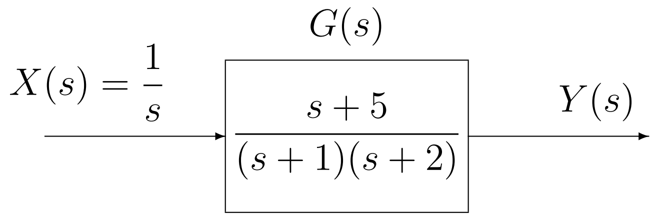
Time response: exercises

Exercise. Compute the unitary step response $y(t)$ of the following system:



The step response $y(t)$ is the inverse Laplace transforma of the following function:

$$Y(s) = G(s)X(s) = \frac{s+5}{s(s+1)(s+2)}$$

The initial value $y(0)$ and the final value $y(\infty)$ of the function $y(t)$ are:

$$y(0) = \lim_{s\to\infty} s\,Y(s) = G(\infty) = 0, \qquad y(\infty) = \lim_{s\to 0} s\,Y(s) = G(0) = \frac{5}{2}$$

Simple fractions decomposition:

$$Y(s) = \frac{s+5}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{(s+1)} + \frac{K_3}{(s+2)}$$

where

$$K_1 = s\,Y(s)|_{s=0} = \left.\frac{s+5}{(s+1)(s+2)}\right|_{s=0} = \frac{5}{2}$$

$$K_2 = (s+1)\,Y(s)|_{s=-1} = \left.\frac{s+5}{s(s+2)}\right|_{s=-1} = -4$$

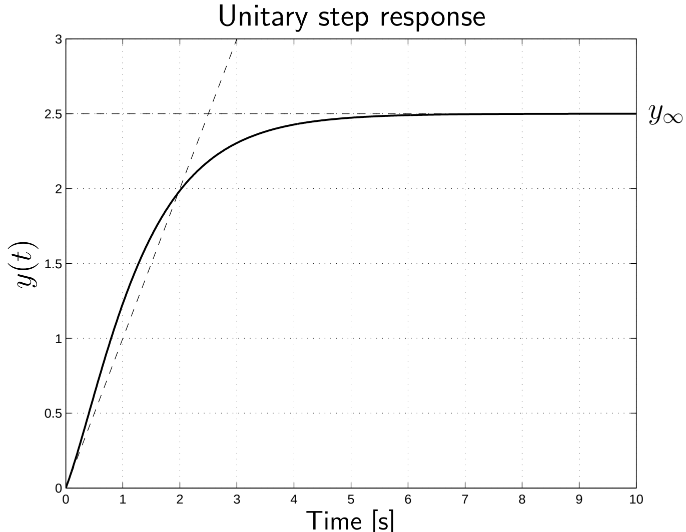
$$K_3 = (s+2)\,Y(s)|_{s=-2} = \left.\frac{s+5}{s(s+1)}\right|_{s=-2} = \frac{3}{2}$$

The unitary step response of system $G(s)$ is therefore the following:

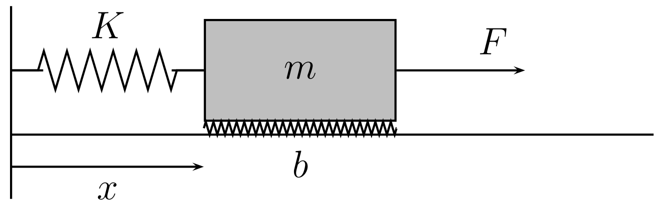
$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{5}{2} - 4e^{-t} + \frac{3}{2}e^{-2t}$$

Note that the slope of function $y(t)$ for $t = 0^+$ is:

$$\dot{y}(0^+) = \lim_{s\to\infty} sG(s) = 1$$



Exercise. Mass-spring-damper system. Calculate the system step response $x(t)$ of the system to the input $F(t) = 10$. Use the parameters $m = 1$, $b = 2$ and $K = 10$.



Differential equation of the system:

$$\frac{d}{dt}[m\dot{x}(t)] = F(t) - b\dot{x}(t) - Kx(t) \qquad \rightarrow \qquad m\ddot{x}(t) + b\dot{x}(t) + Kx(t) = F(t)$$

The transfer function ($m = 1$, $b = 2$ and $K = 10$) is:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + K} = \frac{1}{s^2 + 2\,s + 10} = \frac{1}{(s+1)^2 + 3^2}$$

The step response $X(s)$ to input $F(t) = 10$ is:

$$F(s) = \frac{10}{s} \qquad \rightarrow \qquad X(s) = G(s)F(s) = \frac{10}{s[(s+1)^2 + 3^2]}$$

The simple fractions decomposition can also be obtained as follows:

$$X(s) = \frac{10}{s[(s+1)^2 + 3^2]} = \frac{1}{s} - \frac{\alpha s + \beta}{(s+1)^2 + 3^2}$$

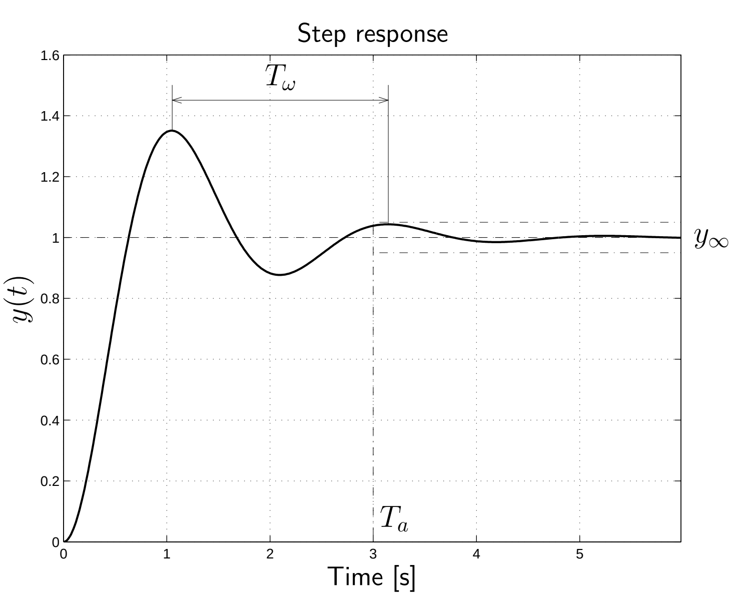
The parameters α and β are determined by imposing the equality between the two functions:

$$X(s) = \frac{1}{s} - \frac{s+2}{(s+1)^2 + 3^2} = \frac{1}{s} - \left[\frac{s+1}{(s+1)^2 + 3^2} + \frac{1}{3}\frac{3}{(s+1)^2 + 3^2}\right]$$

Computing the inverse Laplace transform one obtains:

$$x(t) = 1 - e^{-t}\left[\cos(3\,t) + \frac{1}{3}\sin(3\,t)\right]$$

The time behavior is oscillatory and damped:



Dominant poles of the system:

$$p_{1,2} = -\sigma \pm j\omega = -1 \pm j3$$

The settling time is:

$$T_a = \frac{3}{\sigma} = 3 \text{ s}$$

The oscillation period is:

$$T_\omega = \frac{2\pi}{\omega} \simeq 2.1 \text{ s}$$

Exercise. Consider the following system $G(s)$:

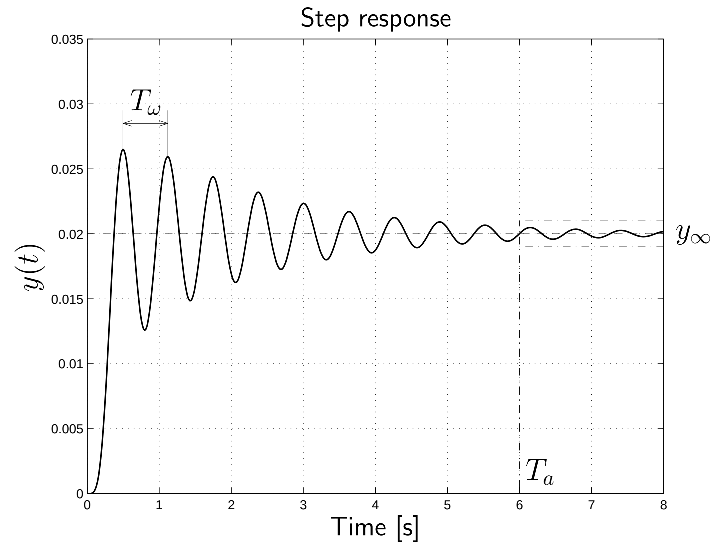
$$G(s) = \frac{800\,(2\,s+30)}{(0.2\,s+3)(2\,s+10)\underbrace{(s^2+s+100)}_{\text{dominant poles}}(s^2+20s+400)}$$

1) Draw the qualitative time behavior $y(t)$ of the step response of system $G(s)$.

Solution. The system is dominated by the complex conjugated poles $p_{1,2} \simeq -0.5 \pm j\,10$. The qualitative time behavior is oscillatory and damped.

2) Calculate the steady-state value y_∞ of the output $y(t)$:

$$y_\infty = G(0) = 0.02.$$



3) Qualitatively estimate the settling time T_a and the period T_ω of the damped oscillation (if present):

$$T_a \simeq \frac{3}{0.5}\ \text{s} = 6\ \text{s},$$
$$T_\omega \simeq \frac{2\pi}{10}\ \text{s} = 0.63\ \text{s}.$$

Exercise. Consider the following system $G(s)$

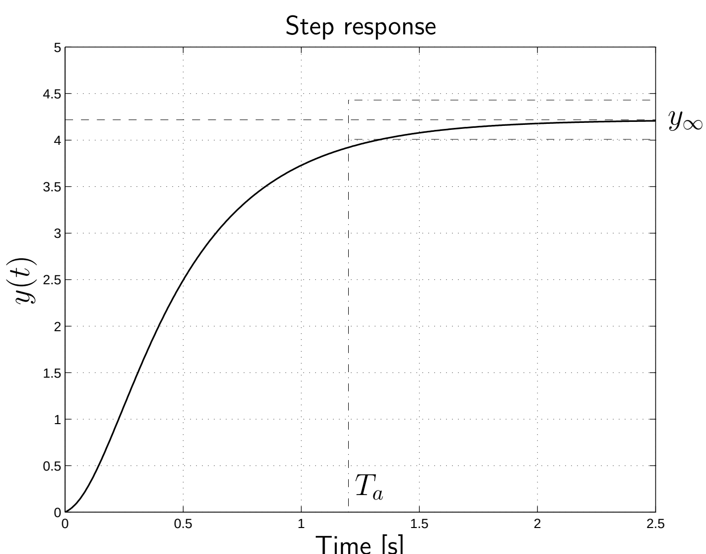
$$G(s) = \frac{(3+0.2\,s)(s^2+60\,s+1800)}{\underbrace{(2+0.8\,s)}_{\text{dominant pole}}(8+0.2\,s)(s^2+16\,s+80)}$$

1) Draw the qualitative time behavior $y(t)$ of the step response of system $G(s)$.

Solution. The system is dominated by a real pole $p = -2.5$. The qualitative time behavior of the system is aperiodic.

2) Calculate the steady-state value y_∞ of the output signal $y(t)$:

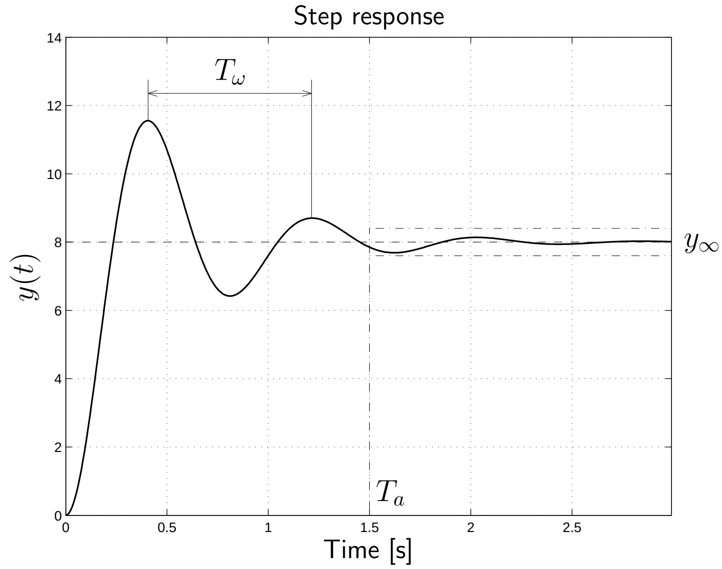
$$y_\infty = G(0) = 4.22.$$



3) Qualitatively estimate the settling time T_a and the period T_ω of the damped oscillation (if present):

$$T_a \simeq \frac{3}{2.5}\ \text{s} = 1.2\ \text{s},$$
$$T_\omega \simeq \nexists.$$

Exercise. The figure shows the output $y(t)$ to the step $x(t) = X_0 = 10$ of a dynamic system $G(s)$ characterized only by 2 stable poles. Calculate:



1) The dominant poles of the system:

$$\sigma = \frac{3}{T_a} \simeq \frac{3}{1.5}, \quad \omega = \frac{2\,\pi}{T_\omega} \simeq \frac{6.28}{0.81},$$

$$p_{1,2} = -\sigma \pm j\,\omega = -2 \pm j\,7.75.$$

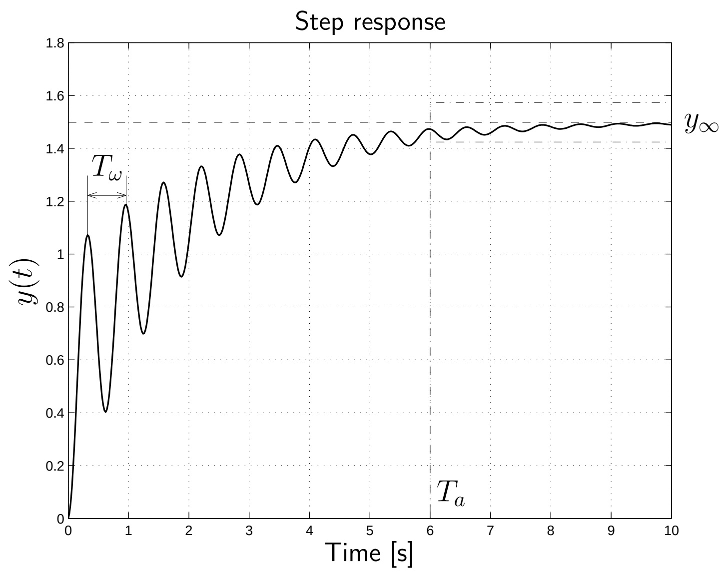
2) The static gain of the system:

$$G_0 = \frac{y_\infty}{X_0} = \frac{8}{10} = 0.8.$$

3) The natural frequency ω_n:

$$\omega_n = \sqrt{\sigma^2 + \omega^2} \simeq 8.$$

Exercise. The figure shows the output $y(t)$ to the unit step of a linear system $G(s)$ characterized by three poles $p_{1,2}$ and p_3 (two complex conjugate e one real) having the same real part. Within the limits of the accuracy of the figure determine:



1) The static gain of the system:

$$G_0 = y_\infty \simeq 1.5.$$

2) The position of the real pole p_3:

$$p_3 = -\sigma = -\frac{3}{T_a} \simeq -0.5$$

3) The imaginary part ω of the conjugated complex poles $p_{1,2}$:

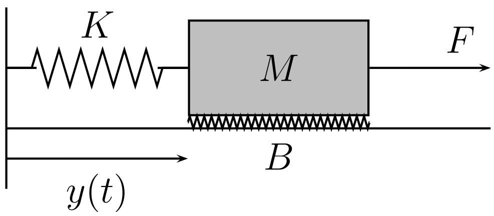
$$\omega = \frac{2\,\pi}{T_\omega} \simeq \frac{6.28}{0.63} \simeq 10.$$

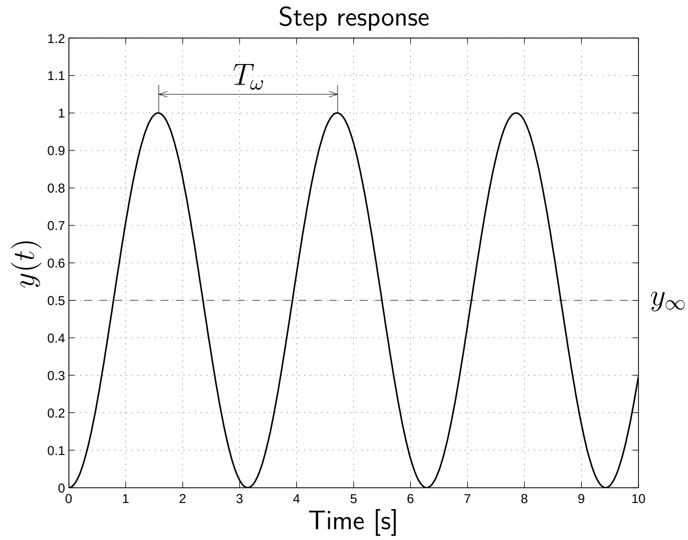
The position of the 3 poles is the following: $p_{1,2} = -0.5 \pm 10\,j$ and $p_3 = -0.5$.

Exercise. Estimate the settling time T_a of the unitary step response of system $G(s)$:

$$G(s) = \frac{(s+37)(s+225)}{(s+350)(10\,s+15)\underbrace{(40\,s+2)}_{\text{dominant pole}}(s^2+2\,s+20)} \qquad \rightarrow \qquad T_a = 60 \text{ s.}$$

Exercise. Let us consider a mass-spring-damper system characterized by the differential equation $M\,\ddot{y}(t) + B\,\dot{y}(t) + K\,y(t) = F(t)$. The output $y(t)$ of the system to a force step $F = 10\,\text{N}$ is shown on the right of the figure.





Within the limits of the accuracy of the figure, determine:

1) The system transfer function $G(s)$ in symbolic form:

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{M\,s^2 + B\,s + K}.$$

2) The frequency ω of the output step response $y(t)$:

$$\omega = \frac{2\pi}{T_\omega} = \frac{2\pi}{3.14} \simeq 2.$$

3) The numerical values of the mass M, the friction coefficient B and the stiffness K are:

$$M = 5, \qquad B = 0, \qquad K = 20.$$

The output step response $y(t)$ clearly shows that the system is simply stable and without dissipations: $B = 0$, $\delta = 0$. Substituting $B = 0$ in the characteristic equation we have:

$$B = 0 \quad \to \quad M\,s^2 + K = 0 \quad \to \quad s_{1,2} = \pm j\sqrt{\frac{K}{M}} = \pm j\,\omega_n = \pm j\,\omega.$$

Being $\delta = 0$, the natural frequency ω_n is equal to frequency ω:

$$\omega_n = \omega = 2.$$

The steady-state value y_∞ of the output signal $y(t)$ is equal to the average value $y_\infty = 0.5$ and it is equal to the product of the input amplitude $F = 10$ and the static gain $G(0)$:

$$y_\infty = F \cdot G(0) \qquad \to \qquad 0.5 = 10 \cdot \frac{1}{K} \qquad \to \qquad K = 20.$$

The value of parameter M can be easily determined by substituting K in the ω expression:

$$\omega = \sqrt{\frac{K}{M}} = 2 \qquad \to \qquad M \simeq \frac{K}{4} = 5.$$