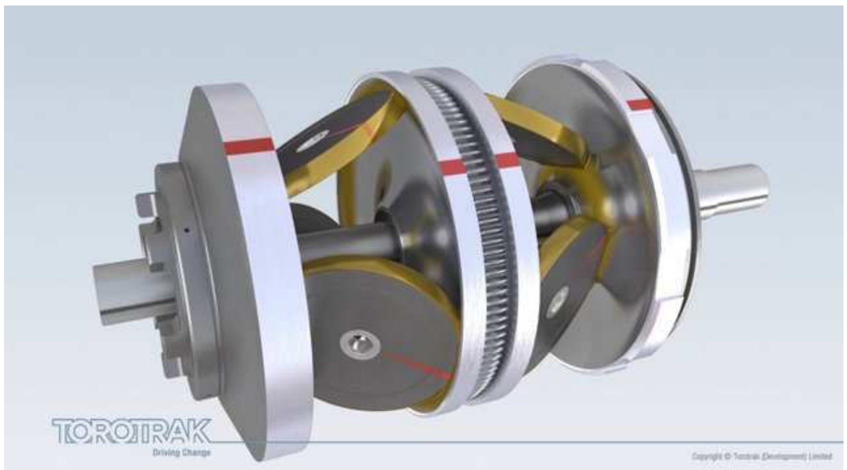
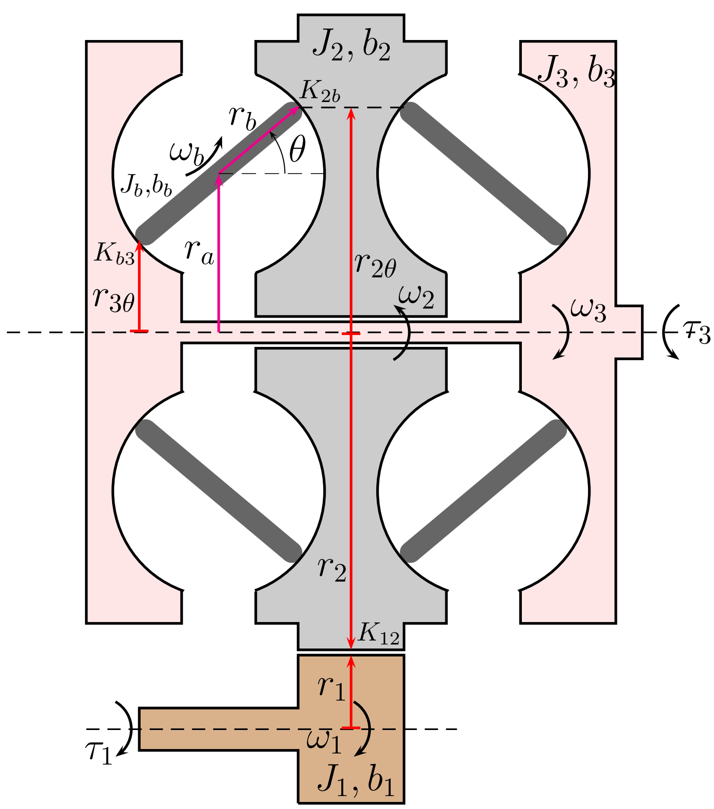
Dynamic Model of a Full Toroidal Variator

- The Full Toroidal Variator (FTV) is the core element in many mechanical hybrid systems for vehicle applications like Kinetic Energy Recovery System and Infinitely Variable Transmission (IVT) because it allows to manage the power transfer with continuous variation of the speed-ratio.



- In KERS applications the FTV manages the power flows between the vehicle and a flywheel (the kinetic energy is stored in the flywheel during braking and then reused under acceleration of the vehicle). A schematic of the considered FTV is the following:



- Meaning of parameters and variables of the system:

J_1	shaft moment of inertia
b_1	shaft linear friction coefficient
ω_1	shaft angular velocity
τ_1	torque acting on the shaft
r_1	gear radius
K_{12}	stiffness coefficient between shaft and inner disk
J_2, J_3	inner and outer disks moments of inertia
b_2, b_3	inner and outer disks linear friction coefficients
ω_2, ω_3	inner and outer disks angular velocities
τ_3	torque acting on the outer disk
r_2	inner disk radius
θ	tilt angle of the rollers
$r_{2\theta}$, $r_{3\theta}$	contact radial positions
K_{2b}, K_{3b}	contact stiffness coefficients
J_b	rollers moment of inertia
b_b	rollers linear friction coefficient
ω_b	rollers angular velocity
r_b	rollers radius
r_a	toroid centerline radius

- The two toroidal cavities contain six rollers (three for each cavity) that transfer the power from the inner disc to the outer disc and viceversa. The variator speed-ratio is a function of the rollers inclination: a change of the tilt angle θ of the rollers causes a speed-ratio variation.

- The contact radial positions $r_{2\theta}$ and $r_{3\theta}$ are related to radii r_a and r_b as follows:

$$r_{2\theta} = r_a + r_b \sin\theta, \qquad\qquad r_{3\theta} = r_a - r_b \sin\theta.$$

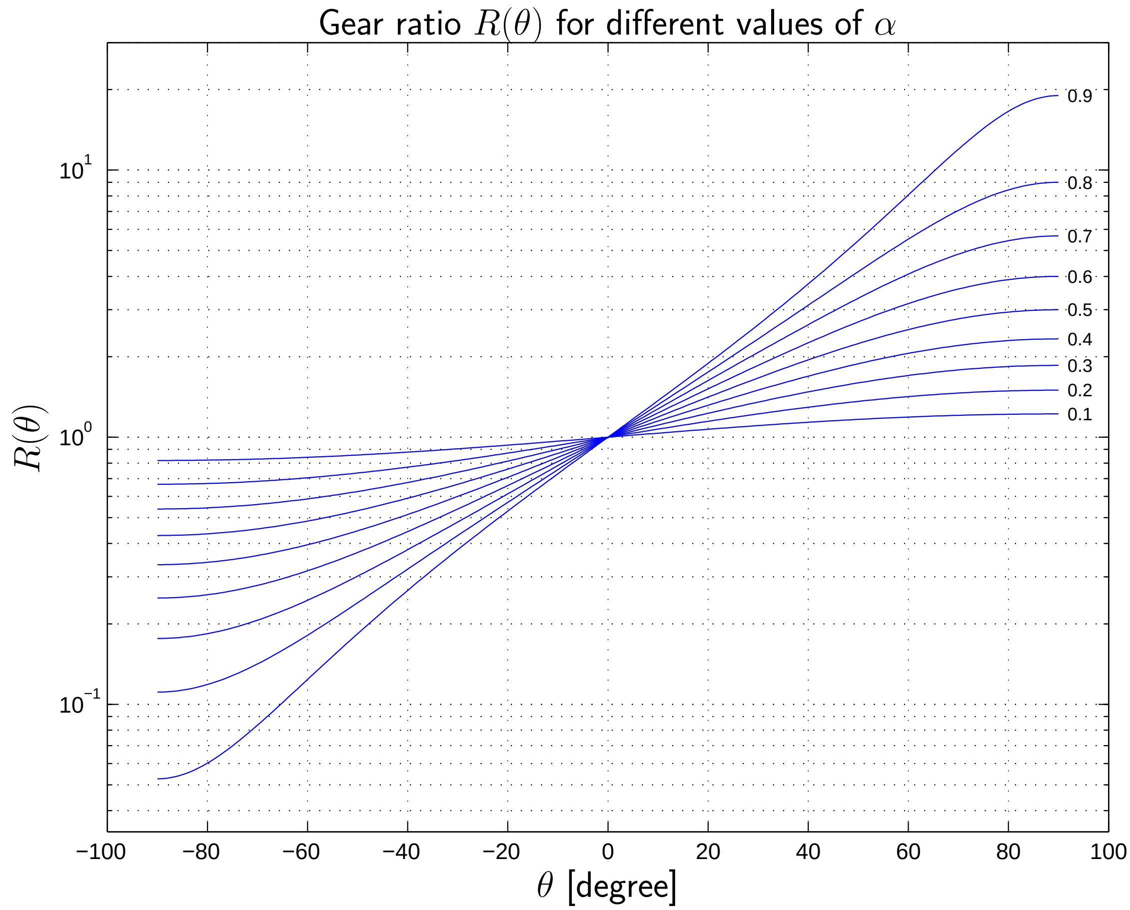
- The gear ratio $R(\theta)$ relating the angular speed ω_2 of the inner disk to the angular speed ω_3 of the outer disk is function of the tilt angle θ as follows:

$$R(\theta) = \frac{\omega_3}{\omega_2} = \frac{r_{2\theta}}{r_{3\theta}} = \frac{r_a + r_b \sin\theta}{r_a - r_b \sin\theta} = \frac{1 + \alpha \sin\theta}{1 - \alpha \sin\theta}$$

where

$$\alpha = \frac{r_b}{r_a}$$

- Plot of the gear ratio $R(\theta)$ for different values of $\alpha \in [0.1 : 0.1 : 0.9]$:



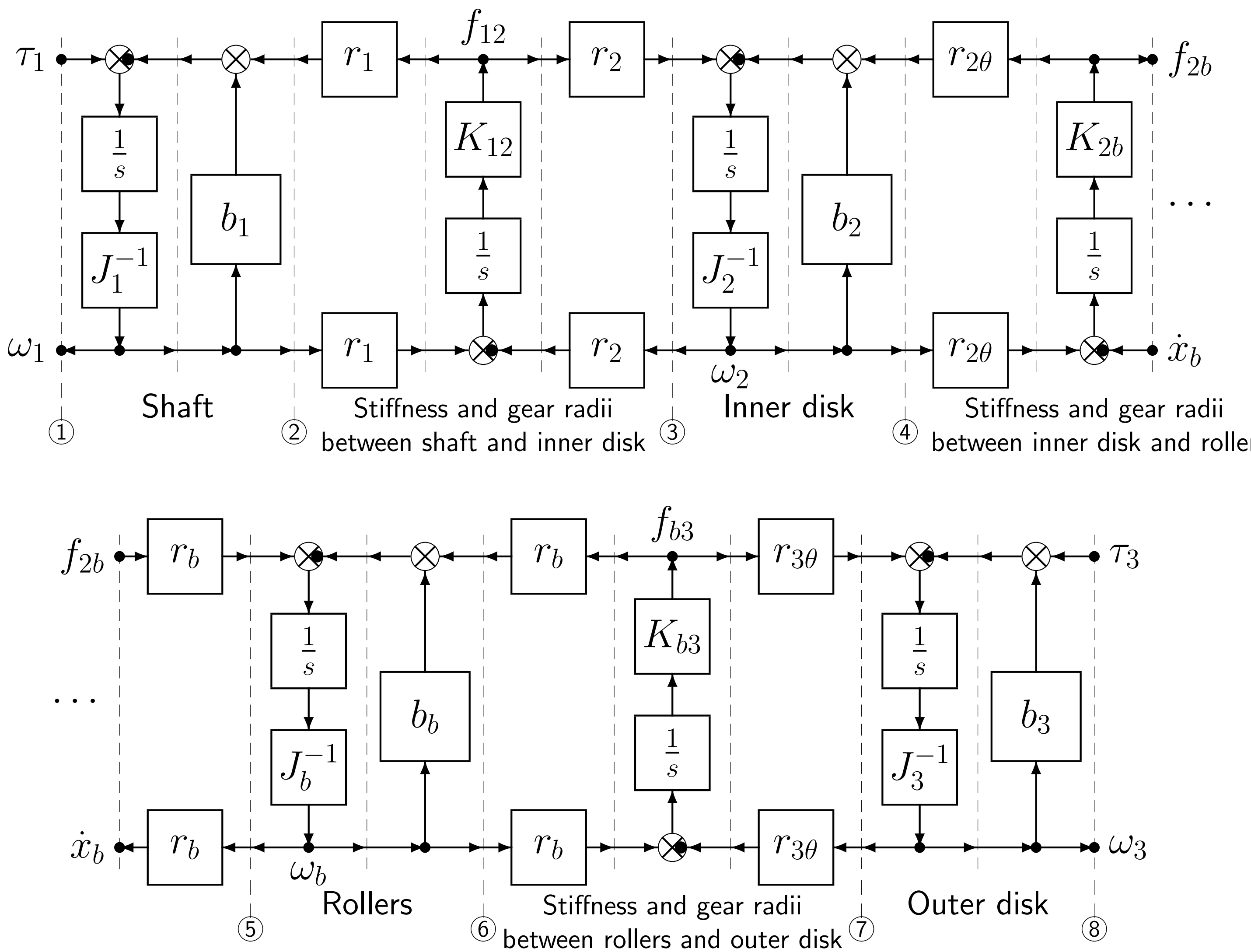
- The FTV is a non-linear and time-variant system and writing its lumped parameter dynamic model is non trivial problem.

- The POG model of this physical system can be easily obtained introducing "fictitious" elastic elements between the inertias of the system. The elastic elements are the only dynamic element that can be put between two masses or inertias respecting the integral causality of the system.

- The presence of these elastic elements simplify the writing down of a dynamic model of the system. The obtained POG model is characterized only by integral causality element suitable for simulation.

- The elastic elements can be afterwards eliminated applying a proper congruent transformation to the original system and so obtaining the POG dynamic model of the reduced system.

- The POG block scheme of the considered FTV system is the following:



- The state space model $\mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\mathbf{u}$ of the POG system is:

$$\underbrace{\begin{bmatrix} J_1\,\dot{\omega}_1 \\ \frac{1}{K_{12}}\,\dot{f}_{12} \\ J_2\,\dot{\omega}_2 \\ \frac{1}{K_{2b}}\,\dot{f}_{2b} \\ J_b\,\dot{\omega}_b \\ \frac{1}{K_{b3}}\,\dot{f}_{b3} \\ J_3\,\dot{\omega}_3 \end{bmatrix}}_{\mathbf{L}\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -b_1 & -r_1 & & & & & \\ r_1 & 0 & -r_2 & & & & \\ & r_2 & -b_2 & -r_{2\theta} & & & \\ & & r_{2\theta} & 0 & -r_b & & \\ & & & r_b & -b_b & -r_b & \\ & & & & r_b & 0 & -r_{3\theta} \\ & & & & & r_{3\theta} & -b_3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \omega_1 \\ f_{12} \\ \omega_2 \\ f_{2b} \\ \omega_b \\ f_{b3} \\ \omega_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \tau_1 \\ \tau_3 \end{bmatrix}}_{\mathbf{u}}$$

- The matrices and vectors $\mathbf{L}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{x}$, $\mathbf{u}$ can be directly read from the POG block scheme.

- When $K_{12} \to \infty$, $K_{2b} \to \infty$ and $K_{b3} \to \infty$ the state variables are linked by the following constraint:

$$\begin{cases} 0 = r_1\,\omega_1 - r_2\,\omega_2 \\ 0 = r_{2\theta}\,\omega_2 - r_b\,\omega_b \\ 0 = r_b\,\omega_b - r_{3\theta}\,\omega_3. \end{cases} \quad \Rightarrow \quad \begin{cases} \omega_2 = \dfrac{r_1}{r_2}\,\omega_1 \\ \omega_b = \dfrac{r_{2\theta}}{r_b}\,\omega_2 = \dfrac{r_{2\theta}\,r_1}{r_b\,r_2}\,\omega_1 \\ \omega_3 = \dfrac{r_b}{r_{3\theta}}\,\omega_b = \dfrac{r_{2\theta}\,r_1}{r_{3\theta}\,r_2}\,\omega_1. \end{cases}$$

- The following state space congruent transformation can be used:

$$\mathbf{x} = \mathbf{T}(\theta)\,\omega_1 \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix} \omega_1 \\ f_{12} \\ \omega_2 \\ f_{2b} \\ \omega_b \\ f_{b3} \\ \omega_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ R_2 \\ 0 \\ R_b(\theta) \\ 0 \\ R_3(\theta) \end{bmatrix}}_{\mathbf{T}(\theta)} \omega_1$$

where gear ratios R_2, $R_b(\theta)$ and $R_3(\theta)$ have the following structure:

$$R_2 = \frac{r_1}{r_2}, \qquad R_b(\theta) = \frac{r_1\,r_{2\theta}}{r_2\,r_b} = \frac{r_1\,(r_a + r_b \sin\theta)}{r_2\,r_b},$$

$$R_3(\theta) = \frac{r_1\,r_{2\theta}}{r_2\,r_{3\theta}} = \frac{r_1\,(r_a + r_b \sin\theta)}{r_2\,(r_a - r_b \sin\theta)}.$$

- Since the transformation matrix $\mathbf{T}(\theta)$ is a column matrix, the transformed and reduced model is a first order dynamic model that can be written in the following *explicit* form:

$$J(\theta)\,\dot{\omega}_1 + N(\theta,\dot{\theta})\,\omega_1 = -b(\theta)\,\omega_1 + \underbrace{\begin{bmatrix} 1 & -R_3(\theta) \end{bmatrix}}_{\bar{\mathbf{B}}} \begin{bmatrix} \tau_1 \\ \tau_3 \end{bmatrix} \tag{1}$$

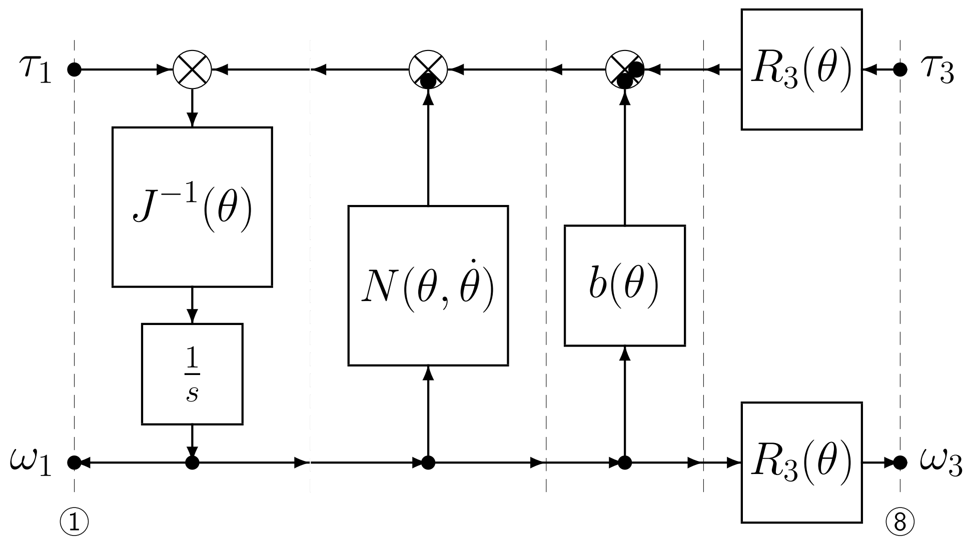
where $\bar{\mathbf{B}} = \mathbf{T}^T\mathbf{B}$, coefficients $J(\theta)$ and $b(\theta)$ are

$$\begin{aligned} J(\theta) &= \mathbf{T}^T\mathbf{L}\mathbf{T} = J_1 + R_2^2\,J_2 + R_b^2(\theta)\,J_b + R_3^2(\theta)\,J_3 \\ b(\theta) &= -\mathbf{T}^T\mathbf{A}\mathbf{T} = b_1 + R_2^2\,b_2 + R_b^2(\theta)\,b_b + R_3^2(\theta)\,b_3 \end{aligned}$$

and

$$N(\theta,\dot\theta) = \mathbf{T}^T\mathbf{L}\dot{\mathbf{T}} = r_{2\theta}\, r_b\, \dot\theta \cos\theta\, R_2^2 \left(\frac{J_b}{r_b^2} + \frac{2\, r_a\, J_3}{r_{3\theta}^3}\right) = \frac{\dot J(\theta)}{2}.$$

- The transformed and reduced system (1) can be graphically represented using the following POG block scheme:



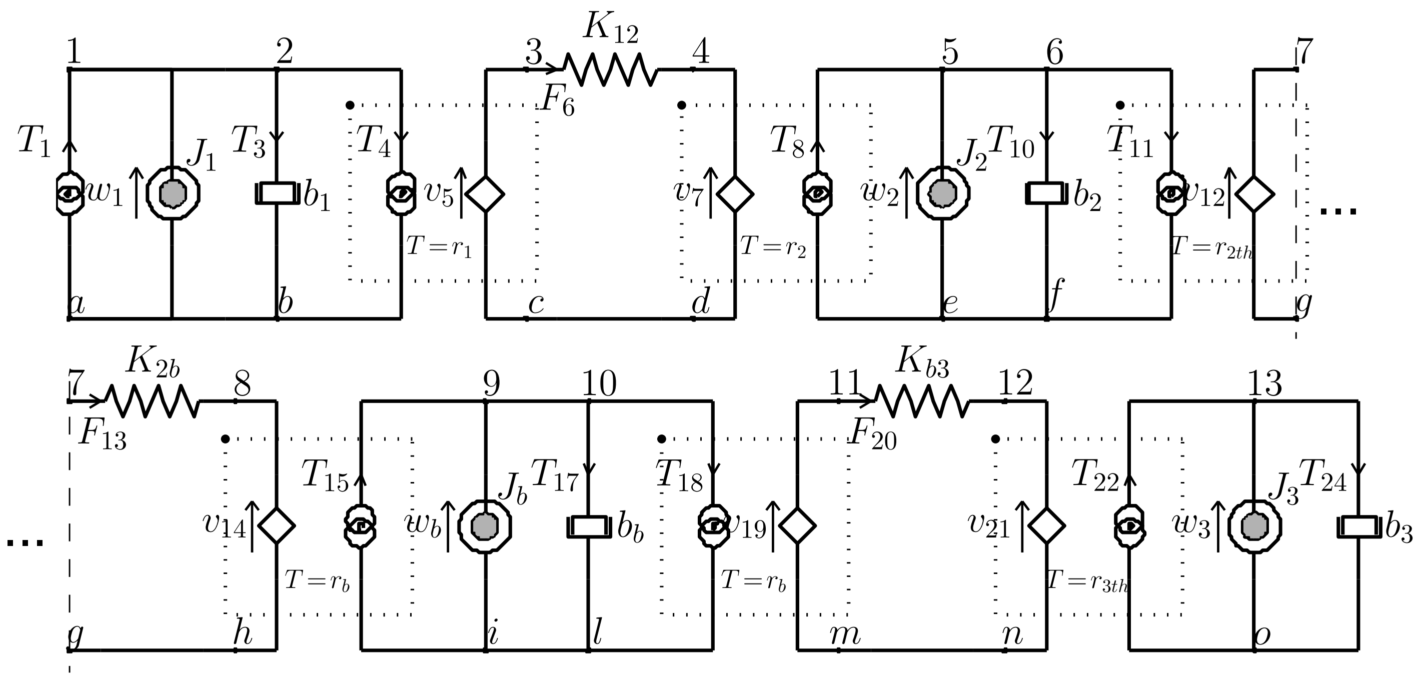
where

$$\begin{aligned}
J(\theta) &= J_1 + R_2^2\, J_2 + R_b^2(\theta)\, J_b + R_3^2(\theta)\, J_3 \\
b(\theta) &= b_1 + R_2^2\, b_2 + R_b^2(\theta)\, b_b + R_3^2(\theta)\, b_3 \\
N(\theta,\dot\theta) &= r_{2\theta}\, r_b\, \dot\theta \cos\theta\, R_2^2 \left(\tfrac{J_b}{r_b^2} + \tfrac{2\, r_a\, J_3}{r_{3\theta}^3}\right)
\end{aligned}$$

and

$$\begin{aligned}
R_2 &= \frac{r_1}{r_2} \\
R_b(\theta) &= \frac{r_1\,(r_a + r_b \sin\theta)}{r_2\, r_b} \\
R_3(\theta) &= \frac{r_1\,(r_a + r_b \sin\theta)}{r_2\,(r_a - r_b \sin\theta)}.
\end{aligned}$$

- The system can be graphically represented by the following POG scheme:

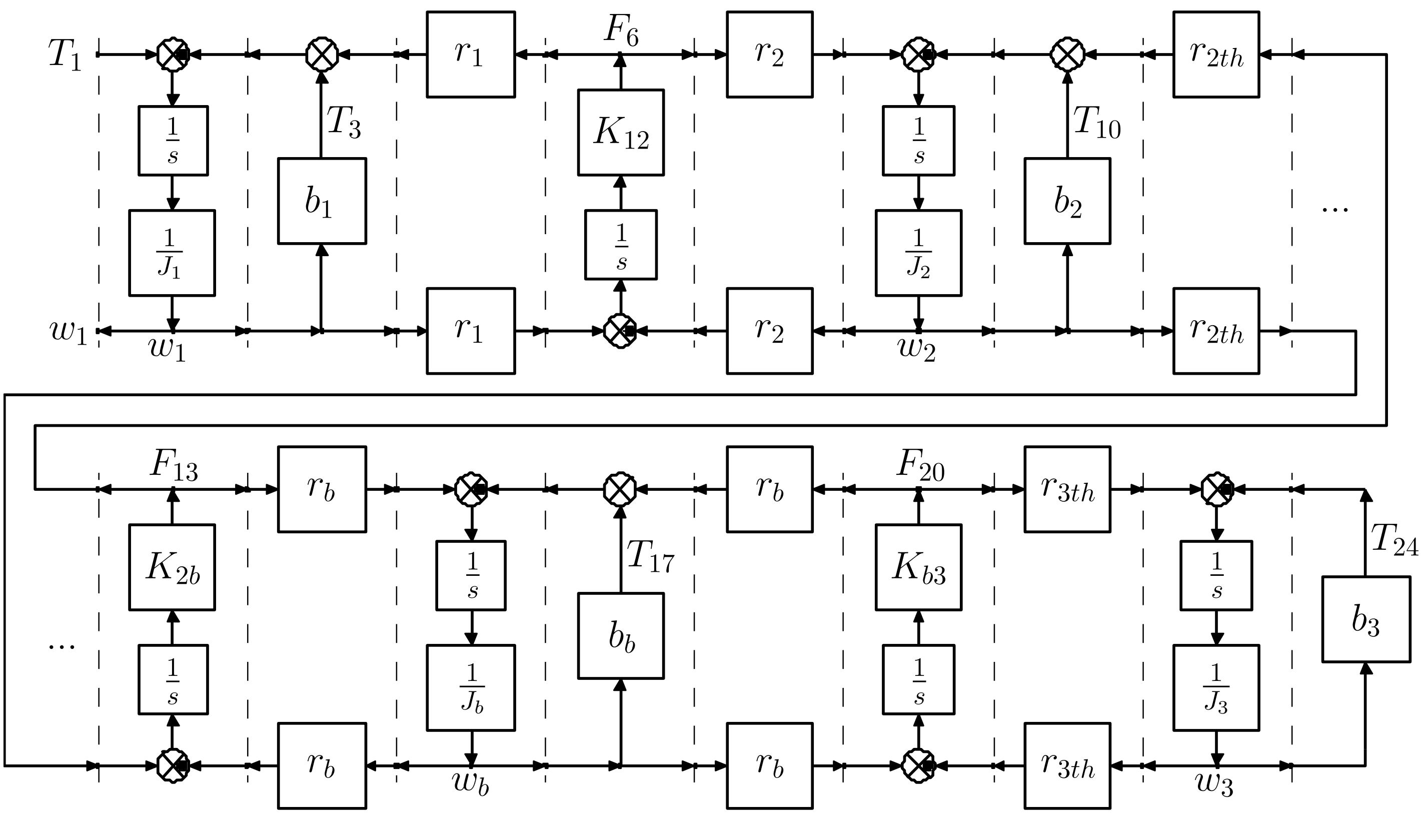


- This scheme has been obtained using the following source code:

```
**, Gr, Si, Sn, Si, As, Si, POG, Si, EQN, Si, SLX, Si
**, pSplit, 42, Sr, No, pSr, No, pEsu, No, Split, 7
rT, 1, a, An,-90, Ln, 1.2
rJ, 1, a, Sh, 0.5, Kn, J_1, En, w_1, Kn0, 13, Qn0, 2
--, 1, 2
--, a, b
rB, 2, b, Kn, b_1
CB, [2;3],[b;c], Kn, E2=r_1*E1, Sh, [0.6;-0.2]
mK, 3, 4, Ln, 0.8, Kn, K_1_2
--, c, d, Ln, 0.8
CB, [4;5],[d;e], Kn, E1=r_2*E2, Sh, [0.2;-0.6]
rJ, 5, e, Kn, J_2, En, w_2
--, 5, 6, Ln, 0.5
--, e, f, Ln, 0.5
rB, 6, f, Kn, b_2
CB, [6;7],[f;g], Kn, E2=r_2_t_h*E1, Sh, [0.6;-0.2]
mK, 7, 8, Ln, 0.8, Kn, K_2_b
--, g, h, Ln, 0.8
CB, [8;9],[h;i], Kn, F2=r_b*F1, Sh, [0.2;-0.6]
rJ, 9, i, Kn, J_b, En, w_b
--, 9, 10, Ln, 0.5
--, i, l, Ln, 0.5
```

```
rB, 10, l, Kn, b_b
CB, [10;11],[l;m], Kn, E2=r_b*E1, Sh, [0.6;-0.2]
mK, 11, 12, Ln, 0.8, Kn, K_b_3
--, m, n, Ln, 0.8
CB, [12;13],[n;o], Kn, F2=r_3_t_h*F1, Sh, [0.2;-0.6]
rJ, 13, o, Kn, J_3, En, w_3
rB, 13, o, Sh, 0.5, Kn, b_3
```

- The POG Modeler provides the following POG block scheme:



- The POG state space equations $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ are:

$$\mathbf{L}\,\dot{\mathbf{x}} = \begin{bmatrix} -b_1 & -r_1 & 0 & 0 & 0 & 0 & 0 \\ r_1 & 0 & -r_2 & 0 & 0 & 0 & 0 \\ 0 & r_2 & -b_2 & -r_{2th} & 0 & 0 & 0 \\ 0 & 0 & r_{2th} & 0 & -r_b & 0 & 0 \\ 0 & 0 & 0 & r_b & -b_b & -r_b & 0 \\ 0 & 0 & 0 & 0 & r_b & 0 & -r_{3th} \\ 0 & 0 & 0 & 0 & 0 & r_{3th} & -b_3 \end{bmatrix} \begin{bmatrix} w_1 \\ F_6 \\ w_2 \\ F_{13} \\ w_b \\ F_{20} \\ w_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} T_1$$

where

$$\mathbf{L} = \begin{bmatrix} J_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{K_{12}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{K_{2b}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{K_{b3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_3 \end{bmatrix}$$

- System parameters:

```
-- Matlab commands ----------- (POG_KERS_Par_SLX_m) -----------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms  J_1 b_1 r_a r_b r_1 r_2 J_2 b_2 r_b J_b b_b J_3 b_3 th
R2=r_1/r_2;
r2th=r_a+r_b*sin(th);   r3th=r_a-r_b*sin(th);
Rbth=r_1*r2th/(r_2*r_b);
R3th=r_1*r2th/(r_2*r3th);
Jth=J_1+J_2*R2^2+J_b*Rbth^2+J_3*R3th^2;
bth=b_1+b_2*R2^2+b_b*Rbth^2+b_3*R3th^2;
dJ_dth=diff(Jth,'th')/2;
%%%%%%%%%%%%   SYSTEM PARAMETERS  %%%%%%%%%%%%%%%%%%%%%%%%%%
J_1=0.026*kg*meters^2;          % 2. Inertia. Internal parameter.
b_1=1.06*Nm/(4000*rpm);         % 3. Angular friction. Internal parameter.
r_1=17.2*mm;                    % 5. Parameter. Transformer/Gyrator.
K_1_2=150*Newton/(0.2*mm);      % 6. Stiffness. Internal parameter.
r_2=51.5*mm;                    % 12. Parameter. Transformer/Gyrator.
J_2=0.675*kg*r_2^2;             % 9. Inertia. Internal parameter.
b_2=2.2*Nm/(4000*rpm);          % 10. Angular friction. Internal parameter.
K_2_b=80*Newton/(0.2*mm);       % 13. Stiffness. Internal parameter.
r_a=34.3*mm;      r_b=27.5*mm;
J_b=1.0*kg*r_b^2;               % 16. Inertia. Internal parameter.
b_b=1.9*Nm/(4000*rpm);          % 17. Angular friction. Internal parameter.
K_b_3=80*Newton/(0.2*mm);       % 20. Stiffness. Internal parameter.
J_3=100*J_2;                    % 23. Inertia. Internal parameter.
b_3=1.3*Nm/(4000*rpm);          % 24. Angular friction. Internal parameter.
%%%%%%%%%%%%   INPUT VALUES  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
th_A=70*degrees;                % Maximum amplitude of theta
th_w=3;                         % Frequency of theta
tt=(0:0.001:10)';               % Time vector
th=th_A*sin(th_w*tt);           % theta(tt)
dot_th=th_w*th_A*cos(th_w*tt);  % dot_theta(tt)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
th_0=th(1);  R2=r_1/r_2;
r_2_t_h=r_a+r_b*sin(th_0);
r_3_t_h=r_a-r_b*sin(th_0);
%%%%%%%%%%%%   INITIAL CONDITIONS  %%%%%%%%%%%%%%%%%%%%%%%%%%
T_1_0=0*Nm;        % 1. Torque. Input value.
w_1_0=2000*rpm;        % 2. Angular momentum. Initial condition.
w_2_0=w_1_0*r_1/r_2;        % 9. Angular momentum. Initial condition.
w_b_0=w_2_0*r_2_t_h/r_b;      % 16. Angular momentum. Initial condition.
w_3_0=w_b_0*r_b/r_3_t_h;      % 23. Angular momentum. Initial condition.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
---------------------------------------------------------------------------
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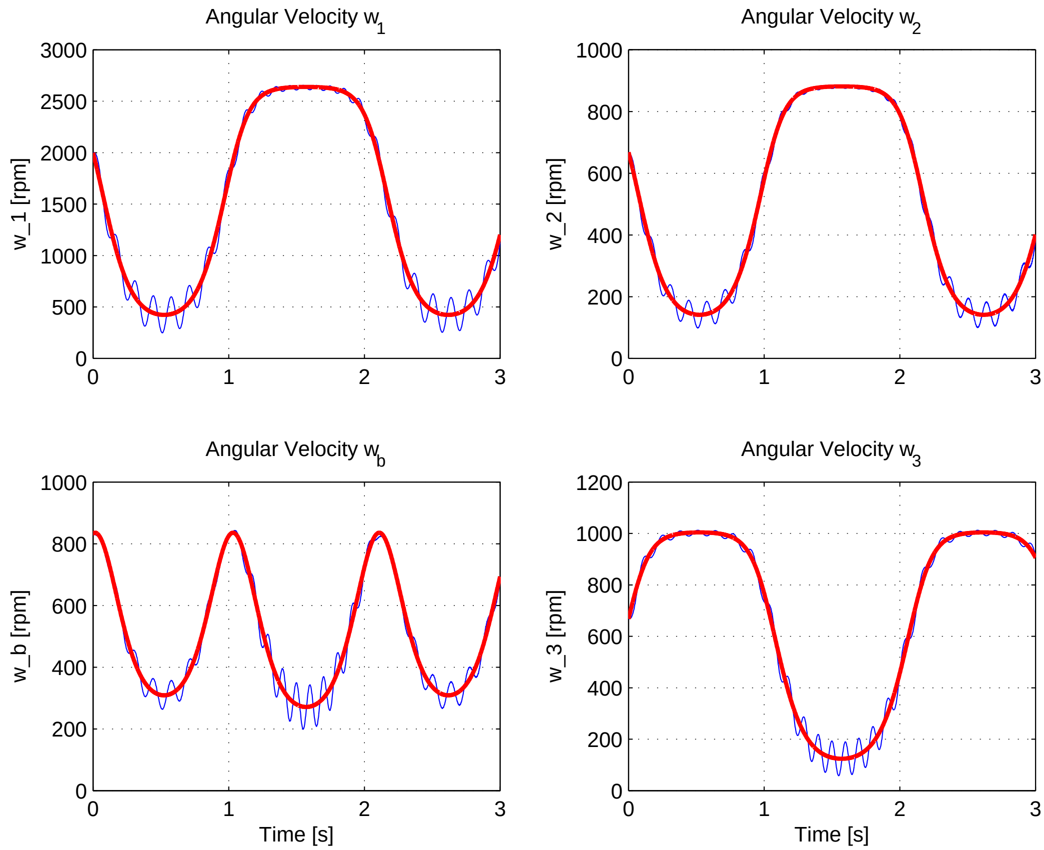
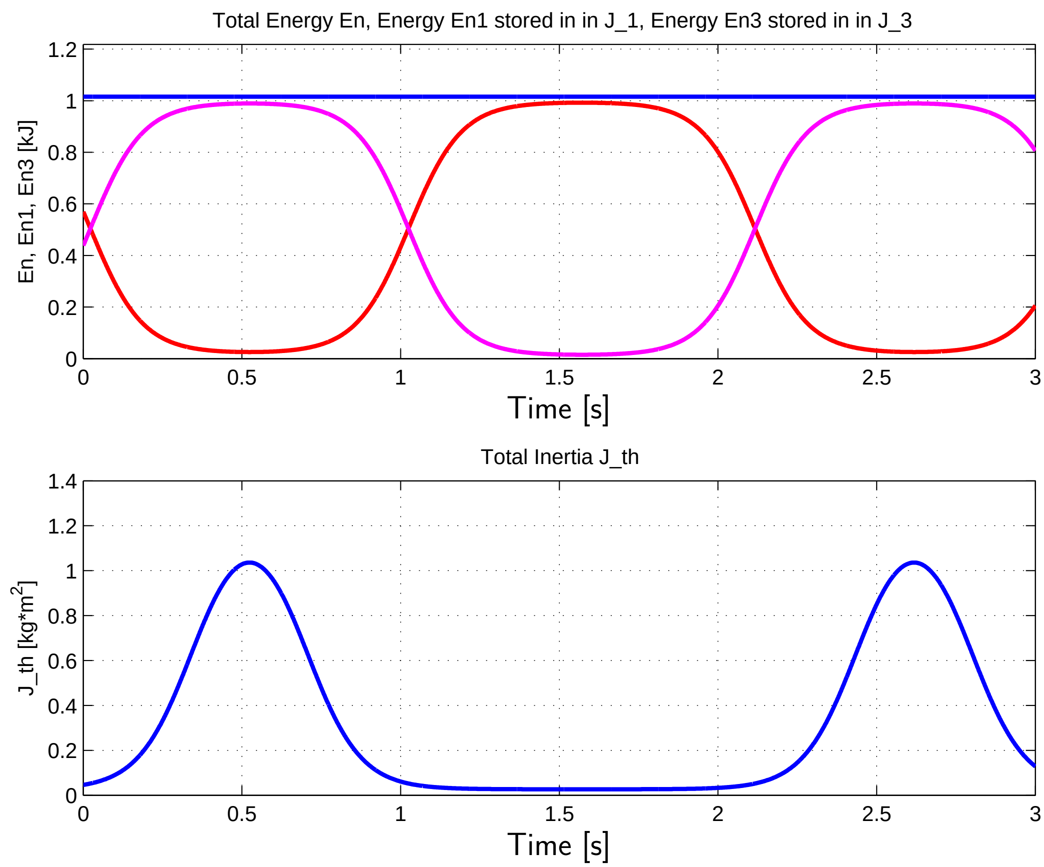
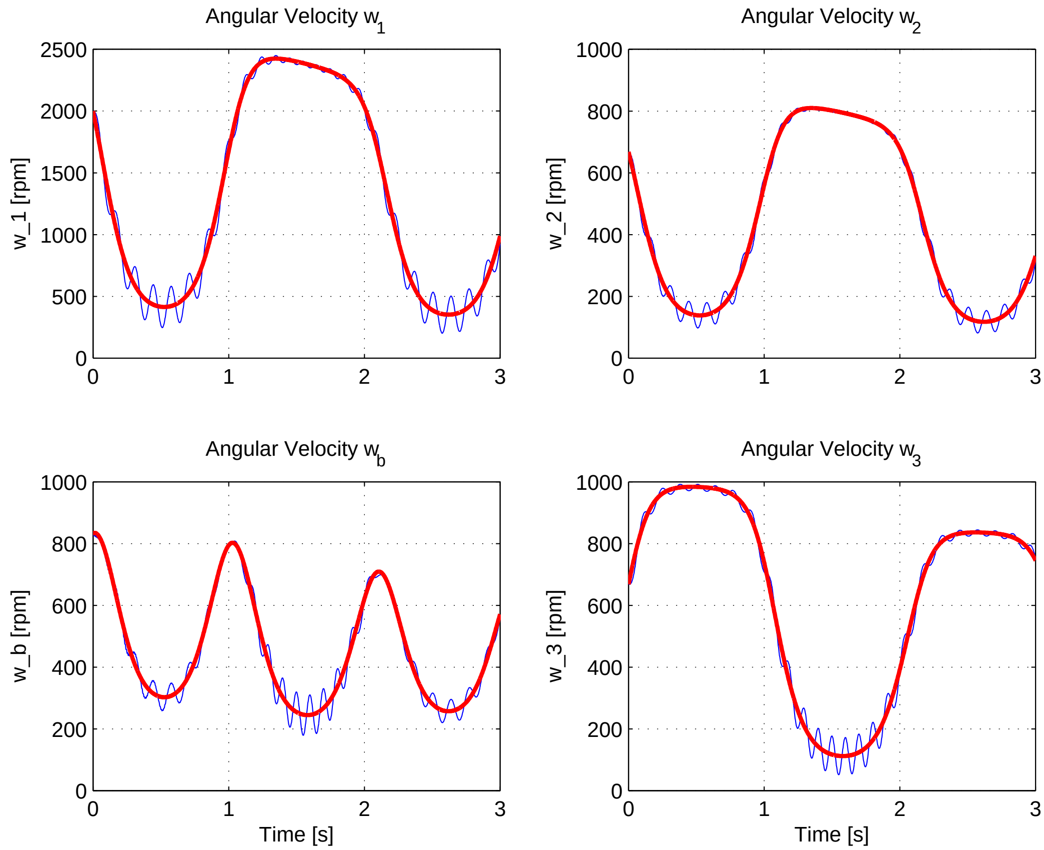
- Angular velocities ω_1, ω_2, ω_b, ω_3 when $b_1 = b_2 = b_b = b_3 = 0$:



- Energy stored in the system when $b_1 = b_2 = b_b = b_3 = 0$:



- Angular velocities ω_1, ω_2, ω_b, ω_3 of the system with dissipations:



- Stored energy of the system with dissipations: