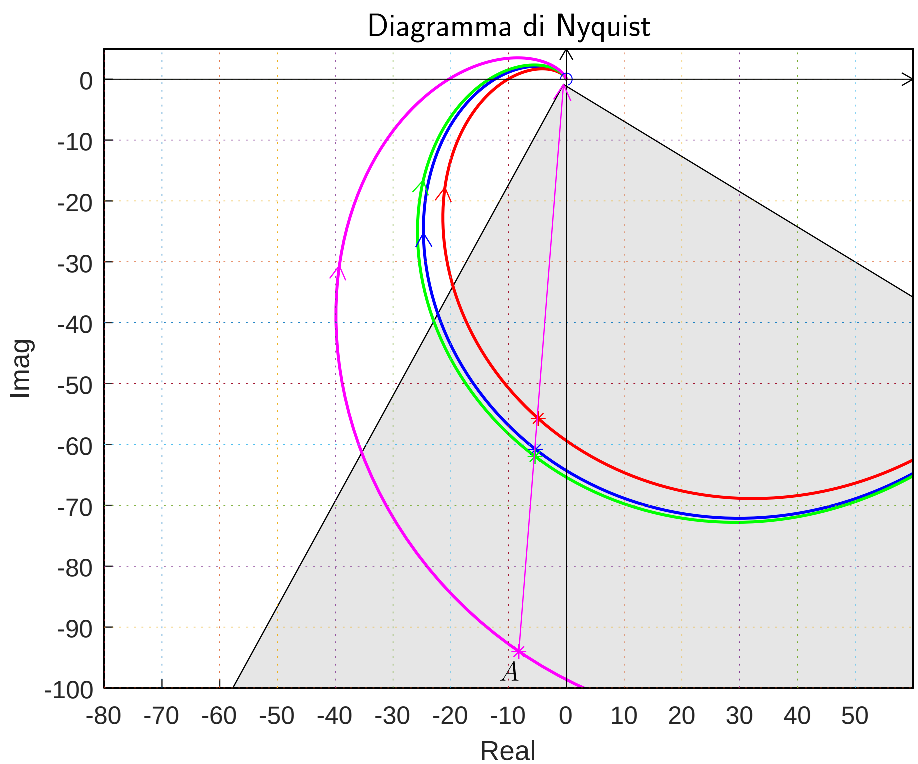
• Let us consider the following minimum-phase systems $G(s)$ of type 0 with relative degree $r = 3$:

$$G_1(s) = \frac{15000}{(s+10)^3}, \quad G_2(s) = \frac{600}{(s+2)(s+5)^2}, \quad G_3(s) = \frac{140(s+3)}{(s+2)^4}, \quad G_4(s) = \frac{400(s+1)}{(s+5)(s+3)^3},$$

• For these systems, the first thing to do is to determine the gain K in order to obtain a small tracking error, for example $e_\infty = 0.01$:

$$K_1 = 6.6, \quad K_2 = 8.25, \quad K_3 = 3.77, \quad K_4 = 33.4.$$

• Nyquist diagrams of functions $K_i\, G_i(s)$:



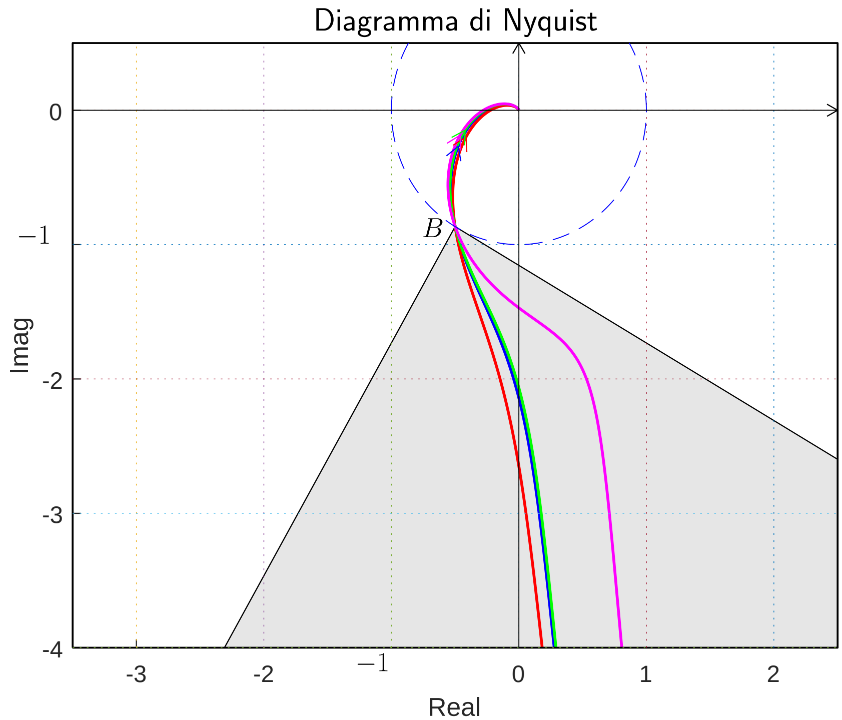
• In this case the points A_i have been chosen at the frequency ω_i such that $\arg(A_i) = \arg(G_i(j\omega_i)) = 180 + M_\varphi + 25$. The frequency ω_i of points A_i are:

$$\omega_1(s) = 6.168, \quad \omega_2(s) = 2.193, \quad \omega_3(s) = 1.098, \quad \omega_4(s) = 3.019.$$

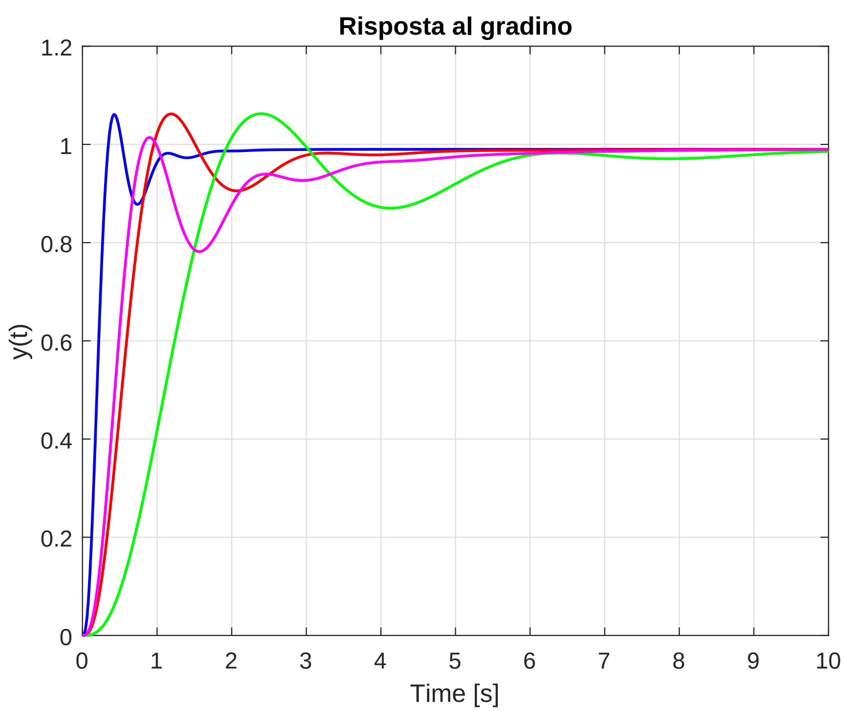
- The next step is choose the phase margin $M_\varphi = 60°$ and design a first order **"Lag Network"** which stabilize the systems $K_1 G_1(s)$, $K_2 G_2(s)$, $K_3 G_3(s)$ and $K_4 G_4(s)$, by moving point A to point B:

$$C_1(s) = \frac{1 + 0.3414\, s}{1 + 23.07\, s}, \qquad C_2(s) = \frac{1 + 0.9586\, s}{1 + 59.39\, s}, \qquad C_3(s) = \frac{1 + 1.918\, s}{1 + 132.1\, s}, \qquad C_4(s) = \frac{1 + 0.702\, s}{1 + 73.26\, s}.$$

- Nyquist diagrams of functions $K_i\, C_i(s)\, G_i(s)$ including the lag network:



- Step response of the controlled systems $K_i\, C(s)\, G_i(s)$:



- Note: the settling time T_s of the step response is inversely proportional to the frequency ω_i of the chosen point A_i.

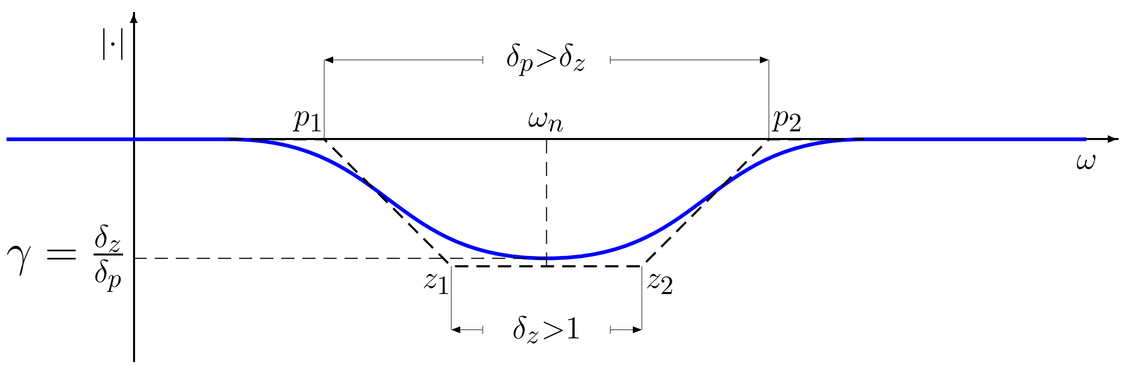
Design of a second order Lead-Lag network

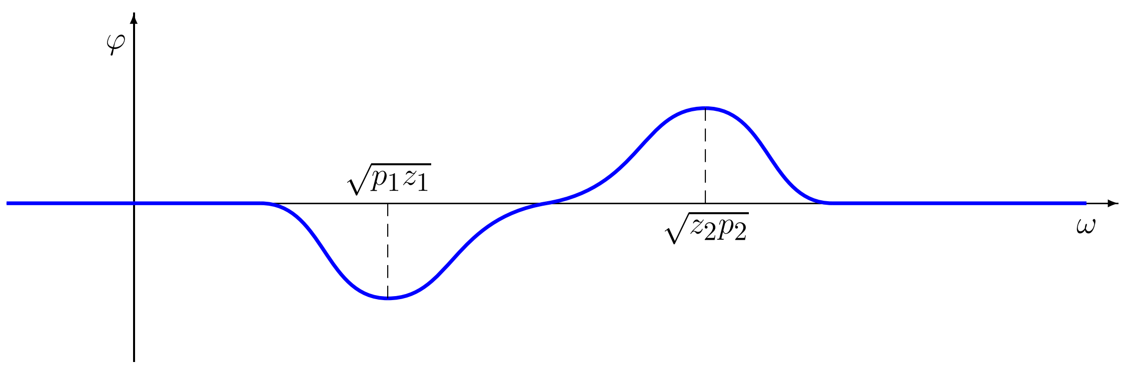
• The Inversion Formulas can also be used to design a second order Lead-Lag network having the following structure:

$$C(s) = \frac{s^2 + 2\delta_z\omega_n s + \omega_n^2}{s^2 + 2\delta_p\omega_n s + \omega_n^2}.$$

where the parameters δ_z, δ_p and ω_n are real and positive. The static and the high frequency gains of controller $C(s)$ are both equal to 1.

• If $\delta_p > \delta_z > 1$, the poles and the zeros of $C(s)$ are real, and the shape of the Bode diagrams of the Lead-Lag network $C(s)$ are the following:





• The zeros z_1, z_2 of the Lead-Lag network $C(s)$ are the following:

$$s^2 + 2\delta_z\omega_n s + \omega_n^2 = 0 \qquad \rightarrow \qquad z_{1,2} = \omega_n\left(-\delta_z \pm \sqrt{\delta_z^2 - 1}\right).$$

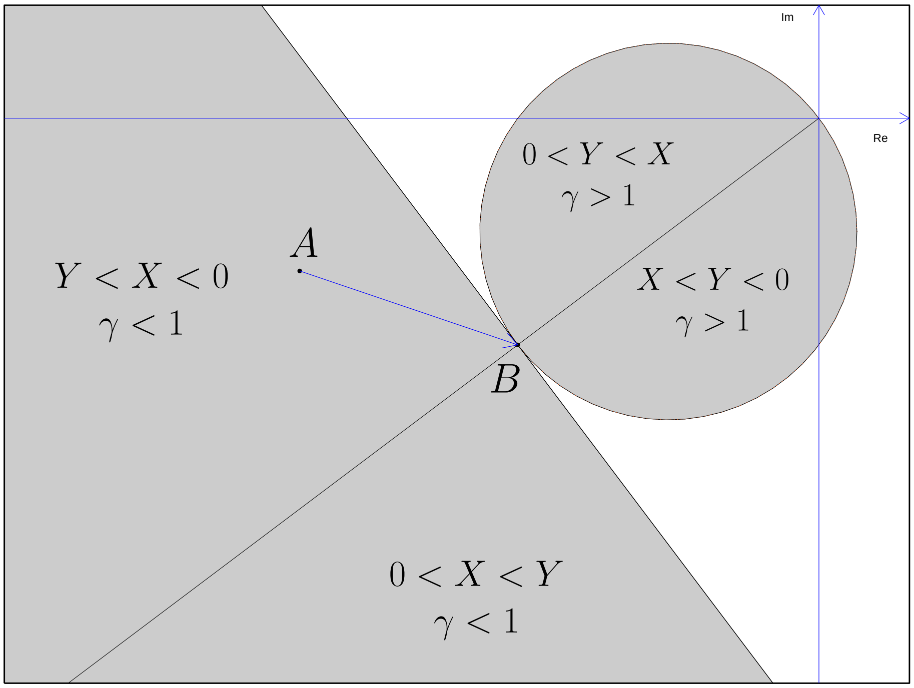
• The poles p_1, p_2 of the Lead-Lag network $C(s)$ are the following:

$$s^2 + 2\delta_p\omega_n s + \omega_n^2 = 0 \qquad \rightarrow \qquad p_{1,2} = \omega_n\left(-\delta_p \pm \sqrt{\delta_p^2 - 1}\right).$$

• The poles p_1, p_2 and the zeros z_1, z_2 satisfy the following relations:

$$p_1 \cdot p_2 = z_1 \cdot z_2 = \omega_n^2.$$

• The grey region of the complex plane shown in the following figure is the **admissible region** $\mathcal{D}_B^-$: the set of all the point A that can be moved to B using the lead/lag network $C(s)$.



• Design procedure:

1) Point B is defined by choosing the phase M_φ or the gain M_α of the system.

2) Point A is chosen within the admissible region $\mathcal{D}_B^-$, and belonging to the frequency response $G(j\omega_A)$ of the considered transfer function $G(s)$.

3) From $A = M_A\, e^{j\varphi_A}$ and $B = M_B\, e^{j\varphi_B}$ compute the parameters M and φ:

$$M = \frac{M_B}{M_A}, \qquad \varphi = \varphi_B - \varphi_A$$

and then the parameters X and Y:

$$X = \frac{M - \cos\varphi}{\sin\varphi}, \qquad Y = \frac{\cos\varphi - \frac{1}{M}}{\sin\varphi}.$$

4) Compute parameters δ_z, δ_p and ω_n of the Lead-Lag network $C(s)$ as follows:

$$\delta_z = \frac{R+1}{2\sqrt{R}}, \qquad \delta_p = \frac{\delta_z\, Y}{X} \qquad \omega_n = \omega_A \left(\frac{\delta_z}{X} + \sqrt{\frac{\delta_z^2}{X^2} + 1} \right)$$

where $R = \frac{|z_2|}{|z_1|}$ is the distance between the two zeros of the network $C(s)$.

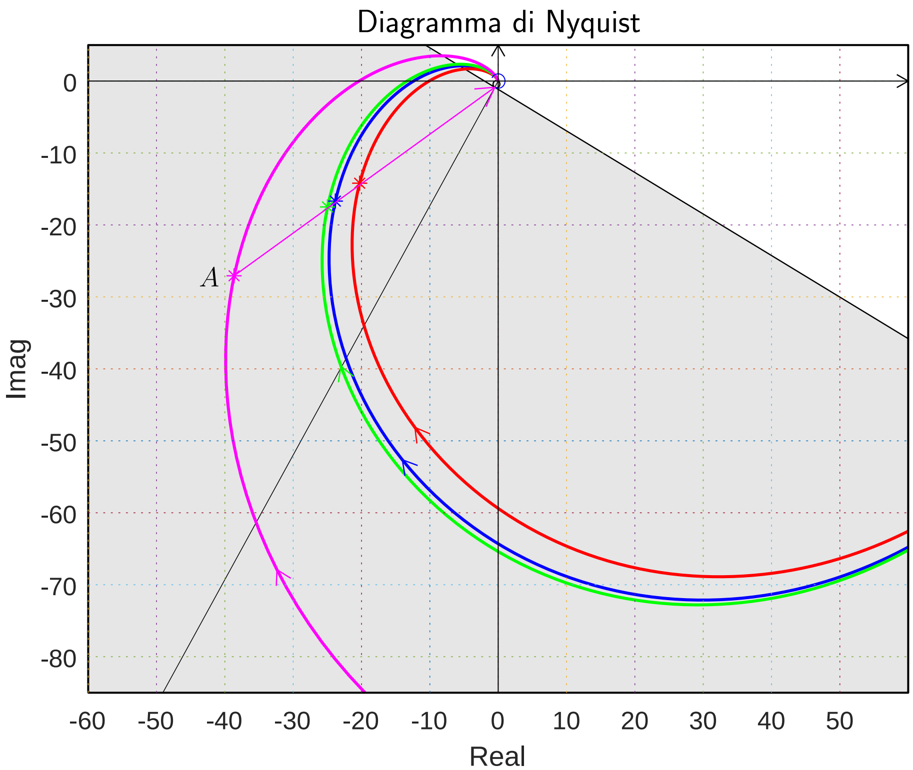
- The gain γ of network $C(s)$ at frequency ω_n can be expressed as follows:

$$\gamma = C(s)|_{s=j\omega_n} = \frac{\delta_z}{\delta_p} = \frac{X}{Y}.$$

- parameter R is a degree of freedom in the design of network $C(s)$. A typical used value for parameter $R = \frac{|z_2|}{|z_1|}$ is $R = 1$. The corresponding value of parameter δ_z is:

$$\delta_z = \left.\frac{R+1}{2\sqrt{R}}\right|_{R=1} = 1$$

- Nyquist diagrams of functions $K_i\, G_i(s)$:



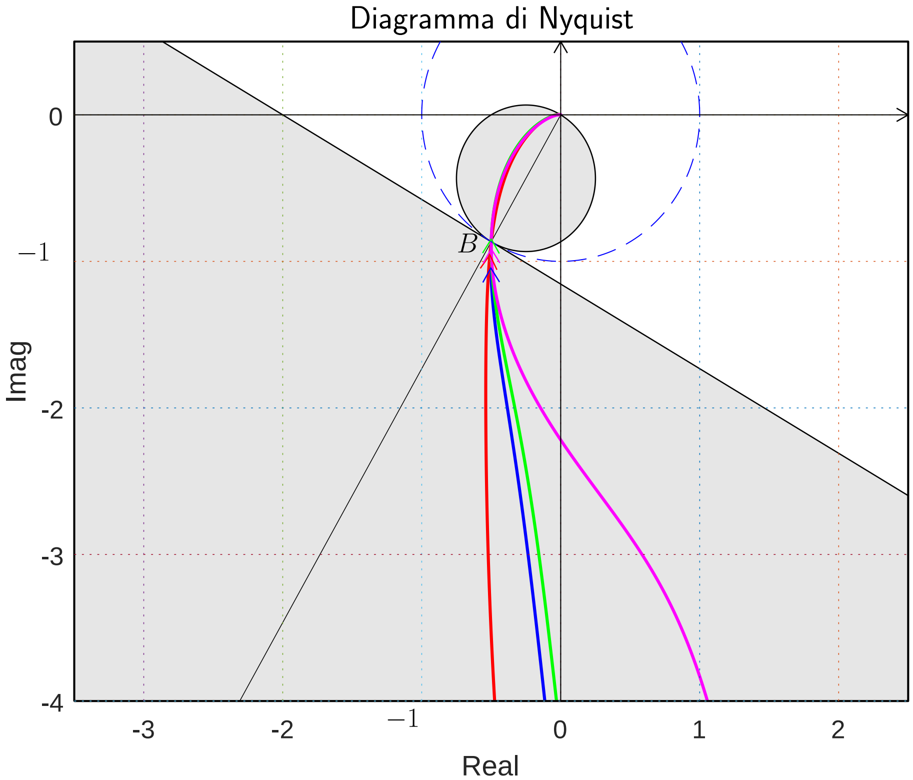
- The points A_i shown in the figure have been chosen at the frequency ω_i such that: $\arg(A_i) = \arg(G_i(j\omega_i)) = 180 + M_\varphi - 25$.

- Lead-Lag Networks $C(s)$ designed to stabilize the systems $K_1\, G_1(s)$, $K_2\, G_2(s)$, $K_3\, G_3(s)$ and $K_4\, G_4(s)$. imposing a phase margin $M_\varphi = 60°$:

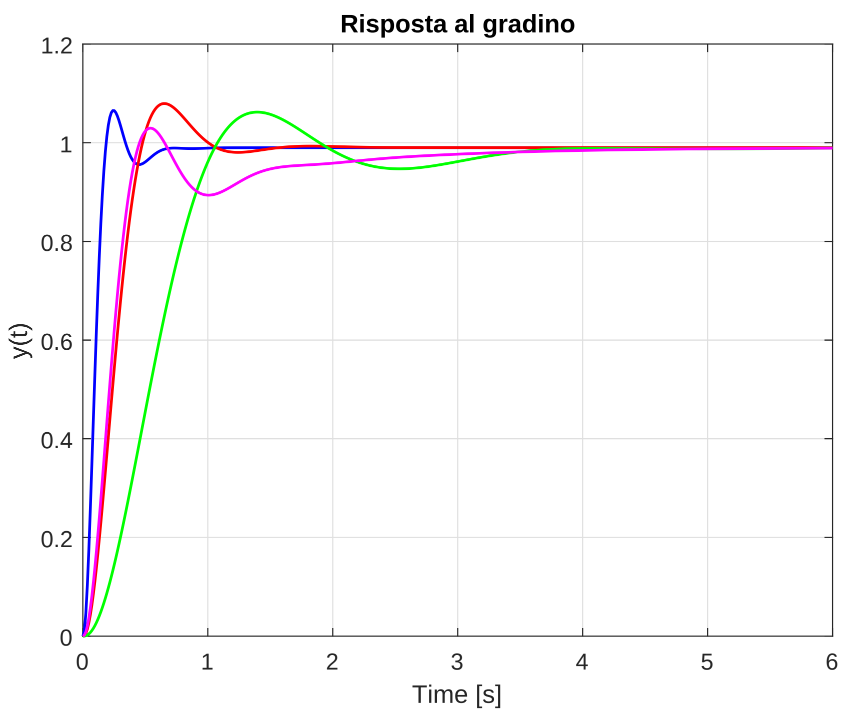
$$C_1(s) = \frac{s^2+14.08s+49.57}{s^2+455.1s+49.57}, \qquad C_2(s) = \frac{s^2+5.274s+6.954}{s^2+145.2s+6.954},$$

$$C_3(s) = \frac{s^2+2.471s+1.527}{s^2+83.72s+1.527}, \qquad C_4(s) = \frac{s^2+6.415s+10.29}{s^2+335.4s+10.29}.$$

- Nyquist diagrams of functions $K_i\, C_i(s)\, G_i(s)$ including the lag network:



- Step response of the controlled systems $K_i\, C(s)\, G_i(s)$:



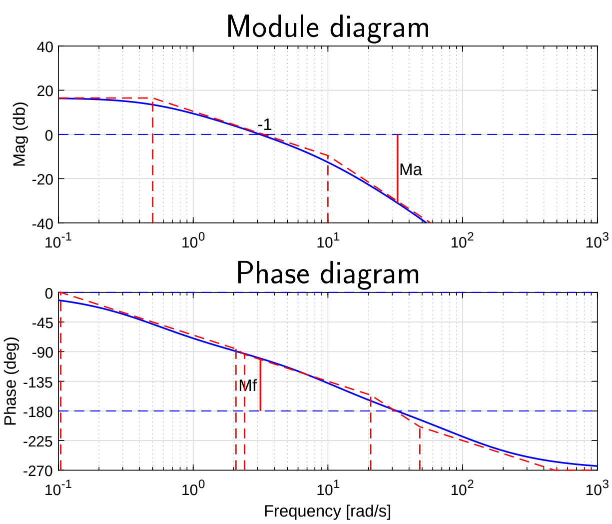
Estimation of the phase margin M_φ on Bode Plots

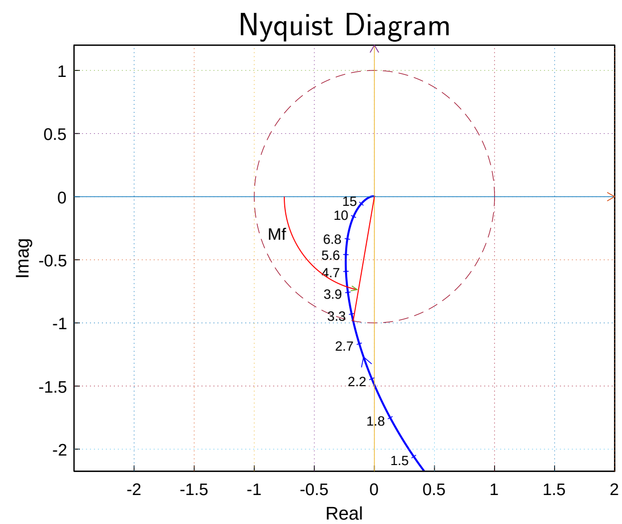
• For minimum phase systems this qualitative rule applies: *the slope "p" with which the module Bode Diagram intersects the unit gain line provides an "estimate" of the system phase margin M_φ:*

$$M_\varphi = (2+p)\frac{\pi}{2}$$

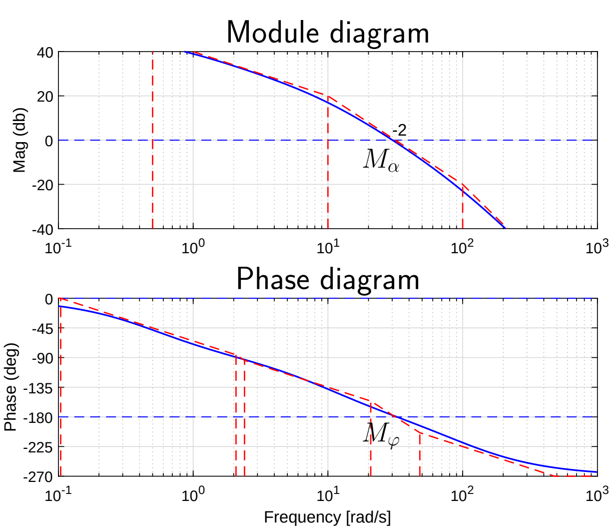
This "qualitative" rule is based on the use of Bode's formula.

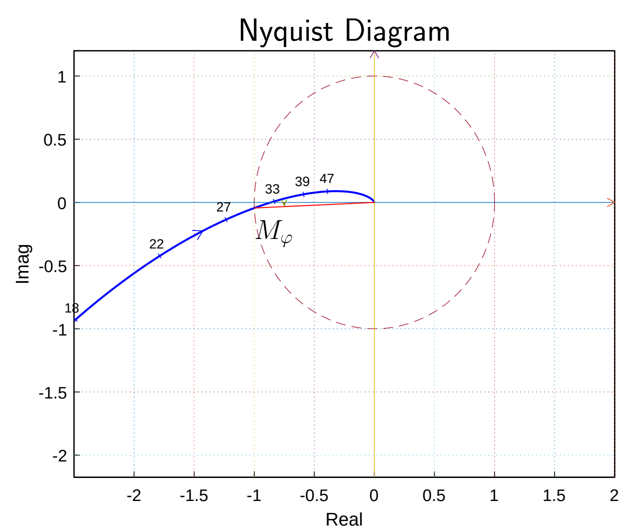
• In order for a feedback system $G(s)$ to be stable with a large margin, it is important that the module Bode Diagram of the function $G(s)$ intersects the unit gain line with slope $p \simeq -1$.



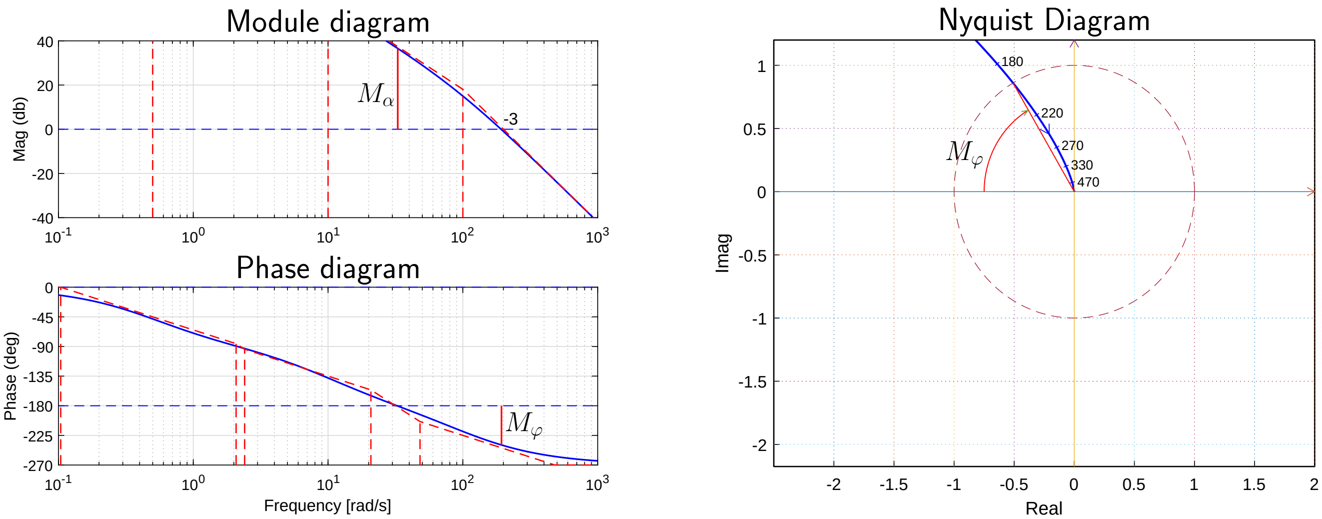


• If the Bode plot of a function $G(s)$ intersects the unit gain line with slope $p \simeq -2$, the corresponding feedback system will be unstable or very close to instability.





• If the Bode plot of a function $G(s)$ intersects the unit gain line with slope $p < -2$, the corresponding feedback system will certainly be unstable.



• The previous considerations provide a simple qualitative rule that can be used on Bode Diagrams to guarantee the stability of a minimum phase system $G(s)$ controlled using a feedback scheme: *the Bode plot of the magnitudes of a function $G(s)$ at minimum phase must intersect the unit gain line with slope between $p \simeq -1$ and $p \simeq -1.5$.*

• Typically this condition can be obtained using:

1) a simple first or second order lead/lag network:

$$C(s) = \frac{1 + \tau_1 s}{1 + \tau_2 s}, \qquad C(s) = \frac{s^2 + 2\gamma\delta_p\omega_n s + \omega_n^2}{s^2 + 2\delta_p\omega_n s + \omega_n^2},$$

2) a PID regulator:

$$C(s) = K_p\left(1 + T_d s + \frac{1}{T_i\, s}\right)$$

Dynamic systems: State Space approach

• The *continuous* and *discrete-time* dynamic systems characterized by m inputs, n states and p outputs can be mathematically represented by using the following vectors:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \qquad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

where $\mathbf{x}(t) \in X = \mathcal{R}^n$ is the state vector, $\mathbf{u}(t) \in U = \mathcal{R}^m$ is the input vector, $\mathbf{y}(t) \in Y = \mathcal{R}^p$ output vector and $t \in \mathcal{T} = \mathcal{R}$ is the time variable.

• The State Space differential equations of a **linear, time-invariant, continuous-time system** are the following:

$$\boxed{\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}} \tag{1}$$

• The continuous-time behaviour of the state space vector $\mathbf{x}(t)$ is the following:

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau}$$

where the symbol $e^{\mathbf{A}t}$ denote the *matrix exponential function*:

$$\mathbf{e}^{\mathbf{A}t} \triangleq \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!}$$

• The Transfer Matrix $\mathbf{H}(s)$ of a continuous-time system is the following:

$$\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s) \qquad \rightarrow \qquad \boxed{\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

• The static gain $\mathbf{H}_0$ is obtained from $\mathbf{H}(s)$ with $s = 0$:

$$\mathbf{H}_0 = \mathbf{H}(s)|_{s=0} = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D} \qquad \rightarrow \qquad \mathbf{y}_0 = \mathbf{H}_0\,\mathbf{u}_0.$$

Example: Let us consider the following second order linear system:

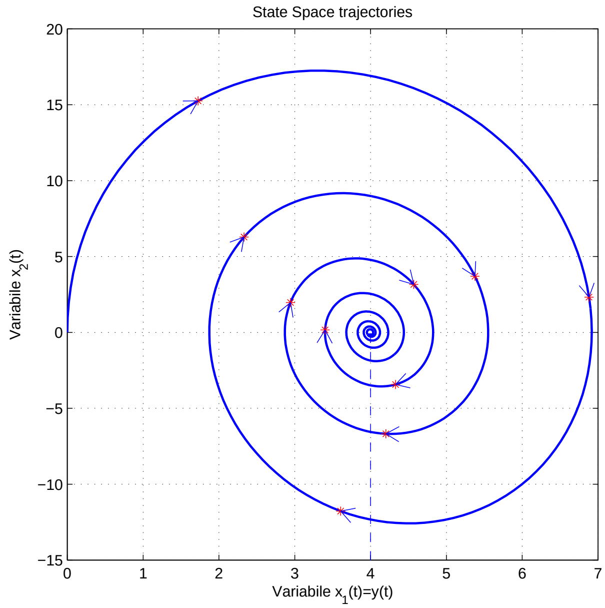
$$G(s) = \frac{100}{s^2+s+25} \quad \Leftrightarrow \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,u \\ y = \mathbf{c}\,\mathbf{x} + d\,u \end{cases}$$

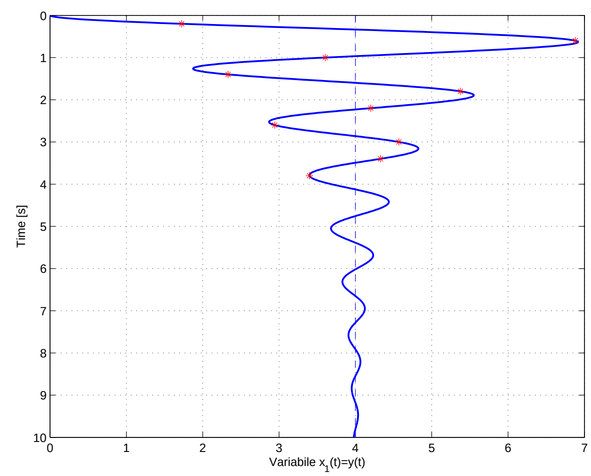
where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -25 & -1 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad d = 0.$$

The state space trajectory of the step response of system $G(s)$ shows that the final state is $\mathbf{x}_f = [y_\infty,\ 0]$, where $y_\infty = 4$.
The real trajectory has a spiral stable behavior in the vicinity of the final state $\mathbf{x}_f$ and the output variable is the projection of this trajectory along the x_1-axis.





Matlab file "Second_Order_Step_Trajectory.m":

```
----------------------------------------------------------------------
function Second_Order_Step_Trajectory
A=[0 1; -25 -1]; b=[0; 100]; c=[1 0]; d=0;
Gs=ss(A,b,c,d);
[yt,t,x]=step(Gs,0:0.01:10);
figure(1); clf
plot(x(:,1),x(:,2),'Linewidth',1.5);     % Plot
hold on; axis square; V=axis; grid on;
dx=(V(2)-V(1))/35; dy=(V(4)-V(3))/35;    % Arrow  width
plot([4 4],[-15 0],'--')
for fr=20:40:400
    plot(x(fr+1,1),x(fr+1,2),'r*')      % Plot the arrow points
    freccia(x(fr,1),x(fr,2),x(fr+1,1),x(fr+1,2),dx,dy)  % Plot the arrows
end
xlabel('Variabile x_1(t)=y(t)')          % Label along axis x
ylabel('Variabile x_2(t)')               % Label along axis y
title('State Space trajectories')        % Title
figure(2); clf;
plot(yt,t,'Linewidth',1.5); hold on;
plot([4 4],[0 10],'--')
set(get(2,'Children'),'Ydir','reverse'); grid on;
for fr=20:40:400
    plot(yt(fr+1),t(fr+1),'r*')          % Draw the arrow points
end
ylabel('Time [s]'); xlabel('Variabile x_1(t)=y(t)')
----------------------------------------------------------------------
```

• The State Space difference equations of a **linear, time-invariant, discrete-time system** are the following:

$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\,\mathbf{x}(k) + \mathbf{D}\,\mathbf{u}(k) \end{cases} \tag{2}$$

• The discrete-time behaviour of the state space vector $\mathbf{x}(k)$ is the following:

$$\boxed{\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)}\mathbf{B}\mathbf{u}(j)}$$

The forced evolution can also be rewritten in the following matrix form:

$$\sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)}\mathbf{B}\mathbf{u}(j) = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \mathbf{u}(k-3) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

• The Transfer Matrix $\mathbf{H}(z)$ of the discrete-time system can be obtained as follows:

$$\mathbf{y}(z) = \mathbf{H}(z)\mathbf{u}(z) \qquad \rightarrow \qquad \boxed{\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

• The static gain $\mathbf{H}_0$ of matrix $\mathbf{H}(z)$ is obtained from $\mathbf{H}(z)$ with $z = 1$:

$$\mathbf{H}_0 = \mathbf{H}(z)|_{z=1} = \mathbf{C}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \qquad \rightarrow \qquad \mathbf{y}_0 = \mathbf{H}_0\,\mathbf{u}_0.$$

• The eigenvalues of matrix $\mathbf{A}$ of the continuous and discrete systems (1) and (2) are equal to the poles of the continuous and discrete transfer matrix $\mathbf{H}(s)$ and $\mathbf{H}(z)$, respectively.

State space transformations

- A state space transformation is a biunivocal linear transformation which links the old state vector $\mathbf{x}$ with the new vector $\overline{\mathbf{x}}$:

$$\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$$

  where $\mathbf{T}$ is a square nonsingular matrix.

- Applying this "similitude" transformation to the continuous-time system (1), or to the discrete-time system (2) one obtains "*a different but equivalent*" mathematical descriptions of the considered systems:

$$\begin{cases} \dot{\overline{\mathbf{x}}}(t) = \overline{\mathbf{A}}\,\overline{\mathbf{x}}(t) + \overline{\mathbf{B}}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \overline{\mathbf{C}}\,\overline{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}, \qquad \begin{cases} \overline{\mathbf{x}}(k+1) = \overline{\mathbf{A}}\,\overline{\mathbf{x}}(k) + \overline{\mathbf{B}}\,\mathbf{u}(k) \\ \mathbf{y}(k) = \overline{\mathbf{C}}\,\overline{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k) \end{cases}$$

- The matrices of the considered systems are linked as follows:

$$\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \qquad \overline{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \qquad \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}$$

- Properly choosing matrix $\mathbf{T}$ it is possible to obtain mathematical descriptions of the given system (the canonical forms) characterized by matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{D}}$ which have particularly simple structures.

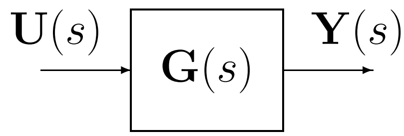
- All these different mathematical models maintain the basic physical properties of the given dynamic system: stability, controllability and observability.

- The "similitude" transformation $\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ satisfies the following property: the matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$ have the same eigenvalues. This means that the original and the transformed systems have the same poles.

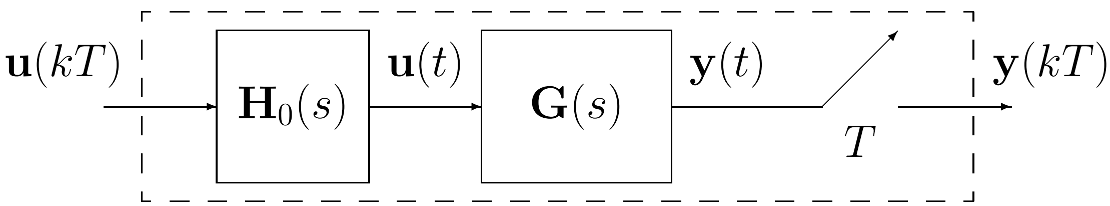
Sampled systems

Let us consider the following continuous-time linear system:

$$\mathbf{G}(s): \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \end{cases}$$

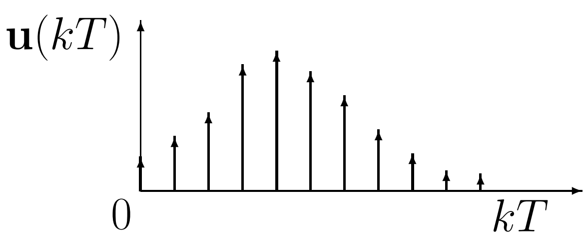


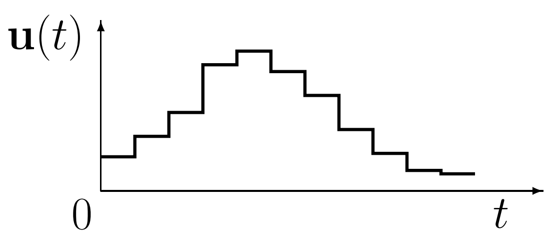
Inserting a zero-order hold $\mathbf{H}_0(s)$ and a sampler (of period T) at the input and at the output of system $\mathbf{G}(s)$, one obtains the following discrete system:



The output $\mathbf{u}(t)$ of the zero order hold $\mathbf{H}_0(s)$ is a piece wise signal:

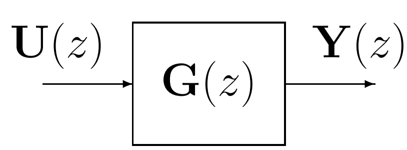
$$\mathbf{u}(t) = \mathbf{u}(kT) \qquad \text{for} \qquad kT \le t \le (k+1)T$$





The signal $\mathbf{y}(t)$ sampled with period T generates the discrete signal $\mathbf{y}(kT)$. The input-output dynamic behavior of the new sampled system can be described using the following discrete time state space model:

$$\mathbf{G}(z): \begin{cases} \mathbf{x}((k+1)T) = \mathbf{F}\,\mathbf{x}(kT) + \mathbf{G}\,\mathbf{u}(kT) \\ \mathbf{y}(kT) = \mathbf{H}\,\mathbf{x}(kT) \end{cases}$$



Matrices $(\mathbf{A},\ \mathbf{B},\ \mathbf{C})$ and matrices $(\mathbf{F},\ \mathbf{G},\ \mathbf{H})$ are linked as follows:

$$\boxed{\mathbf{F} = e^{\mathbf{A}\,T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\,d\sigma, \qquad \mathbf{H} = \mathbf{C}}$$

In Matlab the command which converts a continuous-time system SYS to the corresponding discrete-time system SYSD is the following:

```
SYSD = c2d(SYS,Tsamp,'zoh')
```

where Tsamp is the considered sampling period. A continuous-time system SYS can be discretized using the following methods:

```
'zoh'       Zero-order hold on the inputs.
'foh'       Linear interpolation of inputs (triangle appx.)
'tustin'    Bilinear (Tustin) approximation.
'prewarp'   Tustin approximation with frequency prewarping.
            The critical frequency Wc is specified as fourth
            input by C2D(SYSC,TS,'prewarp',Wc).
'matched'   Matched pole-zero method (for SISO systems only).
```

- Example in Matlab.

```
----------------------------------------------------------------------
clear all; close all
s=tf('s');
Stampa=1; MainString='Gs_Gz';
gs=1000/(s+10)/(s+5)^2/(s^2+2*s+64);    % Linear system
[A,B,C,D]=ssdata(gs);                   % Corresponding matrices
Gs=ss(A,B,C,D);                         % System G(s)
figure(1); clf     %%%%%%%%%%%%%%%%
Tfin=3;                                 % Simulation time range
[y,t]=step(Gs,Tfin);                    % Step response of system G(s)
plot(t,y)                               % Plot of the step response
hold on; grid on
TC_Range=0.05:0.05:0.2;                 % Sampling periods Tc
Color='brgmck';
for ii=1:length(TC_Range)
    Tc=TC_Range(ii);
    Gz=c2d(Gs,Tc);                      % Discrete G(z) system
    [y,t]=step(Gz,Tfin);                % Step response of system G(z)
    stairs(t,y,['.--' Color(ii)])       % Stairs plot of the step response
end
xlim([0 Tfin])
title('Step response of systems G(s) and G(z)')
ylabel('y(k)')
xlabel('Time [s]')
if Stampa;    eval(['print -depsc ' MainString '_' num2str(gcf)]); end
figure(2); clf     %%%%%%%%%%%%%%%%
Tc=0.05;                                % Considered sampling period
ti=0:Tc:Tfin;                           % Sampling instants
ui=sin(10*ti);                          % input signal u(k)
subplot(211)  %%%%%%%%%                 % First subplot
stairs(ti,ui,'b')                       % Plot of the input signal u(k)
grid on
title('Input signal u(k)')
ylabel('u(k)')
subplot(212)  %%%%%%%%%                 % Second subplot
[y,t]=lsim(Gs,ui,ti,'zoh');             % Response of system G(s) to signal u(k)
```

```
plot(t,y)                              % Plot of the step response
hold on; grid on
Gz=c2d(Gs,Tc);                         % Discrete system G(z)
[y,t]=lsim(Gz,ui,ti,'zoh');            % Response of system G(z) to signal u(k)
stairs(t,y,'.--m')                     % Stairs plot of the system response
xlim([0 Tfin])
title('Response of systems G(s) and G(z) to signal u(k)')
ylabel('y(k)')
xlabel('Time [s]')
if Stampa;    eval(['print -depsc ' MainString '_' num2str(gcf)]); end
-----------------------------------------------------------------------
```

- Considered system G(s):

$$G(s) = \frac{1000}{(s+10)(s+5)^2(s^2+2s+64)}$$

- State space matrices of system $G(s)$ ("`[A, B, C, D]=ssdata(Gs)`"):

```
-----------------------------------------------------------------------
 A =                                                       B =

 [ -22    -7.1563    -6.9531    -4.1504    -1.9531]          [ 1/4 ]
 [  32          0          0          0          0]          [   0 ]
 [   0     8.0000          0          0          0]          [   0 ]
 [   0          0     8.0000          0          0]          [   0 ]
 [   0          0          0     4.0000          0]          [   0 ]

 C = [ 0, 0, 0, 0, 125/256 ]                               D = 0
-----------------------------------------------------------------------
```

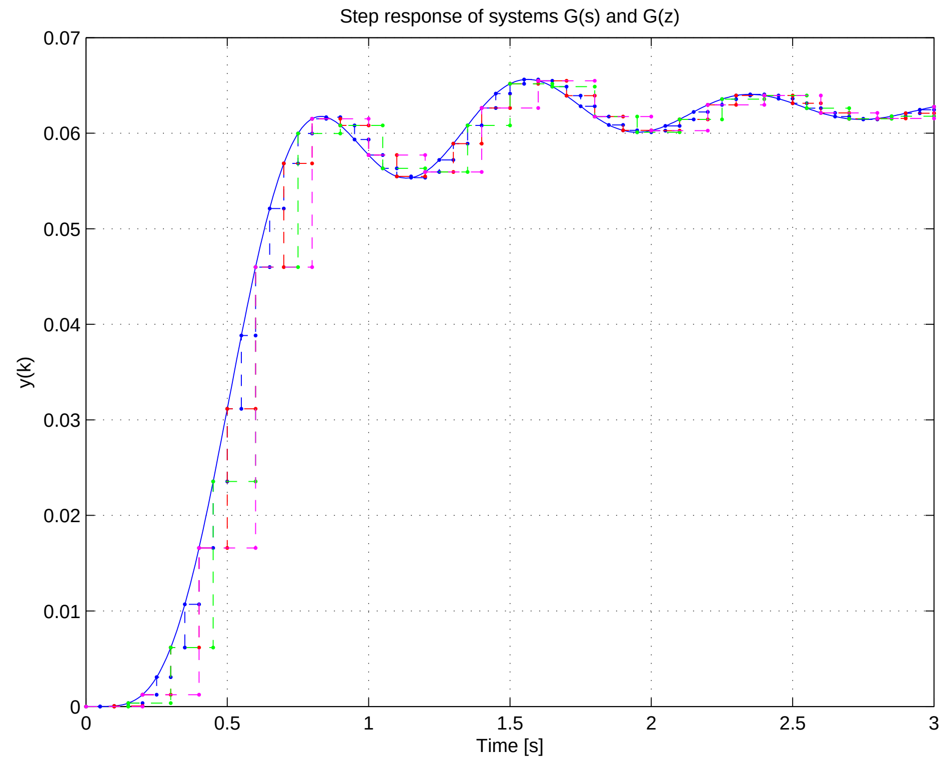
- Sampled system $G(z)$ when $T_c = 0.05$:

```
-----------------------------------------------------------------------
  Gz =

    2.164e-06 z^4 + 4.667e-05 z^3 + 9.853e-05 z^2 + 3.237e-05 z + 1.04e-06
    ----------------------------------------------------------------------
          z^5 - 3.919 z^4 + 6.253 z^3 - 5.048 z^2 + 2.049 z - 0.3329
-----------------------------------------------------------------------
```

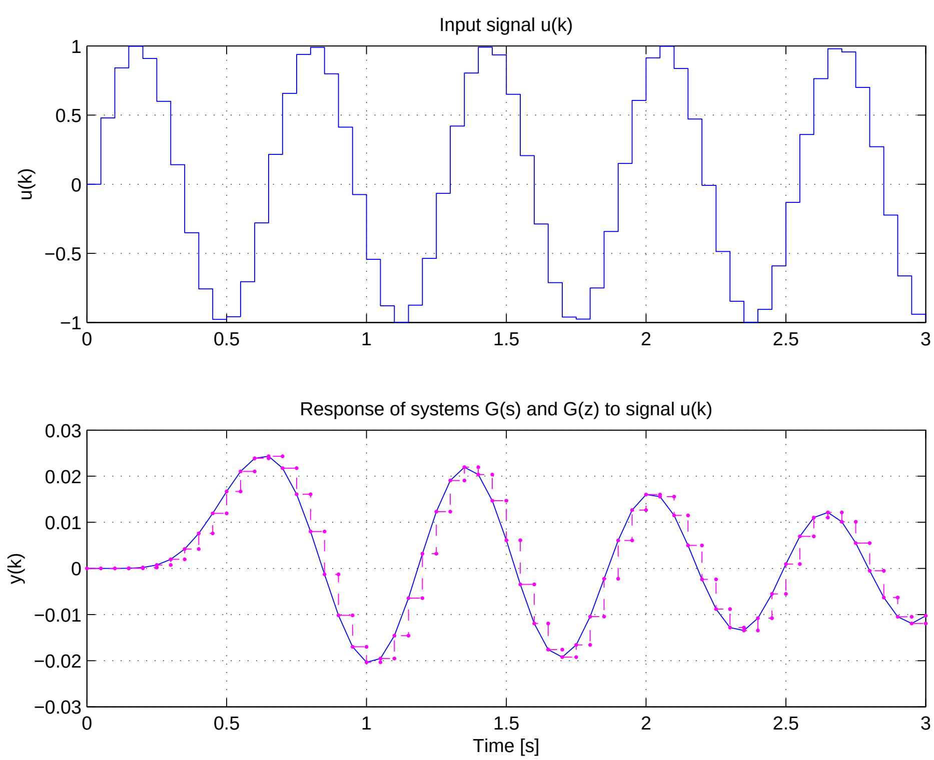
- State space matrices of system $G(z)$ ("`[F, G, C, D]=ssdata(Gz)`"):

```
-----------------------------------------------------------------------
 F =                                                               G =

 [    0.1754     -0.2468     -0.2188     -0.1198     -0.05324]     [  0.006814 ]
 [    0.8722       0.775     -0.2068     -0.1171     -0.05328]     [  0.006819 ]
 [    0.2182      0.3681      0.9702    -0.01712     -0.00786]     [  0.001006 ]
 [    0.0322     0.07669      0.3969      0.9982   -0.0008333]     [ 0.0001067 ]
 [  0.001707    0.005198     0.03987      0.1999            1]     [ 4.432e-06 ]

 C  = [ 0        0        0        0   0.488 ]                     D = 0
-----------------------------------------------------------------------
```

- At the sampling instants $t_k = kT_c$ the outputs of systems $G(s)$ and $G(z)$ are exactly the same if the input signal $u(k)$ is the same for the two systems.
- Step response of systems $G(s)$ and $G(z)$ for different values of the sampling period $T_c = [0.05,\ 0.1,\ 0.15,\ 0.2]$:



- Input signal $u(k) = \sin(10\,k\,T_c)$ and output signal $y(k)$ of the two systems $G(s)$ and $G(z)$ when $T_c = 0.05$:



Reachability

Let us consider the following continuous and discrete time-invariant systems:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k).$$

- *A system is reachable* if the subspace $\mathcal{X}^+$ of all the reachable states from the origin is equal to the whole state space $\mathbf{X}$:

$$\boxed{\mathcal{X}^+ = \mathbf{X}}$$

- *Necessary and sufficient* condition for a system to be reachable is:

$$\boxed{\mathrm{rank}(\mathcal{R}^+) = n}$$

where $\mathcal{R}^+$ is the **Reachability matrix** of the system:

$$\boxed{\mathcal{R}^+ \stackrel{\triangle}{=} \mathcal{R}^+(k)\big|_{k=n} = \mathcal{R}^+(n) = [\mathbf{B}\ \mathbf{AB}\ \ldots\ \mathbf{A}^{n-1}\mathbf{B}]}$$

Example. Compute the reachability matrix $\mathcal{R}^+$ of the following system:

$$\left\{ \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u(t) \right.$$

Reachability matrix $\mathcal{R}^+$ and computation of the subspace $\mathcal{X}^+$:

$$\mathcal{R}^+ = \left[\ \mathbf{b}\ \ \mathbf{Ab}\ \ \mathbf{A}^2\mathbf{b}\ \right] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \qquad \mathcal{X}^+ = \mathrm{Im}\mathcal{R}^+ = \mathrm{Im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system is reachable.

Example. Compute the reachability matrix $\mathcal{R}^+$ of the following system:

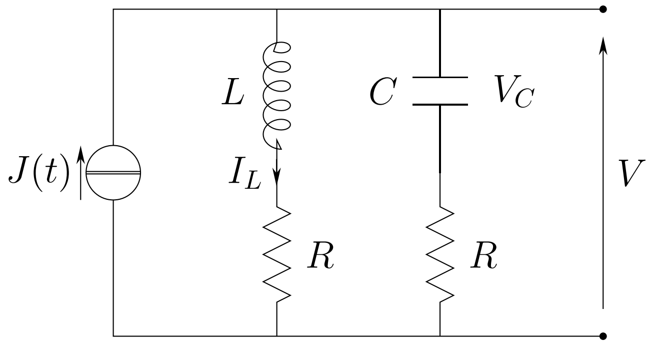
$$\left\{ \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(t) \right.$$

Reachability matrix $\mathcal{R}^+$ and computation of the subspace $\mathcal{X}^+$:

$$\mathcal{R}^+ = \left[\ \mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B}\ \right] = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right], \qquad \mathcal{X}^+ = \mathrm{Im}\left[\mathcal{R}^+\right] = \mathrm{Im} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The system is NOT completely reachable

Example. Let us consider the following electrical network:



The dynamic equations of the systems are:

$$\begin{cases} L\dfrac{d\,I_L}{dt} = V_C + R(J - I_L) - R\,I_L \\ C\dfrac{d\,V_C}{dt} = J - I_L \\ V = V_C + R(J - I_L) \end{cases}$$

where I_L is the current which flows in the inductance, V_C is the voltage across the capacitor, J is the input current and V is the output voltage. In matrix form, the system dynamics can be represented as follows:

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} \frac{-2R}{L} & \frac{1}{L} \\ \frac{-1}{C} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{R}{L} \\ \frac{1}{C} \end{bmatrix} J \\ V = \begin{bmatrix} -R & 1 \end{bmatrix}\mathbf{x} + [\,R\,]J \end{cases} \qquad \mathbf{x} = \begin{bmatrix} I_L \\ V_C \end{bmatrix}$$

The reachability matrix of the system is

$$\mathcal{R}^+ = \begin{bmatrix} \frac{R}{L} & \frac{1}{LC} - \frac{2R^2}{L^2} \\ \frac{1}{C} & -\frac{R}{LC} \end{bmatrix}, \qquad \det \mathcal{R}^+ = \frac{1}{LC}\left[\frac{R^2}{L} - \frac{1}{C}\right]$$

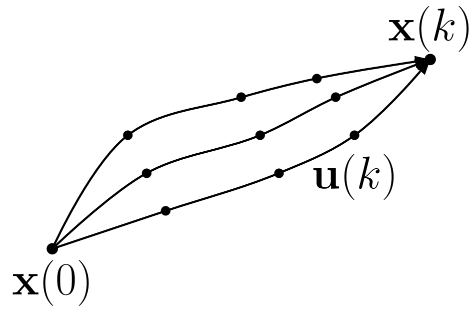
The system is reachable only if $\mathcal{R}^+$ is a full rank matrix. The system is NOT completely reachable if:

$$R^2 = \frac{L}{C} \qquad \leftrightarrow \qquad RC = \frac{L}{R}$$

that is if the inductance time constant $\frac{L}{R}$ is equal to the capacitor time constant RC. In this case the two system eigenvalues are coincident: $\lambda_{1,2} = -\frac{1}{\sqrt{LC}}$.

Discrete systems: point to point control

Control problem: Given the states $\mathbf{x}(0)$ and $\mathbf{x}(k)$, compute the input sequence $\mathbf{u}(0), \ldots, \mathbf{u}(k-1)$ which stirs the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$.



This control problem can be solved by solving the following equation:

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)} \mathbf{B} \mathbf{u}(j)$$

with respect to the unknown sequence $\mathbf{u}(j)$ for $j \in [0,\ k-1]$.

Property. The control problem has a solution if and only if:

$$\boxed{\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0) \in \mathcal{X}^+(k)}$$

that is if the vector $\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0)$ is reachable from the zero state in k steps.

If the problem has a solution, the solution can be determined solving the following non homogeneous linear system with respect to the unknown sequence $\mathbf{u}(j)$ for $j \in [0,\ k-1]$:

$$\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0) = \underbrace{[\mathbf{B}\ \mathbf{AB}\ \ldots\ \mathbf{A}^{k-1}\mathbf{B}]}_{\mathcal{R}^+(k)} \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix} = \mathcal{R}^+(k)\mathbf{u}$$

where $\mathbf{u}$ denotes the following unknown vector:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix} = \begin{bmatrix} u_1(k-1) \\ \vdots \\ u_m(k-1) \\ \vdots \\ u_1(0) \\ \vdots \\ u_m(0) \end{bmatrix}$$

The solution $\mathbf{u}$ of the given problem is NOT unique. All the possible solutions $\mathbf{u}$ can be obtained adding the kernel $\overline{\mathbf{v}}$ of matrix $\mathcal{R}^+(k)$ to a particular solution $\overline{\mathbf{u}}$:

$$\mathbf{u} = \bar{\mathbf{u}} + \bar{\mathbf{v}}, \qquad \bar{\mathbf{v}}: \quad \mathcal{R}^+(k)\bar{\mathbf{v}} = 0$$

For brevity, in the following we will indicate $\mathcal{R}_k^+ = \mathcal{R}^+(k)$.

Property. It can be proved that among all the possible solutions $\mathbf{u}$, the one that minimizes the Euclidean norm

$$||\mathbf{u}|| = \sqrt{\sum_{i=0}^{k-1} \mathbf{u}^T(i)\,\mathbf{u}(i)}$$

is the following:

$$\boxed{\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]}$$

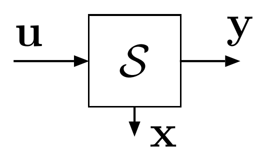
This solution *requires a perfect knowledge of the system parameters to be controlled* (matrices $\mathbf{A}$ and $\mathbf{B}$) and no disturbances acting on the system.

Note. If $\mathcal{R}_k^+$ is a full rank matrix, then $(\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}$ is the speudo-inverse matrix of the rectangular matrix $\mathcal{R}_k^+$.

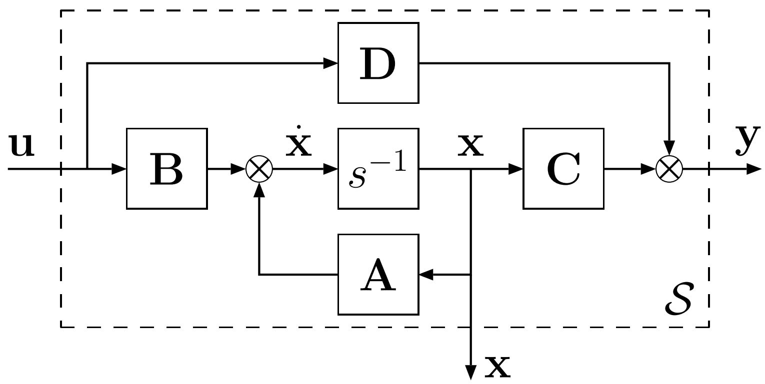
Static state feedback

- Let us consider an invariant linear system $\mathcal{S} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ (discrete o continuous-time) and *let us suppose that all the components of the state vector* $\mathbf{x}$ *are* *<u>known</u>*.

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} & = & \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

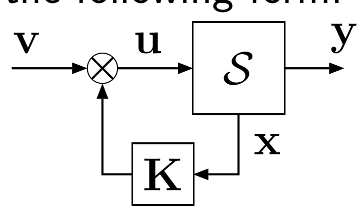


- The system (in this case a continuous-time system) can be graphically represented in the following way:



- The <u>static state feedback</u> control law has the following form:

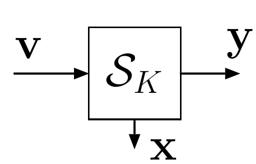
$$\mathbf{u}(t) = \mathbf{K}\,\mathbf{x}(t) + \mathbf{v}(t)$$



where $\mathbf{K} \in \mathcal{R}^{m \times n}$ is the design a matrix and $\mathbf{v}(t)$ is un additional input.

- Applying the control law $\mathbf{u} = \mathbf{K}\,\mathbf{x} + \mathbf{v}$ to the previous continuous-time system (the same holds for a discrete-time system) one obtains:

$$\begin{cases} \dot{\mathbf{x}} & = & (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{v} \\ \mathbf{y} & = & (\mathbf{C} + \mathbf{D}\mathbf{K})\mathbf{x} + \mathbf{D}\mathbf{v} \end{cases}$$



The new dynamic system $\mathcal{S}_K$ characterized by the following matrices:

$$\mathbf{A} + \mathbf{B}\,\mathbf{K}, \qquad \mathbf{B}, \qquad \mathbf{C} + \mathbf{D}\,\mathbf{K}, \qquad \mathbf{D}.$$

Pole placement

- *Property. Let $\mathcal{S} = (\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d})$ be a time-invariant linear system of dimension n, completely reachable and with only one input ($m = 1$). For each monic polynomial $p(\lambda)$ of degree n:*

$$p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0.$$

it exists a vector $\mathbf{k}^T \in \mathcal{R}^{1\times n}$ such that the characteristic polynomial of the matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^T$ of the feedback system $\mathcal{S}_\mathbf{k}$ is equal to $p(\lambda)$.
Vector $\mathbf{k}^T$ can be computed as follows:

$$\mathbf{k}^T = \mathbf{k}_c^T \left\{ \begin{bmatrix} \mathbf{b}, & \mathbf{A}\mathbf{b}, & \ldots, & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & 1 \\ \alpha_2 & \ldots & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \right\}^{-1}$$

where $\mathbf{k}_c^T$ is the following:

$$\mathbf{k}_c^T = \begin{bmatrix} \alpha_0 - d_0, & \alpha_1 - d_1, & \ldots, & \alpha_{n-1} - d_{n-1} \end{bmatrix}$$

and α_i $(i = 0, \ldots, n-1)$ are the coefficients of the characteristic polynomial $\Delta_\mathbf{A}(\lambda)$ of matrix $\mathbf{A}$:

$$\Delta_\mathbf{A}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$$

Ackerman formula

- *Let us consider a continuous or discrete-time linear system with only one input ($m = 1$):*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t) \qquad \text{or} \qquad \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}\,u(k)$$

and let $p(\lambda)$ be a polynomial freely chosen:

$$p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$$

If the couple $(\mathbf{A}, \mathbf{b})$ *is reachable, then the gain vector* $\mathbf{k}^T$ *such that* $\Delta_{\mathbf{A}+\mathbf{b}\mathbf{k}^T}(\lambda) = p(\lambda)$ *can be computed using the following* Ackerman formula*:*

$$\boxed{\mathbf{k}^T = -\mathbf{q}^T p(\mathbf{A})}$$

where $\mathbf{q}^T$ *is the last row of the inverse of the reachabilily matrix* $\mathcal{R}^+$ *:*

$$\mathbf{q}^T = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} (\mathcal{R}^+)^{-1}$$

and $p(\mathbf{A})$ *denotes the matrix obtained from polynomial* $p(\lambda)$ *when parameter* λ *is substituted by matrix* $\mathbf{A}$*.*

- The advantage of using the Ackerman formula is that it does not require the knowledge of the characteristic polynomial $\Delta_{\mathbf{A}}(\lambda)$ of matrix $\mathbf{A}$.

Example. Let us refer to the previous example where it was $p(\lambda) = \lambda^3$. The gain vector $\mathbf{k}^T$ can also be computed as follows:

$$\begin{aligned} \mathbf{k}^T &= -\mathbf{q}^T p(\mathbf{A}) = -\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (\mathcal{R}^+)^{-1} \mathbf{A}^3 \\ &= -\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^3 \\ &= -\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ -2 & 5 & -3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix} \end{aligned}$$

Example. Given the following system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\,u(t)$:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

compute the gain vector $\mathbf{k}^T$ of the static feedback control law $u(t) = \mathbf{k}^T\mathbf{x}$ which places in -1, -2 and -2 the three eigenvalues of the feedback system.

The system is completely reachable:

$$\mathcal{R}^+ = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{q}^T = -\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$$

The desired polynomial is:

$$p(s) = (s+2)^2(s+1) = s^3 + 5s^2 + 8s + 4$$

Being:

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^3 = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

one obtains that:

$$\begin{aligned} p(\mathbf{A}) &= (\mathbf{A}+2\mathbf{I})^2(\mathbf{A}+\mathbf{I}) = \mathbf{A}^3 + 5\mathbf{A}^2 + 8\mathbf{A} + 4\mathbf{I} \\ &= \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}^2 \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 30 & 12 \\ 0 & 9 & 9 \\ 0 & 9 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 5\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

So, the following gain vector is obtained:

$$\mathbf{k}^T = -\mathbf{q}^T p(\mathbf{A}) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 18 & 30 & 12 \\ 0 & 9 & 9 \\ 0 & 9 & 9 \end{bmatrix} = \begin{bmatrix} -9 & -6 & 3 \end{bmatrix}$$

The same result can also be obtained using the following formula:

$$\mathbf{k}^T = \mathbf{k}_c^T[\mathcal{R}^+(\mathcal{R}_c^+)^{-1}]^{-1}$$

In this case the characteristic polynomial of matrix $\mathbf{A}$ must be computed:

$$\det(s\mathbf{I}-\mathbf{A}) = \begin{vmatrix} s-1 & -2 & 0 \\ 0 & s & -1 \\ 0 & -1 & s \end{vmatrix} = (s+1)(s-1)^2 = s^3 - s^2 - s + 1$$

The desired polynomial is

$$p(s) = (s+2)^2(s+1) = s^3 + 5s^2 + 8s + 4$$

Vector $\mathbf{k}^T$ is obtained as follows

$$\begin{aligned}
\mathbf{k}^T &= \mathbf{k}_c^T \{\mathcal{R}^+ (\mathcal{R}_c^+)^{-1}\}^{-1} \\
&= \begin{bmatrix} -3 & -9 & -6 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}^{-1} \\
&= \begin{bmatrix} -3 & -9 & -6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -3 & -9 & -6 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{2} \\
&= \begin{bmatrix} -9 & -6 & 3 \end{bmatrix}
\end{aligned}$$