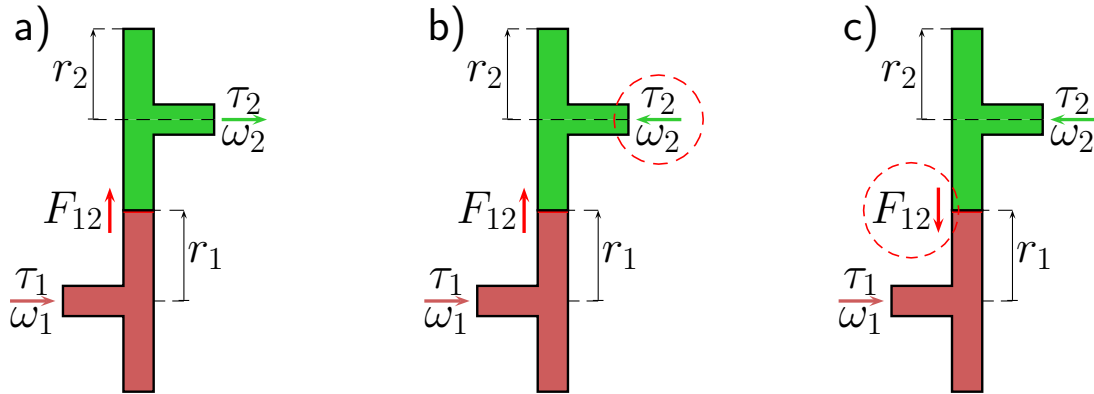


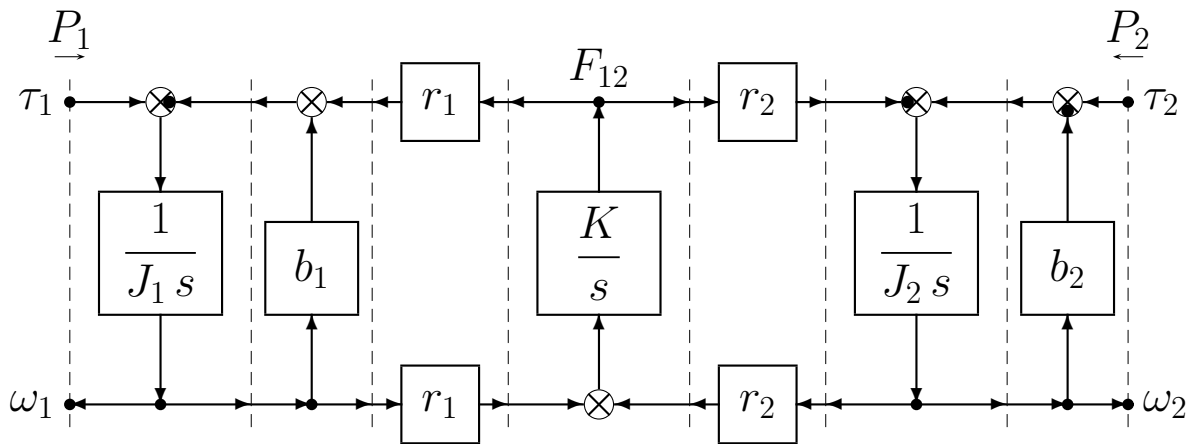
# Planetary Gears: fast and direct modeling

- Let us consider the following gear transmission, case a):



The figure shows the positive directions of the system variables: the angular velocities of the two gears,  $\omega_1$  and  $\omega_2$ , and the tangential force  $F_{12}$  that gear 1 transmits to gear 2.

- The POG block scheme:



- The corresponding POG state space equations:

$$\left\{ \begin{array}{l} \underbrace{\begin{bmatrix} J_1 & 0 & | & 0 \\ 0 & J_2 & | & 0 \\ 0 & 0 & | & K^{-1} \end{bmatrix}}_{\mathbf{L}} \dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -b_1 & 0 & | & -r_1 \\ 0 & -b_2 & | & -r_2 \\ r_1 & r_2 & | & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ F \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}}_{\mathbf{u}} \\ \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \mathbf{u} \end{array} \right. \quad (1)$$

- The signs of the parameters within system (1) are a direct consequence of the choices about the positive direction of the system state variables  $\omega_1$ ,  $\omega_2$  and  $F_{12}$ .
- Powers  $P_1 = \omega_1 \tau_1$  and  $P_2 = \omega_2 \tau_2$  flowing through the input and output power sections are positive if they enter the system.
- One can easily prove that, if the positive direction of variables  $\omega_2$  and  $\tau_2$  is reversed, (see case b), the state space equations remain the same except for matrix  $\mathbf{A}$ , which turns into:

$$\mathbf{A}_1 = \left[ \begin{array}{cc|c} -b_1 & 0 & -r_1 \\ 0 & -b_2 & r_2 \\ \hline r_1 & -r_2 & 0 \end{array} \right].$$

In this case, only the sign of parameter  $r_2$  is changed.

- Starting from case b) and reversing the positive direction of variable  $F_{12}$ , (see case c), the state space equations (1) remain unchanged once again except for matrix  $\mathbf{A}$ , which now turns into:

$$\mathbf{A}_2 = \left[ \begin{array}{cc|c} -b_1 & 0 & -r_1 \\ 0 & -b_2 & r_2 \\ \hline -r_1 & r_2 & 0 \end{array} \right]$$

In this case, the signs of both parameters  $r_1$  and  $r_2$  within matrix  $\mathbf{A}_2$  are changed with respect to matrix  $\mathbf{A}_1$ .

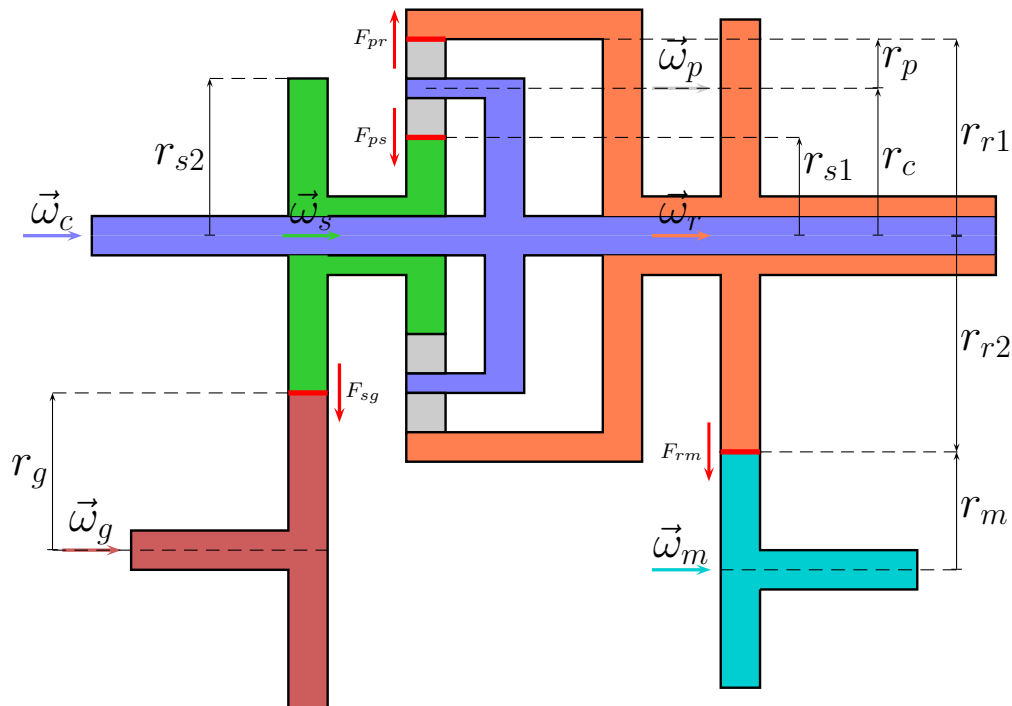
- By rewriting matrix  $\mathbf{A}$  in the following form:

$$\mathbf{A} = \begin{bmatrix} -\mathbf{B}_J & \mathbf{R}^T \\ \mathbf{R} & 0 \end{bmatrix}, \quad \mathbf{B}_J = \begin{bmatrix} -b_1 & 0 \\ 0 & -b_2 \end{bmatrix}, \quad \mathbf{R} = [r_1 \ r_2]$$

and focusing on the previous two cases, one can state that: 1) matrix  $\mathbf{B}_J$  does not change even if the system variables  $\omega_1$ ,  $\omega_2$  and  $F_{12}$  change the signs of their positive directions; 2) if the angular velocity  $\omega_i$  changes the sign of its positive direction, then all the coefficients of the  $i$ -th column of matrix  $\mathbf{R}$  change their signs; 3) if the tangential force  $F_{12}$  changes the sign of its positive direction, then all the coefficients of matrix  $\mathbf{R}$  change their signs.

## Modeling a one-stage planetary gear

- Let us refer to the following planetary gear:



- Each gear is characterized by a specific color and by a one-digit subscript “ $i$ ”, where  $i \in \{c, p, s, r, g, m\}$ .
- Each gear rotates around a horizontal axis, denoted in the figure by a short arrow (i.e. “ $\rightarrow$ ”), having the same color of the considered gear. The direction of the short arrow indicates the positive direction of the associated angular velocity  $\vec{\omega}_i$  of gear “ $i$ ”.
- The one-digit string “ $i$ ” also denotes all the parameters and variables associated with gear “ $i$ ”:  $J_i$  is the moment of inertia,  $b_i$  is the linear friction coefficient and  $\tau_i$  is the input torque applied to the gear.
- The gears interact each other by means of their teeth. The contact points of the gears are denoted by short red lines (i.e. “ $-$ ”) representing the stiffness of the teeth where the gears interact each other.
- The tangential forces  $F_{ij}$  transmitted by the contact points are denoted by short red vertical arrows (i.e. “ $\downarrow$ ”) representing the positive directions of the contact forces themselves.



- Velocity vector  $\omega$ , input torque vector  $\tau$  and force vector  $\mathbf{F}$ :

$$\omega = \begin{bmatrix} \omega_c \\ \omega_p \\ \omega_s \\ \omega_r \\ \omega_g \\ \omega_m \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_c \\ \tau_p \\ \tau_s \\ \tau_r \\ \tau_g \\ \tau_m \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_{ps} \\ F_{pr} \\ F_{sg} \\ F_{rm} \end{bmatrix}. \quad (4)$$

- The inertia matrix  $\mathbf{J}$  and the inertia friction matrix  $\mathbf{B}_J$  are:

$$\mathbf{J} = \begin{bmatrix} J_c & 0 & 0 & 0 & 0 & 0 \\ 0 & J_p & 0 & 0 & 0 & 0 \\ 0 & 0 & J_s & 0 & 0 & 0 \\ 0 & 0 & 0 & J_r & 0 & 0 \\ 0 & 0 & 0 & 0 & J_g & 0 \\ 0 & 0 & 0 & 0 & 0 & J_m \end{bmatrix}, \quad \mathbf{B}_J = \begin{bmatrix} b_c & 0 & 0 & 0 & 0 & 0 \\ 0 & b_p & 0 & 0 & 0 & 0 \\ 0 & 0 & b_s & 0 & 0 & 0 \\ 0 & 0 & 0 & b_r & 0 & 0 \\ 0 & 0 & 0 & 0 & b_g & 0 \\ 0 & 0 & 0 & 0 & 0 & b_m \end{bmatrix}.$$

- The stiffness matrix  $\mathbf{K}$  and the stiffness friction matrix  $\mathbf{B}_K$  are:

$$\mathbf{K} = \begin{bmatrix} K_{ps} & 0 & 0 & 0 \\ 0 & K_{pr} & 0 & 0 \\ 0 & 0 & K_{sg} & 0 \\ 0 & 0 & 0 & K_{rm} \end{bmatrix}, \quad \mathbf{B}_K = \begin{bmatrix} b_{ps} & 0 & 0 & 0 \\ 0 & b_{pr} & 0 & 0 \\ 0 & 0 & b_{sg} & 0 \\ 0 & 0 & 0 & b_{rm} \end{bmatrix}.$$

- The position of the angular velocities  $\omega_i$  within vector  $\omega$  and of the tangential forces  $F_{ij}$  within vector  $\mathbf{F}$ , see (4), completely and uniquely defines the position of parameters  $J_i$ ,  $b_i$ ,  $K_{ij}$  and  $b_{ij}$  within diagonal matrices  $\mathbf{J}$ ,  $\mathbf{B}$ ,  $\mathbf{K}$  and  $\mathbf{B}_K$  respectively.
- The only matrix in the state space equations (2) that fully defines the internal structure of the given planetary gear is the radius matrix  $\mathbf{R}$ .
- For the considered system, the radius matrix  $\mathbf{R}$  is defined as follows:

$$\mathbf{R} = \begin{bmatrix} r_c & -r_p & -r_{s1} & 0 & 0 & 0 \\ r_c & r_p & 0 & -r_{r1} & 0 & 0 \\ 0 & 0 & -r_{s2} & 0 & -r_g & 0 \\ 0 & 0 & 0 & -r_{r2} & 0 & -r_m \end{bmatrix}. \quad (5)$$

- The radii present in the radius matrix  $\mathbf{R}$  are constrained as follows:  $r_c = r_p + r_{s1}$  and  $r_{r1} = 2r_p + r_{s1}$ .

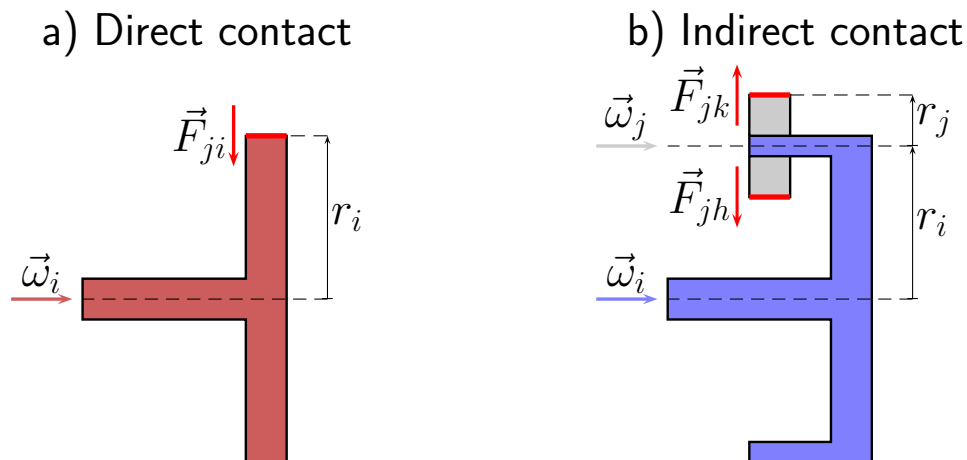
### Direct derivation of the radius matrix $R$ .

- Let  $r_{ij,i}$  denote the generic coefficient of matrix  $\mathbf{R} = [r_{ij,i}]$  in (5). Subscripts  $ij \in \{ps, pr, sg, rm\}$  and  $i \in \{c, p, s, r, g, m\}$  indicate that  $r_{ij,i}$  is the coefficient that links the tangential force  $\vec{F}_{ij}$  to the angular velocity  $\vec{\omega}_i$ .
- Generalizing the rules provided at the end of the previous section one can easily prove that the coefficient  $r_{ij,i}$  can be expressed as follows:

$$r_{ij,i} = S_{F_{ij}} S_{\omega_i} r_i$$

where  $r_i$ ,  $S_{F_{ij}}$  and  $S_{\omega_i}$  have the following meaning:

- 1)  $r_i$  is the **effective radius** of angular velocity  $\vec{\omega}_i$ . Two different cases must be considered (see the figure): a) if velocity  $\vec{\omega}_i$  directly affects the force vector  $\vec{F}_{ij}$ , then the *effective radius*  $r_i$  coincides with the radius of the gear associated with velocity  $\vec{\omega}_i$ ; b) if angular velocity  $\vec{\omega}_i$  affects the force vector  $\vec{F}_{ij}$  through an intermediate gear “ $j$ ”, then the *effective radius*  $r_i$  coincides with the distance between the rotation axes of the two angular velocities  $\vec{\omega}_i$  and  $\vec{\omega}_j$ .



2)  $S_{F_{ij}}$  is the sign of the positive direction of vector  $\vec{F}_{ij}$ :

$$S_{F_{ij}} = \begin{cases} 1 & \text{if } \vec{F}_{ij} \text{ is oriented from gear } i \text{ to gear } j \\ -1 & \text{if } \vec{F}_{ji} \text{ is oriented from gear } j \text{ to gear } i \end{cases}$$

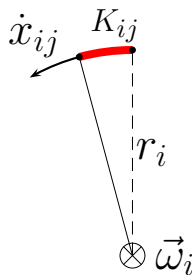
3)  $S_{\omega_i}$  is related to the sign of the velocity vector  $\vec{\omega}_i$ :

$$S_{\omega_i} = \begin{cases} 1 & \text{if } \vec{F}_{ij} \text{ is on the left of vector } \vec{\omega}_i \\ -1 & \text{if } \vec{F}_{ij} \text{ is on the right of vector } \vec{\omega}_i \end{cases}$$

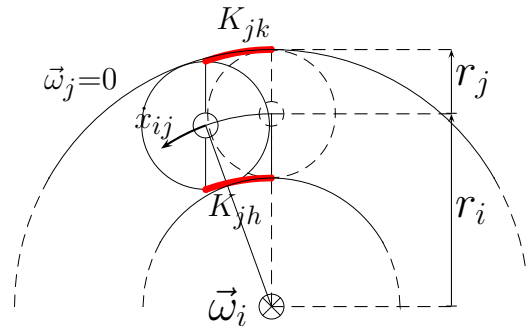
The left and right half-planes of vector  $\vec{\omega}_i$  are determined by moving in the positive direction along vector  $\vec{\omega}_i$ .

- From (2), it follows that  $\dot{x}_{ij} = r_{ij,i} \omega_i$  is the tangential velocity of one of the two terminals of spring  $K_{ij}$  when the angular velocity  $\vec{\omega}_i$  moves along its positive direction. Since the sign of velocity  $\dot{x}_{ij}$  directly affects the sign of the force vector  $\vec{F}_{ij}$ , it is evident that  $\dot{x}_{ij}$  must change sign both when the velocity vector  $\vec{\omega}_i$  and when the force vector  $\vec{F}_{ij}$  change their positive directions.
- The following figure graphically shows why the *effective radius* associated to an angular velocity  $\vec{\omega}_i$  is equal to  $r_i$  for both case a) and b) of direct and indirect contact.

a) Direct contact



b) Indirect contact



- In particular, in the indirect contact case b) the tangential velocity  $\dot{x}_{ij} = r_i \omega_i$  is the same for the two elastic elements  $K_{jh}$  and  $K_{jk}$  because, when  $\vec{\omega}_i$  moves along its positive direction, the angular velocity  $\vec{\omega}_j$  is kept equal to zero.

## Rigid and reduced model when $\mathbf{K} \rightarrow \infty$ .

- When  $\mathbf{K} \rightarrow \infty$ , from state space model (2) one obtains:

$$\mathbf{R}\omega = 0 \Leftrightarrow \begin{cases} r_c \omega_c - r_p \omega_p - r_{s1} \omega_s = 0 \\ r_c \omega_c + r_p \omega_p - r_{r1} \omega_r = 0 \\ -r_g \omega_g - r_{s2} \omega_s = 0 \\ -r_m \omega_m - r_{r2} \omega_r = 0 \end{cases} \quad (6)$$

- System (6) provides four static constraints between the six components  $\omega_i$  of the angular velocity vector  $\omega$ , meaning that vector  $\omega$  can be expressed as a function of “two” of the six composing angular velocities:  $\omega = \mathbf{Q}_1 \mathbf{x}_1$ , where  $\mathbf{x}_1 = [\omega_i \ \omega_j]^T$ .
- By choosing  $\mathbf{x}_1 = [\omega_c \ \omega_r]^T$ , one obtains the following congruent state space transformation:

$$\underbrace{\begin{bmatrix} \omega \\ \mathbf{F} \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} \omega_c \\ \omega_p \\ \omega_s \\ \omega_r \\ \omega_g \\ \omega_m \\ F_{ps} \\ F_{pr} \\ F_{sg} \\ F_{rm} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{r_c}{r_p} & \frac{r_{r1}}{r_p} \\ \frac{2r_c}{r_{s1}} & -\frac{r_{r1}}{r_{s1}} \\ 0 & 1 \\ -\frac{2r_c r_{s2}}{r_g r_{s1}} & \frac{r_{r1} r_{s2}}{r_g r_{s1}} \\ 0 & -\frac{r_{r2}}{r_m} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{T}_1} \underbrace{\begin{bmatrix} \omega_c \\ \omega_r \end{bmatrix}}_{\mathbf{x}_1} = \underbrace{\begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{0} \end{bmatrix}}_{\mathbf{T}_1} \mathbf{x}_1.$$

- By applying transformation  $\mathbf{x} = \mathbf{T}_1 \mathbf{x}_1$  to system (2), one obtains the following reduced system  $\mathbf{L}_1 \dot{\mathbf{x}}_1 = \mathbf{A}_1 \mathbf{x}_1 + \mathbf{B}_1 \mathbf{u}$ :

$$\underbrace{\begin{bmatrix} J_1 & J_3 \\ J_3 & J_2 \end{bmatrix}}_{\mathbf{L}_1} \underbrace{\begin{bmatrix} \dot{\omega}_c \\ \dot{\omega}_r \end{bmatrix}}_{\dot{\mathbf{x}}_1} = \underbrace{\begin{bmatrix} b_1 & b_3 \\ b_3 & b_2 \end{bmatrix}}_{\mathbf{A}_1} \underbrace{\begin{bmatrix} \omega_c \\ \omega_r \end{bmatrix}}_{\mathbf{x}_1} + \underbrace{\mathbf{Q}_1^T}_{\mathbf{B}_1} \underbrace{\tau}_{\mathbf{u}} \quad (7)$$

where matrices  $\mathbf{L}_1$ ,  $\mathbf{A}_1$  and  $\mathbf{B}_1$  have the following structure:

$$\begin{cases} \mathbf{L}_1 = \mathbf{T}_1^T \mathbf{L} \mathbf{T}_1 = \mathbf{Q}_1^T \mathbf{J} \mathbf{Q}_1 \\ \mathbf{A}_1 = \mathbf{T}_1^T \mathbf{A} \mathbf{T}_1 = -\underbrace{\mathbf{Q}_1^T \mathbf{R}^T \mathbf{B}_K \mathbf{R} \mathbf{Q}_1}_0 - \mathbf{Q}_1^T \mathbf{B}_J \mathbf{Q}_1 \\ \mathbf{B}_1 = \mathbf{T}_1^T \mathbf{B} = \mathbf{Q}_1^T \end{cases}$$

- Term  $\mathbf{Q}_1^T \mathbf{R}^T \mathbf{B}_K \mathbf{R} \mathbf{Q}_1$  is equal to zero because  $\mathbf{Q}_1 \in \ker(\mathbf{R})$ :

$$\mathbf{R} \omega = 0 \Leftrightarrow \mathbf{R} \mathbf{Q}_1 \mathbf{x}_1 = 0 \Leftrightarrow \mathbf{R} \mathbf{Q}_1 = 0.$$

- The structure of elements  $J_1$ ,  $J_3$  and  $J_2$  in matrix  $\mathbf{L}_1$  is:

$$\begin{aligned} J_1 &= J_c + \frac{J_p r_c^2}{r_p^2} + \frac{4J_s r_c^2}{r_{s1}^2} + \frac{4J_g r_c^2 r_{s2}^2}{r_g^2 r_{s1}^2} \\ J_3 &= -\frac{J_p r_c r_{r1}}{r_p^2} - \frac{2J_s r_c r_{r1}}{r_{s1}^2} - \frac{2J_g r_c r_{r1} r_{s2}^2}{r_g^2 r_{s1}^2} \\ J_2 &= J_r + \frac{J_m r_{r2}^2}{r_m^2} + \frac{J_p r_{r1}^2}{r_p^2} + \frac{J_s r_{r1}^2}{r_{s1}^2} + \frac{J_g r_{r1}^2 r_{s2}^2}{r_g^2 r_{s1}^2} \end{aligned}$$

- The structure of elements  $b_1$ ,  $b_3$  and  $b_2$  in matrix  $\mathbf{A}_1$  is:

$$\begin{aligned} b_1 &= -b_c - \frac{b_p r_c^2}{r_p^2} - \frac{4b_s r_c^2}{r_{s1}^2} - \frac{4b_g r_c^2 r_{s2}^2}{r_g^2 r_{s1}^2} \\ b_3 &= \frac{b_p r_c r_{r1}}{r_p^2} + \frac{2b_s r_c r_{r1}}{r_{s1}^2} + \frac{2b_g r_c r_{r1} r_{s2}^2}{r_g^2 r_{s1}^2} \\ b_2 &= -b_r - \frac{b_m r_{r2}^2}{r_m^2} - \frac{b_p r_{r1}^2}{r_p^2} - \frac{b_s r_{r1}^2}{r_{s1}^2} - \frac{b_g r_{r1}^2 r_{s2}^2}{r_g^2 r_{s1}^2} \end{aligned}$$

- **Property 1.** *The force vector  $\mathbf{F}$  is not present in the reduced system (7), but its time behavior can still be obtained using the following relation:*

$$\mathbf{F} = (\mathbf{R}\mathbf{R}^T)^{-1} \mathbf{R}(\tau - \mathbf{J}\mathbf{Q}_1 \dot{\mathbf{x}}_1 - \mathbf{B}_J \mathbf{Q}_1 \mathbf{x}_1) \quad (8)$$

*Proof.* From (2) and (7), one obtains:

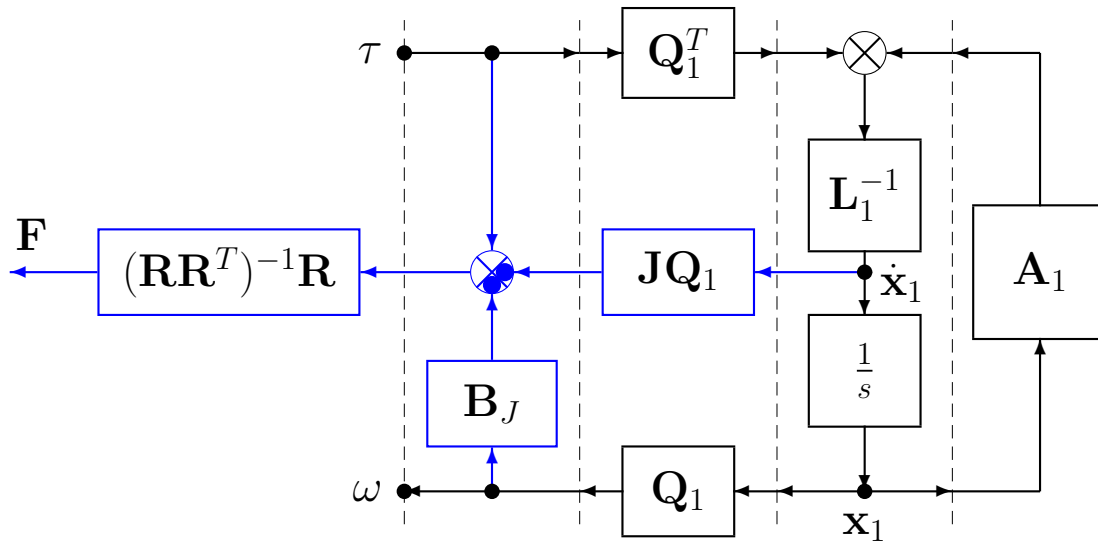
$$\mathbf{R}^T \mathbf{F} = \tau - \mathbf{J} \dot{\omega} - (\mathbf{B}_J + \mathbf{R}^T \mathbf{B}_k \mathbf{R}) \omega$$

When  $\mathbf{K} \rightarrow \infty$ , it is  $\omega = \mathbf{Q}_1 \mathbf{x}_1$ . By substituting in the previous relation, it results:

$$\mathbf{R}^T \mathbf{F} = \tau - \mathbf{J} \mathbf{Q}_1 \dot{\mathbf{x}}_1 - \mathbf{B}_J \mathbf{Q}_1 \mathbf{x}_1 - \underbrace{\mathbf{R}^T \mathbf{B}_k \mathbf{R} \mathbf{Q}_1}_0 \mathbf{x}_1. \quad (9)$$

Relation (8) can be directly obtained from (9) by applying the pseudoinverse of matrix  $\mathbf{R}$ , that is  $(\mathbf{R}\mathbf{R}^T)^{-1}\mathbf{R}$ , on the left of relation (9) and recalling that  $\mathbf{R}\mathbf{Q}_1 = 0$ .  $\square$

- The following modified POG block scheme combines in the same figure the computation of the force vector  $\mathbf{F}$  (blue blocks) with the POG reduced system (7) (black blocks).



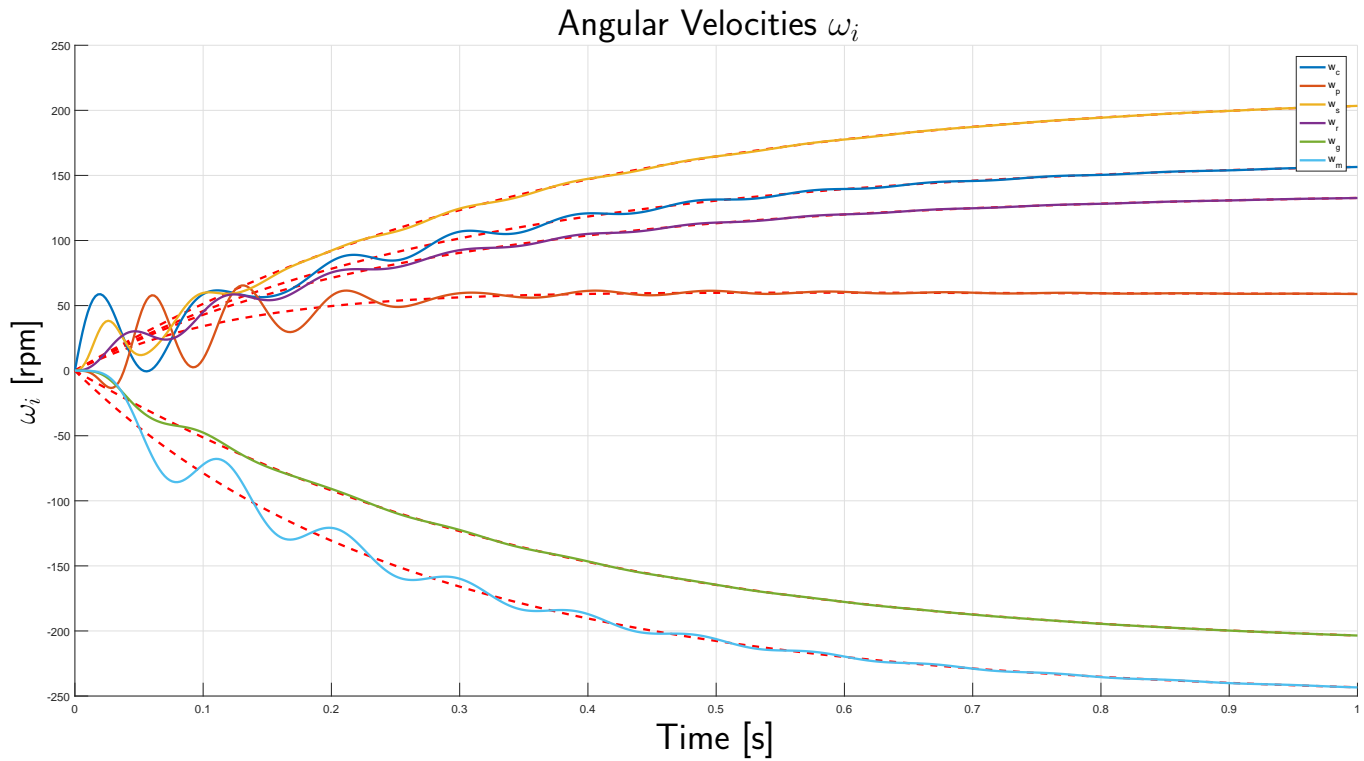
- Both the full elastic model (2) and the reduced rigid model (7) have been simulated in Matlab/Simulink using the following parameters:

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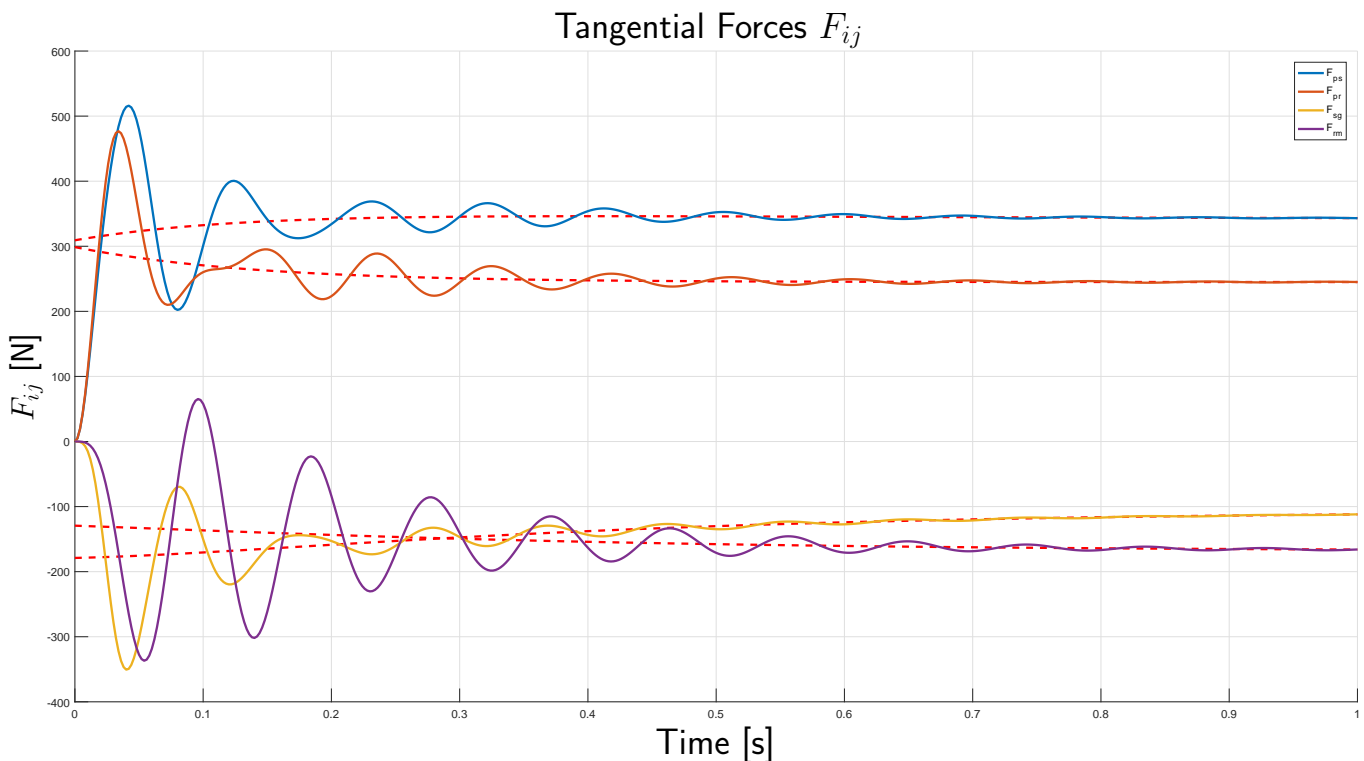
r_s1 = 10*cm;           % Sun_1: radius
r_s2 = 16*cm;           % Sun_2: radius
r_p  = 4.8*cm;          % Planetary: radius
r_g  = 16*cm;           % Generator ring: radius
r_m  = 12*cm;           % Motor ring 1: radius
r_r2 = 22*cm;           % Motor ring 2: radius
r_c  = 14.8*cm;         % Carrier: radius
r_r1 = 19.6*cm;         % Ring: radius
J_c  = 0.17648*kg*meters^2; % Carrier: inertia
J_p  = 0.009756*kg*meters^2; % Planetary: inertia
J_r  = 0.54251*kg*meters^2; % Ring: inertia
J_s  = 0.038617*kg*meters^2; % Sun: inertia
J_g  = 0.48197*kg*meters^2; % Generator: inertia
J_m  = 0.15263*kg*meters^2; % Motor ring: inertia
b_c  = 0.08*Nm/rpm;     % Carrier: friction coeff.
b_p  = 0.08*Nm/rpm;     % Planetary: friction coeff.
b_r  = 0.08*Nm/rpm;     % Ring: friction coefficient
b_s  = 0.08*Nm/rpm;     % Sun: friction coefficient
b_g  = 0.08*Nm/rpm;     % Generator: friction coeff.
b_m  = 0.08*Nm/rpm;     % Motor: friction coefficient
K_ps = 30*Newton/mm;    % Planetary-Sun: stiffness
K_pr = 30*Newton/mm;    % Planetary-Ring 1: stiffness
K_sg = 30*Newton/mm;    % Sun-Generator : stiffness
K_rm = 30*Newton/mm;    % Ring 2-Motor : stiffness
b_ps = 0.4*Newton*s/cm; % Planetary-Sun: fric. coeff.
b_pr = 0.4*Newton*s/cm; % Planetary-Ring: fric. coeff.
b_sg = 0.4*Newton*s/cm; % Sun-Generator: fric. coeff.
b_rm = 0.4*Newton*s/cm; % Ring 2-Motor: fric. coeff.
Tau  = [100 0 0 0 0 0]*Nm; % Input Torques
W_0  = [0 0 0 0 0 0]*rpm; % Angular velocities: In.Cond.
F_0  = [0 0 0 0]*Newton; % Tangential forces: In.Cond.

```

- Simulation results: the angular velocities  $\omega_i$ , for  $i \in \{c, p, s, r, g, m\}$ :



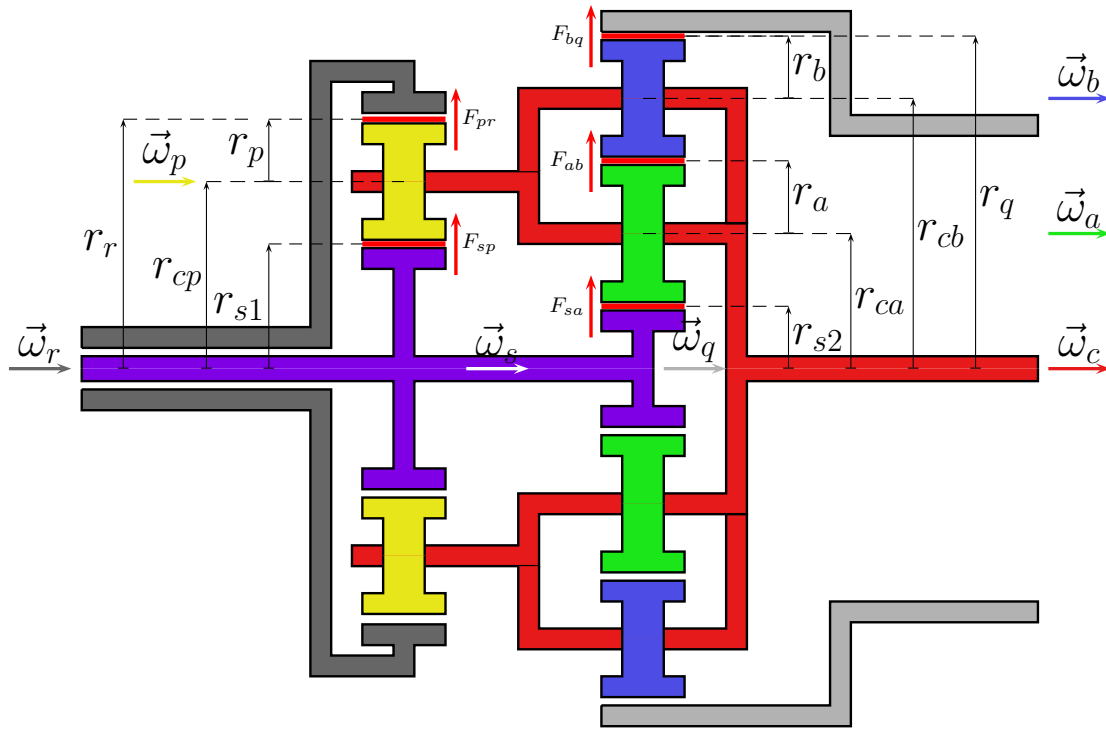
- Simulation results: the tangential forces  $F_{ij}$ , for  $ij \in \{ps, pr, sg, rm\}$ :



- The time behaviors plotted by using dashed red lines refer to the simulation results obtained by using the rigid reduced system (7). The time behaviors plotted using colored lines refer to the simulation results obtained by using the extended POG system (2) taking the elasticities into account.

## Modeling a two-stages planetary gear

- Let us refer to the following two-stages planetary gear:



- The corresponding POG state space model can be obtained by means of the same equations and the same symbols used in (2) and in (3) for the previously considered one-stage planetary gear.
- In this case, the velocity vector  $\omega$ , the input torque vector  $\tau$  and the force vector  $\mathbf{F}$  are defined as follows:

$$\omega = \begin{bmatrix} \omega_s \\ \omega_c \\ \omega_p \\ \omega_a \\ \omega_r \\ \omega_b \\ \omega_q \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_s \\ \tau_c \\ \tau_p \\ \tau_a \\ \tau_r \\ \tau_b \\ \tau_q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_{sp} \\ F_{sa} \\ F_{pr} \\ F_{ab} \\ F_{bq} \end{bmatrix}.$$

- The inertia matrix  $\mathbf{J}$  and the inertia friction matrix  $\mathbf{B}_J$  are:

$$\mathbf{J} = \begin{bmatrix} J_s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & J_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_q \end{bmatrix}, \quad \mathbf{B}_J = \begin{bmatrix} b_s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_q \end{bmatrix}.$$

- The stiffness matrix  $\mathbf{K}$  and the stiffness friction matrix  $\mathbf{B}_K$  are:

$$\mathbf{K} = \begin{bmatrix} K_{sp} & 0 & 0 & 0 & 0 \\ 0 & K_{sa} & 0 & 0 & 0 \\ 0 & 0 & K_{pr} & 0 & 0 \\ 0 & 0 & 0 & K_{ab} & 0 \\ 0 & 0 & 0 & 0 & K_{bq} \end{bmatrix}, \quad \mathbf{B}_K = \begin{bmatrix} b_{sp} & 0 & 0 & 0 & 0 \\ 0 & b_{sa} & 0 & 0 & 0 \\ 0 & 0 & b_{pr} & 0 & 0 \\ 0 & 0 & 0 & b_{ab} & 0 \\ 0 & 0 & 0 & 0 & b_{bq} \end{bmatrix}.$$

- The radius matrix  $\mathbf{R}$  is defined as follows:

$$\mathbf{R} = \begin{bmatrix} r_{s1} & -r_{cp} & r_p & 0 & 0 & 0 & 0 \\ r_{s2} & -r_{ca} & 0 & r_a & 0 & 0 & 0 \\ 0 & r_{cp} & r_p & 0 & -r_r & 0 & 0 \\ 0 & r_{ca} - r_{cb} & 0 & r_a & 0 & r_b & 0 \\ 0 & r_{cb} & 0 & 0 & 0 & r_b & -r_q \end{bmatrix}. \quad (10)$$

- Radii  $r_i$  or  $r_{ij}$  present in matrix  $\mathbf{R}$  are constrained as follows:

$$\begin{aligned} r_{cp} &= r_p + r_{s1}, & r_{cb} &= 2r_a + r_b + r_{s2}, \\ r_{ca} &= r_a + r_{s2}, & r_q &= 2r_a + 2r_b + r_{s2}. \end{aligned}$$

- The radii within matrix  $\mathbf{R}$  defined in (10) have been computed by means of the rules defined in the previous section.
- Let us consider, for example, the coefficient  $r_{ab,c}$ . In this case the angular velocity  $\vec{\omega}_c$  affects the force vector  $\vec{F}_{ab}$  through two different intermediate gears, characterized by moments of inertia  $J_a$  and  $J_b$ .



- When  $\mathbf{K} \rightarrow \infty$ , see (2), the angular velocities satisfy relation  $\mathbf{R}\omega = 0$  and are constrained as follows:

$$\left\{ \begin{array}{l} r_p \omega_p - r_{cp} \omega_c + r_{s1} \omega_s = 0 \\ r_a \omega_a - r_{ca} \omega_c + r_{s2} \omega_s = 0 \\ r_{cp} \omega_c + r_p \omega_p - r_r \omega_r = 0 \\ r_a \omega_a + r_b \omega_b + \omega_c (r_{ca} - r_{cb}) = 0 \\ r_b \omega_b + r_{cb} \omega_c - r_q \omega_q = 0 \end{array} \right. \quad (11)$$

- In this case, by choosing  $\mathbf{x}_1 = [\omega_c \ \omega_r]^T$ , from (11) one obtains the following congruent state space transformation:

$$\underbrace{\begin{bmatrix} \omega_s \\ \omega_c \\ \omega_p \\ \omega_a \\ \omega_r \\ \omega_b \\ \omega_q \\ F_{sp} \\ F_{sa} \\ F_{pr} \\ F_{ab} \\ F_{bq} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{r_{s1}}{2r_{cp}} & \frac{r_r}{2r_{cp}} \\ -\frac{r_{s1}}{2r_p} & \frac{r_r}{2r_p} \\ \frac{r_{ca}r_{s1} - 2r_{cp}r_{s2}}{2r_a r_{cp}} & \frac{r_{ca}r_r}{2r_a r_{cp}} \\ 0 & 1 \\ \frac{r_{cb}r_{s1} - 2r_{ca}r_{s1} + 2r_{cp}r_{s2}}{2r_b r_{cp}} & -\frac{2r_{ca}r_r - r_{cb}r_r}{2r_b r_{cp}} \\ \frac{r_{cb}r_{s1} - r_{ca}r_{s1} + r_{cp}r_{s2}}{r_{cp}r_q} & -\frac{r_{ca}r_r - r_{cb}r_r}{r_{cp}r_q} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{T}_1} \underbrace{\begin{bmatrix} \omega_s \\ \omega_r \end{bmatrix}}_{\mathbf{x}_1}$$

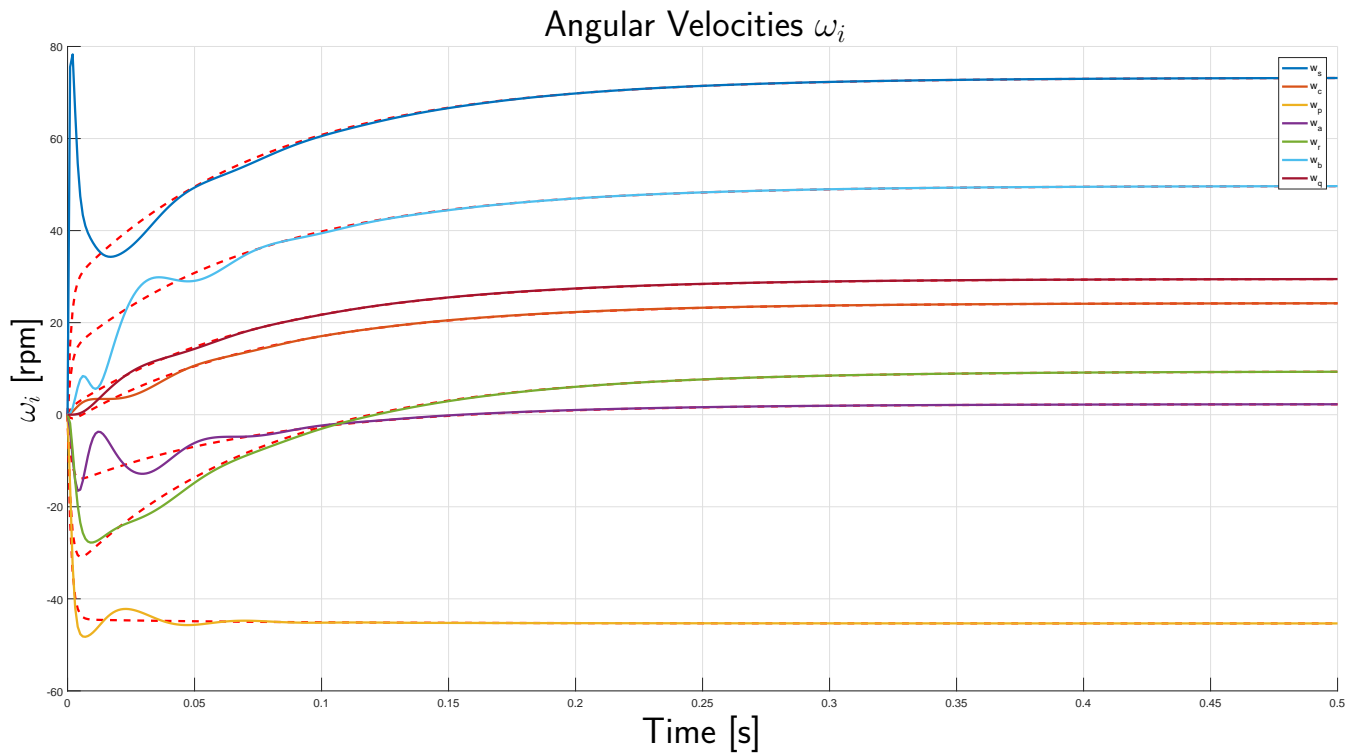
- By applying the congruent transformation  $\mathbf{x} = \mathbf{T}_1 \mathbf{x}_1$  to system (2), one obtains a reduced system having the same structure of the reduced model (7) obtained in the previous example.
- Also in this case, the force vector  $\mathbf{F}$  can be recovered from the reduced POG model (7) by means of relation (8).
- The two-stages planetary gear has been simulated in Matlab/Simulink. The following parameters have been used:

```

r_r = 5.4*cm;           % Ring: radius
r_s1 = 5.4*cm;         % Sun_1: radius
r_s2 = 2.6*cm;         % Sun_2: radius
r_p = 3.8*cm;          % Planetary: radius
r_a = 5.8*cm;          % Planetary_a: radius
r_b = 5*cm;            % Planetary_b: radius
r_cp = 9.2*cm;         % Center Planetary_p: radius
r_ca = 8.4*cm;         % Center Planetary_a: radius
r_cb = 19.2*cm;        % Center Planetary_b: radius
r_q = 24.2*cm;         % Ring_2: radius
J_c = 1.891*kg*meters^2; % Carrier: inertia
J_p = 0.0038321*kg*meters^2; % Planetary_p: inertia
J_r = 0.0046882*kg*meters^2; % Ring: inertia
J_s = 0.0062509*kg*meters^2; % Sun: inertia
J_a = 0.020798*kg*meters^2; % Planetary_a: inertia
J_b = 0.011486*kg*meters^2; % Planetary_b: inertia
J_q = 2.5213*kg*meters^2; % Ring_2: inertia
b_c = 0.64*Nm/rpm;     % Carrier: friction coeff.
b_p = 0.64*Nm/rpm;     % Planetary_p: fric. coeff.
b_r = 0.64*Nm/rpm;     % Ring: friction coefficient
b_s = 0.64*Nm/rpm;     % Sun: friction coefficient
b_a = 0.64*Nm/rpm;     % Planetary_a: fric. coeff.
b_b = 0.64*Nm/rpm;     % Planetary_b: fric. coeff.
b_q = 0.64*Nm/rpm;     % Ring_2: fric. coeff.
K_sp = 1000*Newton/mm; % Sun-Plan_p: stiffness
K_sa = 1000*Newton/mm; % Sun-Plan_a: stiffness
K_pr = 1000*Newton/mm; % Plan_p-Ring: stiffness
K_ab = 1000*Newton/mm; % Plan_a-Plan_b: stiffness
K_bq = 1000*Newton/mm; % Plan_b-Ring_2: stiffness
b_sp = 8*Newton*s/cm;  % Sun-Plan_p: fric. coeff.
b_sa = 8*Newton*s/cm;  % Sun-Plan_a: fric. coeff.
b_pr = 8*Newton*s/cm;  % Plan_p-Ring: fric. coeff.
b_ab = 8*Newton*s/cm;  % Plan_a-Plan_b: fric. coeff.
b_bq = 8*Newton*s/cm;  % Plan_b-Ring_2: fric. coeff.
Tau = [100 0 0 0 0 0 0]*Nm; % Input Torques
W_0 = [0 0 0 0 0 0 0]*rpm; % Angular velocities: In. Co.
F_0 = [0 0 0 0 0]*Newton; % Tangential forces: In. Co.

```

- Simulation results: the angular velocities  $\omega_i$ , for  $i \in \{c, p, s, r, g, m\}$ :



- Simulation results: the tangential forces  $F_{ij}$ , for  $ij \in \{ps, pr, sg, rm\}$ :

