# Step response of a first order dynamic system

- Let us consider the following first order dynamic system:

$$G(s) = \frac{1}{1+\tau\, s}$$

- The system is characterized only by one parameter: the *time constant* $\tau$.
- If $\tau > 0$ the system pole $p$ has negative real part (the system is stable):

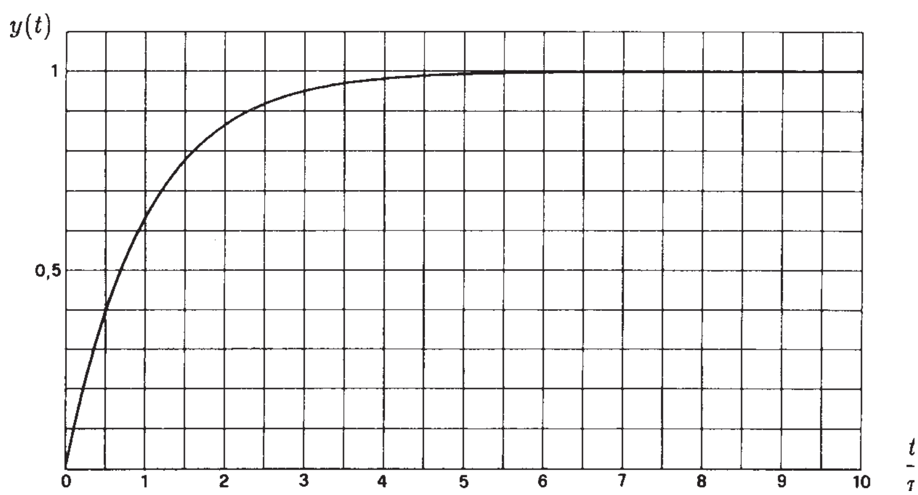
$$p = -\frac{1}{\tau}$$

- The unitary step response of the considered system is the following:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s\,(1+\tau\, s)}\right] = \frac{1}{\tau}\mathcal{L}^{-1}\left[\frac{1}{s\,(s+\frac{1}{\tau})}\right] \\ &= \frac{1}{\tau}\mathcal{L}^{-1}\left[\frac{\tau}{s} - \frac{\tau}{s+\frac{1}{\tau}}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right] = 1 - e^{-\frac{t}{\tau}} \end{aligned}$$

- The time behavior (the time axis is normalized with respect to $\tau$):

$t = \tau \;\rightarrow\; 63,2\%$
$t = 2\tau \;\rightarrow\; 86.5\%$
$\underline{t = 3\tau} \;\rightarrow\; \underline{95.0\%}$
$t = 5\tau \;\rightarrow\; 99.3\%$
$t = 7\tau \;\rightarrow\; 99.9\%$




- Then step response $y(t)$ of a first order dynamic system is always of *aperiodic* type: the final value is reached without overshoot.
- After three time constants the system has reached the 95% of the final value. The settling time $T_a$ of the system is:

$$\boxed{T_a = 3\,\tau = \frac{3}{|p|}}$$

**Example.** Compute the unitary step response $y(t)$ of the following differential equation:

$$a\,\dot{y}(t) + b\,y(t) = c\,x(t)$$

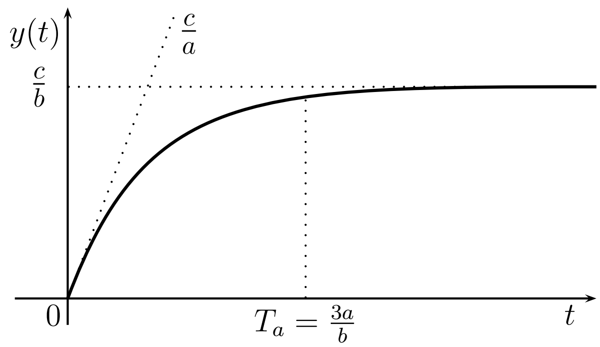
The transfer function $G(s)$ associated to the differential equation is:

$$G(s) = \frac{c}{a\,s+b} \qquad \rightarrow \qquad G(s) = \frac{c}{b}\,\frac{1}{(\tau\,s+1)} \qquad \text{where} \qquad \tau = \frac{a}{b}$$

The step response $y(t)$ of system $G(s)$ is therefore the following:

$$y(t) = \frac{c}{b}\left(1 - e^{-\frac{b}{a}\,t}\right)$$

The time behavior of function $y(t)$ is aperiodic and exponential:



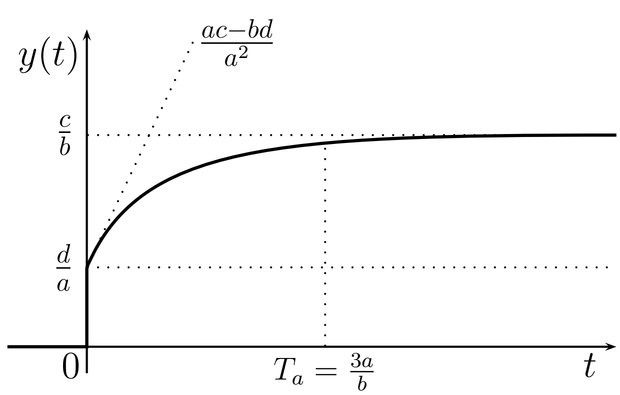

**Example.** Compute the step response $y(t)$ of the following transfer function $G(s)$:

$$G(s) = \frac{d\,s+c}{a\,s+b} \qquad \rightarrow \qquad G(s) = \frac{d}{a} + \left(\frac{c}{b} - \frac{d}{a}\right)\frac{1}{(\tau\,s+1)}$$

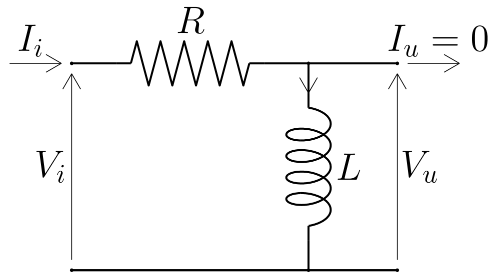
where $\tau = \frac{a}{b}$. The step response $y(t)$ of function $G(s)$ is therefore the following:

$$y(t) = \frac{d}{a} + \frac{ac-bd}{ab}\left(1 - e^{-\frac{b}{a}\,t}\right)$$

The time behavior of function $y(t)$ is aperiodic and exponential:



**Example.** Electric system RL:



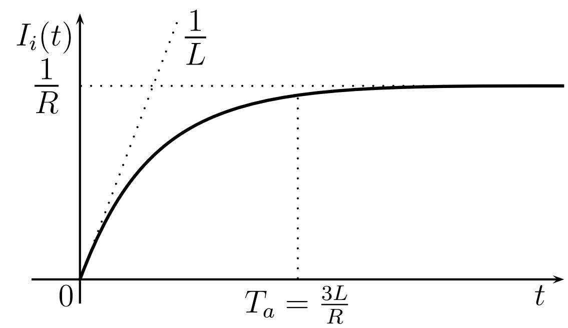

Differential equation:

$$L\,\dot{I}_i(t) + R\,I_i(t) = V_i(t)$$

Transfer function:

$$G(s) = \frac{I_i(s)}{V_i(s)} = \frac{1}{L\,s + R}$$

Step response when $V_i(t) = 1$:



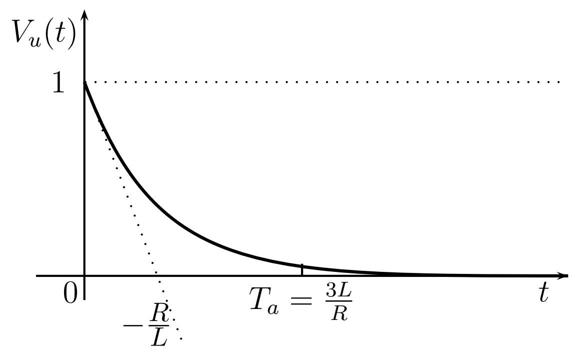

Differential equation:

$$L\dot{V}_u(t) + R\,V_u(t) = L\dot{V}_i(t)$$

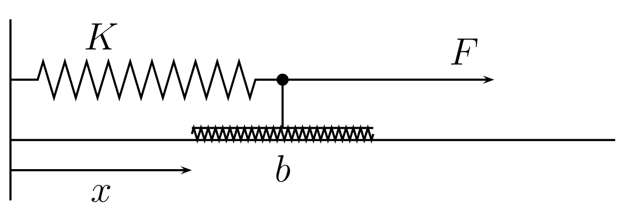
Transfer function:

$$G(s) = \frac{V_u(s)}{V_i(s)} = \frac{L\,s}{L\,s + R} = 1 - \frac{R}{L\,s + R}$$

Step response when $V_i(t) = 1$:




**Example.** Spring-damper system:



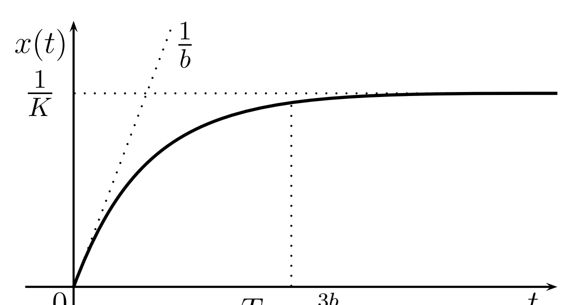

Differential equation:

$$b\,\dot{x}(t) + K\,x(t) = F(t)$$

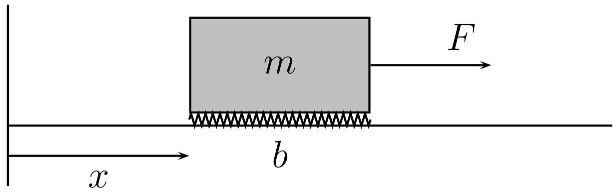
Transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{b\,s + K}$$

Step response when $F(t) = 1$:



**Example.** Mass-damper system:



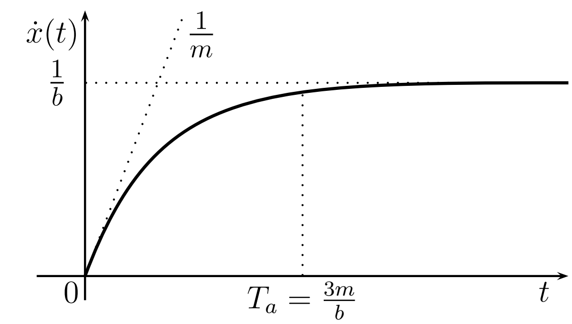

Differential equation:

$$m\ddot{x}(t) + b\,\dot{x}(t) = F(t)$$

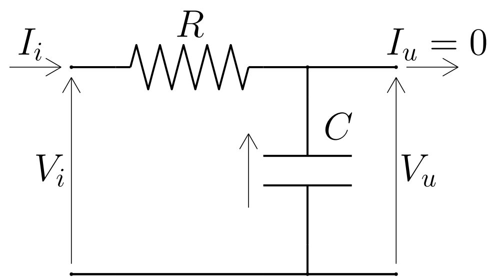
Transfer function:

$$G(s) = \frac{\dot{X}(s)}{F(s)} = \frac{1}{m\,s + b}$$

Step response when $F(t) = 1$:




**Example.** Electric system RC:



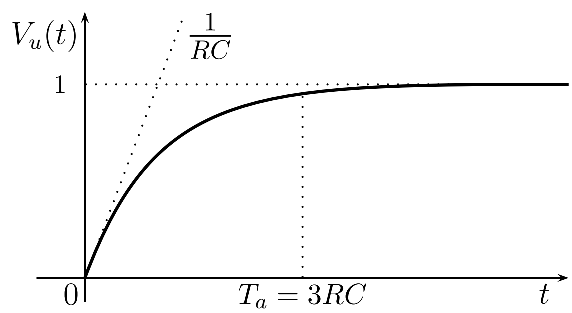

Differential equation:

$$RC\,\dot{V}_u(t) + V_u(t) = V_i(t)$$

Transfer function:

$$G(s) = \frac{V_u(s)}{V_i(s)} = \frac{1}{R\,C\,s + 1}$$

Step response when $V_i(t) = 1$:



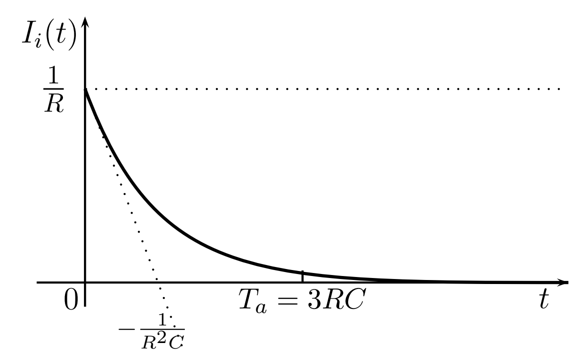

Differential equation:

$$RC\dot{I}_i(t) + I_i(t) = C\dot{V}_i(t)$$

Transfer function:

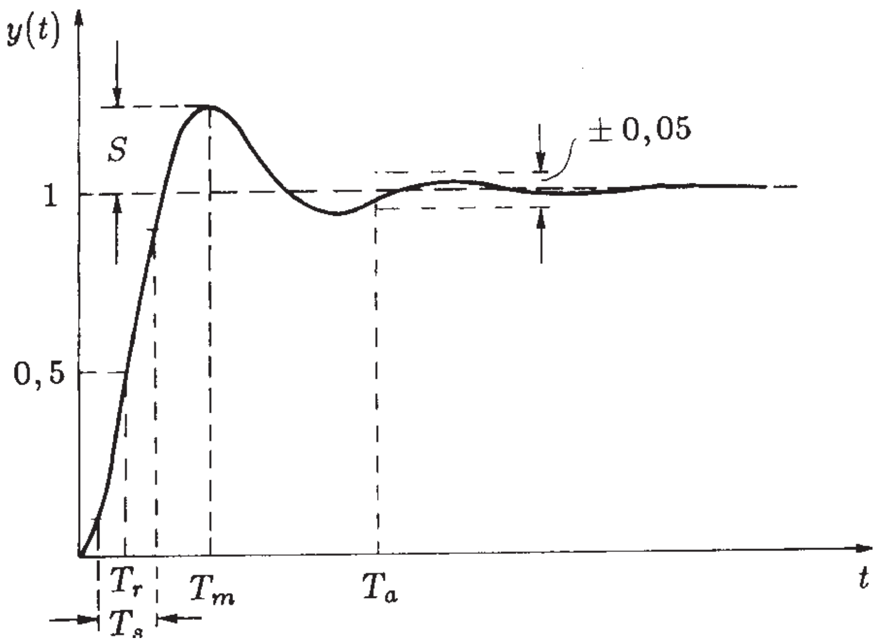
$$G(s) = \frac{I_i(s)}{V_i(s)} = \frac{Cs}{RCs + 1} = \frac{1}{R} - \frac{1}{R(RCs + 1)}$$

Step response when $V_i(t) = 1$:



# Second order dynamic system

- Very often the linear dynamic systems show a step response similar to the time step response of a second order system. This happens for systems having two *dominant poles*, i.e. systems characterized by two complex conjugate poles which are near to the imaginary axis.
- The influence of the dominant poles on the time step response of the system is much higher than that of all the other poles.




- Main parameters which describe the time behavior of the step response:
  1. *Maximum overshoot* $S$: difference between the maximum and the steady-state values of the output signal $y(t)$. This parameter is expressed in % of the final value.
  2. *Delay time* $T_r$: is the time it takes for the system transient to reach the $50$% of the final value.
  3. *Rise time* $T_s$: is the time it takes for the system transient to move from $10$% to $90$% of the final value.
  4. *Settling time* $T_a$: is the time it takes for the system transient to enter and stay within $\pm 5$% of the final value.
  5. *Instant of the maximum overshoot* $T_m$: instant at which the maximum overshoot is reached.

- Typical transfer function of a second order system:

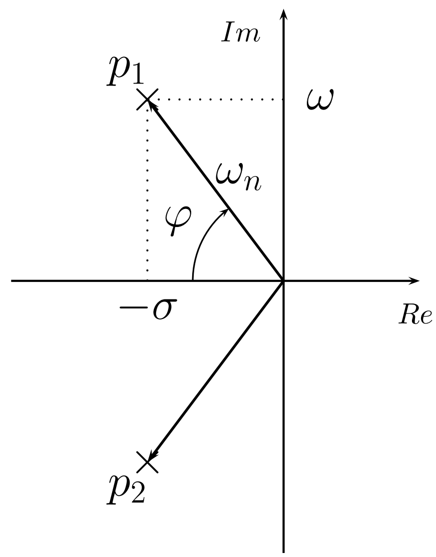
$$G(s) = \frac{1}{1 + 2\,\delta\,\dfrac{s}{\omega_n} + \dfrac{s^2}{\omega_n^2}} = \frac{\omega_n^2}{s^2 + 2\,\delta\,\omega_n\,s + \omega_n^2}$$

where $\delta = \cos\varphi$ is the *damping coefficient* and $\omega_n$ is the *natural frequency* of the system.

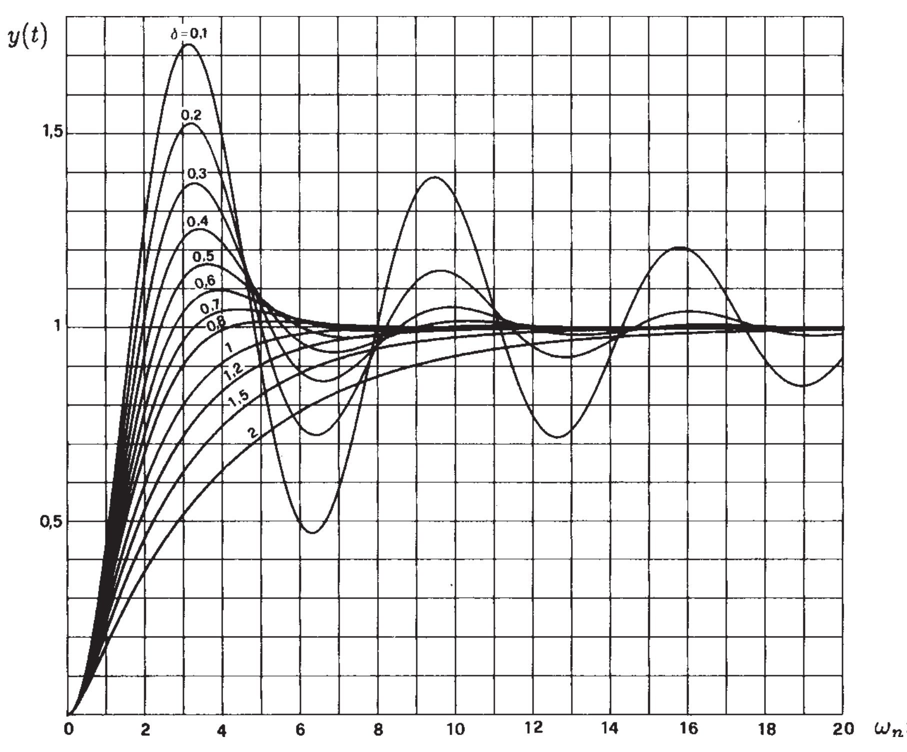
- The unitary step response of the given system is:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{\omega_n^2}{s\,(s^2 + 2\,\delta\,\omega_n\,s + \omega_n^2)}\right] \\ &= 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\,\sin(\omega\,t + \varphi) \end{aligned}$$

$$\begin{aligned} \omega &:= \omega_n\sqrt{1-\delta^2} \\ \sigma &:= \delta\omega_n \\ \varphi &:= \arccos\delta = \arctan\frac{\sqrt{1-\delta^2}}{\delta} \end{aligned}$$




- The time behavior of the time response $y(t)$ as a function of parameter $\delta$ is the following (the time axis is normalized with respect to $\omega_n$):



- When $\delta = 1$, the overshoot $S$ is zero: $y(t)$ asymptotically tends to the final value without exceeding it.

- Let us compute the maximum and minimum points. Let $A = \frac{1}{\sqrt{1-\delta^2}}$ and then compute the time-derivative of function $y(t)$:

$$\frac{dy(t)}{dt} = -A\, e^{-\delta\omega_n t}\omega\, \cos(\omega\, t + \varphi) + A\,\delta\,\omega_n\, e^{-\delta\omega_n t}\, \sin(\omega t + \varphi)$$

When the time-derivative is equal to zero, one obtains:

$$-\omega_n\sqrt{1-\delta^2}\,\cos(\omega t + \varphi) + \delta\,\omega_n\, \sin(\omega t + \varphi) = 0$$

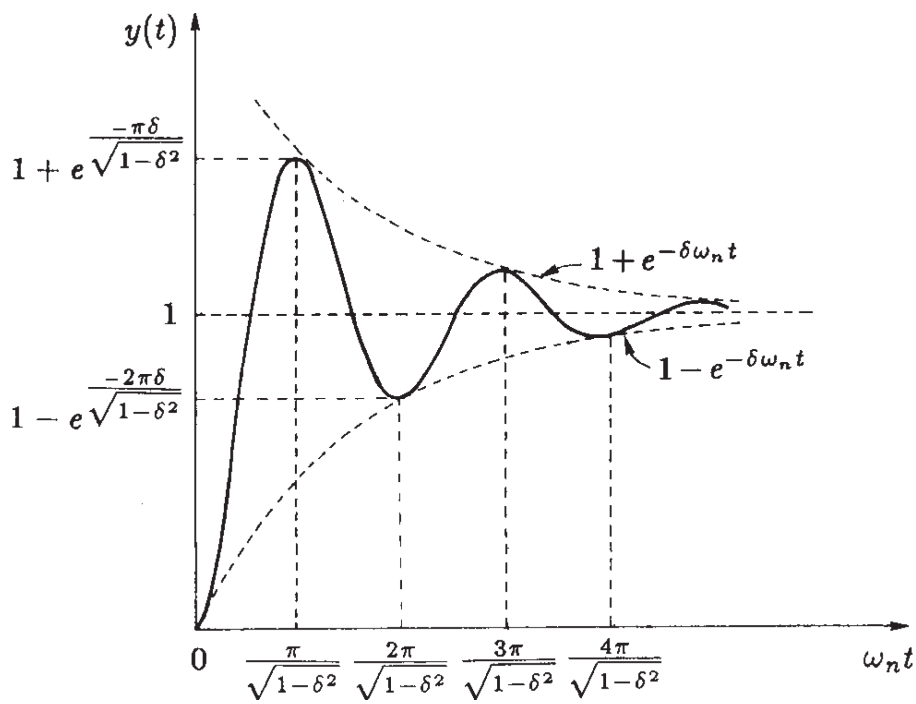
from which it follows that

$$\tan(\omega t + \varphi) = \frac{\sqrt{1-\delta^2}}{\delta} \qquad \leftrightarrow \qquad \omega t = n\,\pi$$

(for $n = 0, 1, \ldots$), that is:

$$\boxed{t = \frac{n\,\pi}{\omega_n\sqrt{1-\delta^2}} = \frac{n\,\pi}{\omega}}$$

- The time behavior of the maximum and minimum points is the following:



- Output values in correspondence of the maximum and minimum points:

$$y(t)\Big|_{\substack{\max\\ \min}} = 1 - \frac{e^{\frac{-n\pi\delta}{\sqrt{1-\delta^2}}}}{\sqrt{1-\delta^2}}\,\sin\left(n\,\pi + \varphi\right)$$

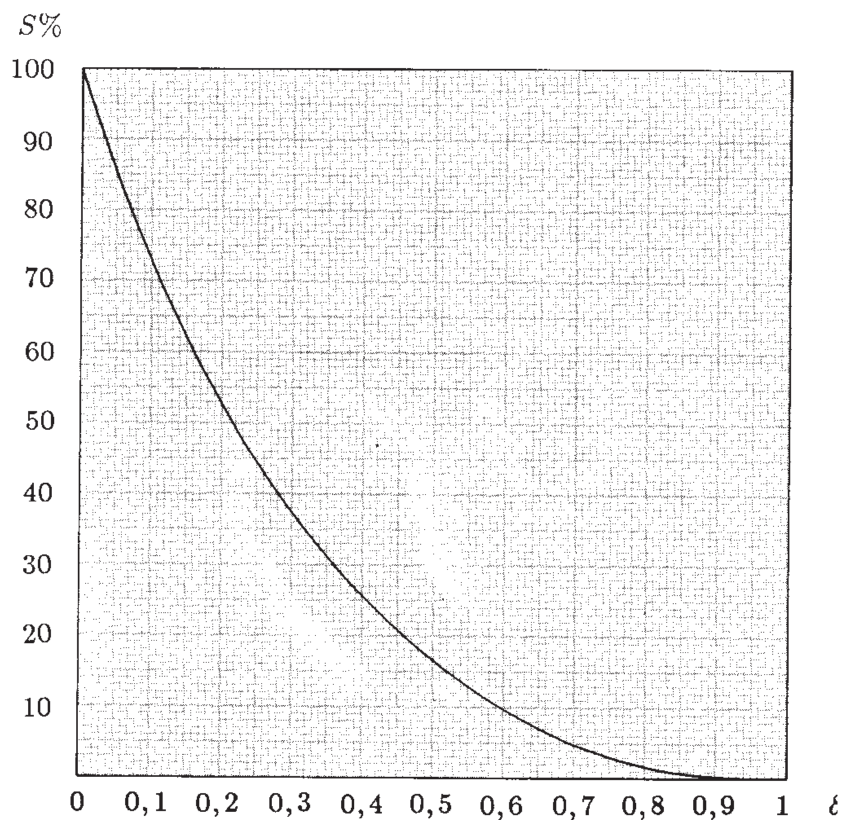
from which it follows that

$$y(t)\Big|_{\substack{\max\\ \min}} = 1 - (-1)^n\, e^{\frac{-n\pi\delta}{\sqrt{1-\delta^2}}}$$

- The maximum overshoot $S$ satisfies the following relation:

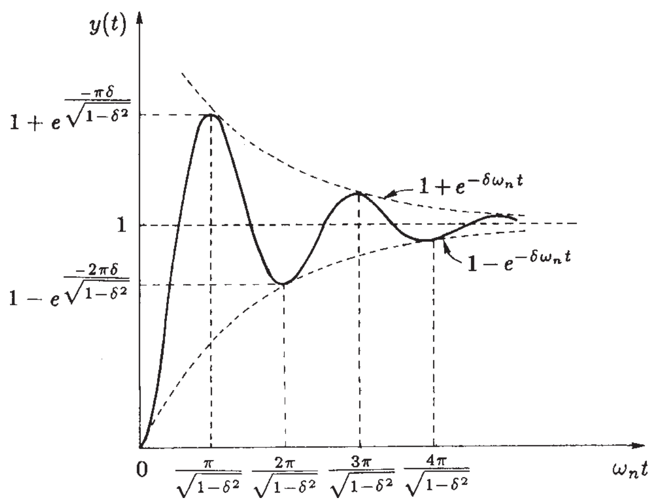
$$S = 100\,\frac{(y_{\max} - y_\infty)}{y_\infty}, \qquad \boxed{S = 100\,e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}}$$

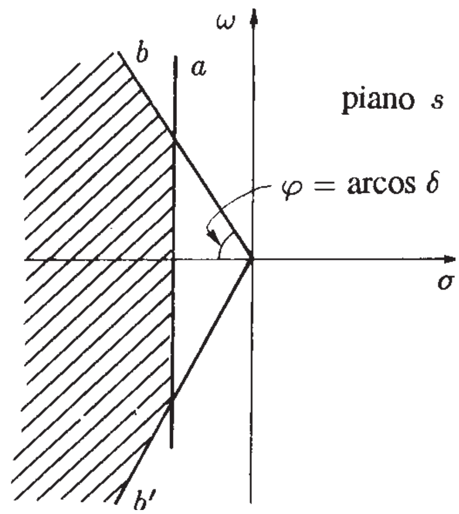
- The maximum overshoot $S$ depends only on the damping coefficient $\delta$, and it is equal to $100\%$ when $\delta = 0$




- The natural frequency $\omega_n$ does not affect the maximum overshoot $S$.

- The maximum overshoot does not exceed an assigned value if the poles of the system are included within the sector delimited by two lines $b$ and $b'$ uniquely determined by the damping coefficient $\delta$.




- An upper bound for the settling time $T_a$ can be obtained from the relation

$$e^{-\delta\omega_n T_a} = 0.05$$

from which it follows

$$\delta\,\omega_n\,T_a = 3, \qquad \text{that is} \qquad \boxed{T_a = \frac{3}{\delta\,\omega_n} = \frac{3}{|\sigma|}}$$

- The settling time is not higher than the assigned value $T_a$ if

$$\delta\,\omega_n \geq \frac{3}{T_a}$$

where $\delta\,\omega_n$ is the modulus of the real part $\sigma$ of the system's poles.

- The constraint on settling time is satisfied if all the poles of the system are located to the left of a vertical line $a$.

- Both specifications, settling time and maximum overshoot, are satisfied if all the poles of the system are located within the dashed area.

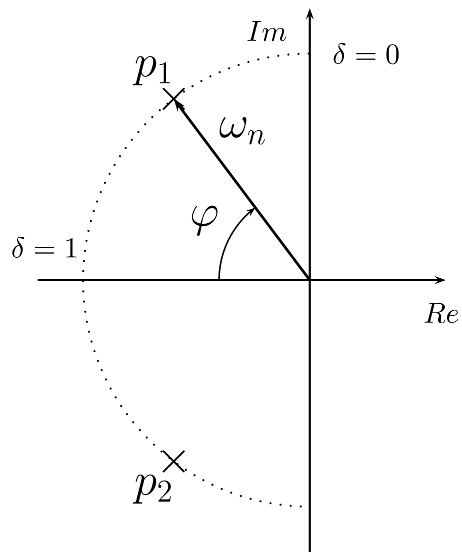
## Constant natural frequency $\omega_n$

- Keeping $\omega_n$ constant and changing $\delta$ means moving the poles of the system along a circumference with radius $\omega_n$:

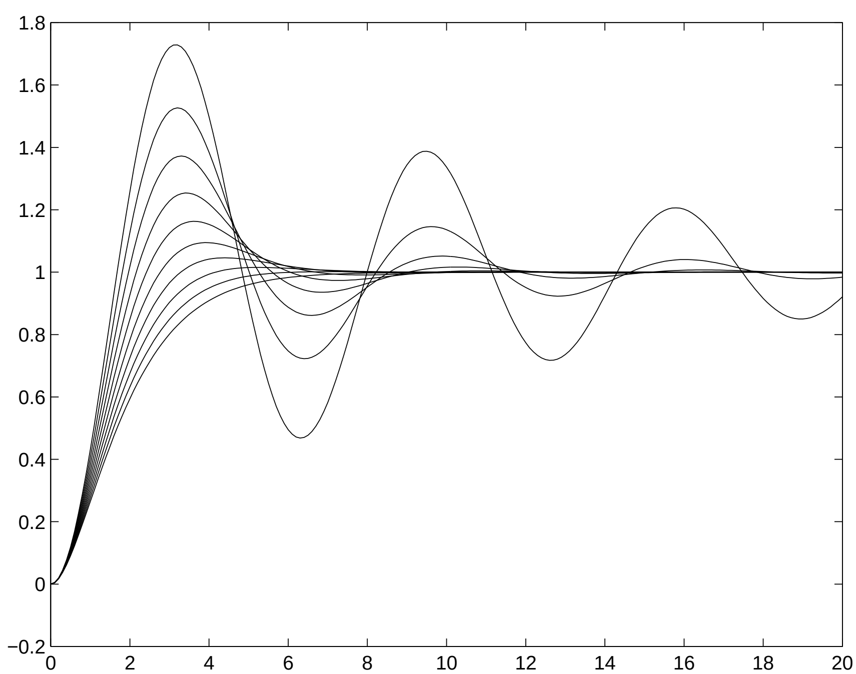
$0 < \delta < 1$

$\frac{\pi}{2} > \varphi > 0$

$\omega_n$ constant




- The unitary step response of system $G(s)$ when $\delta \in [0.1,\ 0.2,\ \ldots,\ 1]$ is shown in the following graph:




- The $\delta$ coefficient directly affects the maximum overshoot $S\%$:

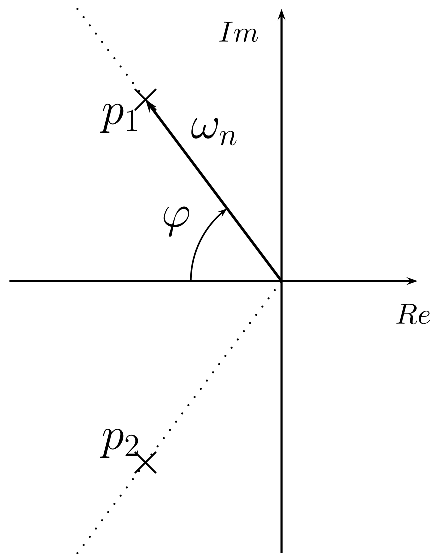
$$S\% = 100\, e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}}$$

## Constant damping coefficient $\delta$

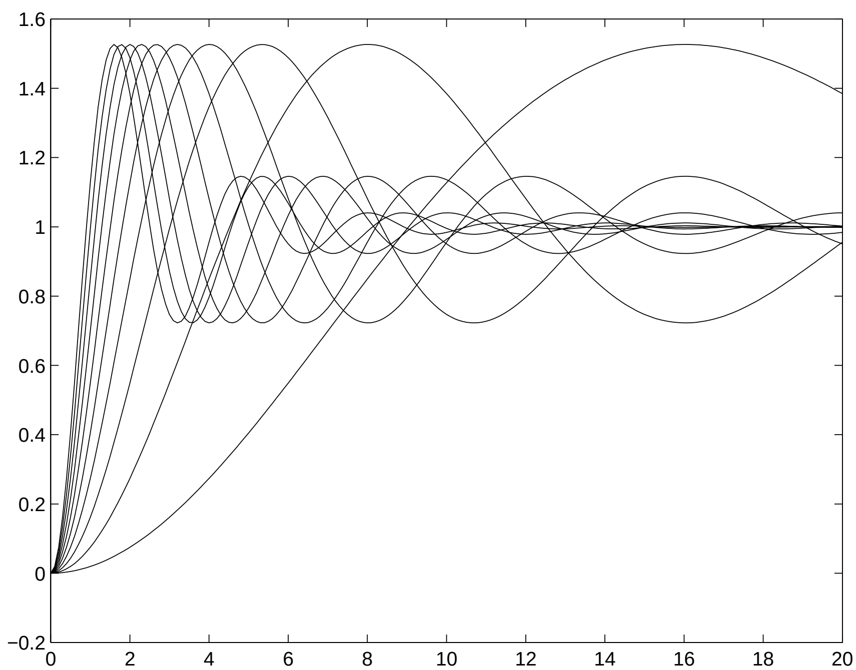
- Keeping coefficient $\delta$ constant and varying $\omega_n$ means moving the poles of the system along a line that forms an angle $\varphi = \arccos \delta$ with the real negative semiaxis:

$\delta$ constant

$\omega_n > 0$



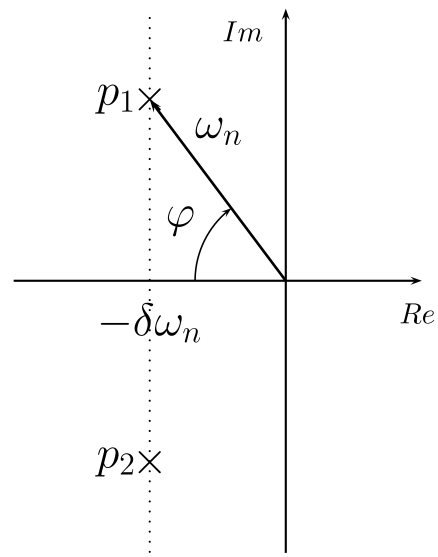

- If you choose $\delta = 0.2$ and change $\omega_n \in [0.2,\ 0.4,\ \ldots,\ 2]$ you get the following time behaviors:



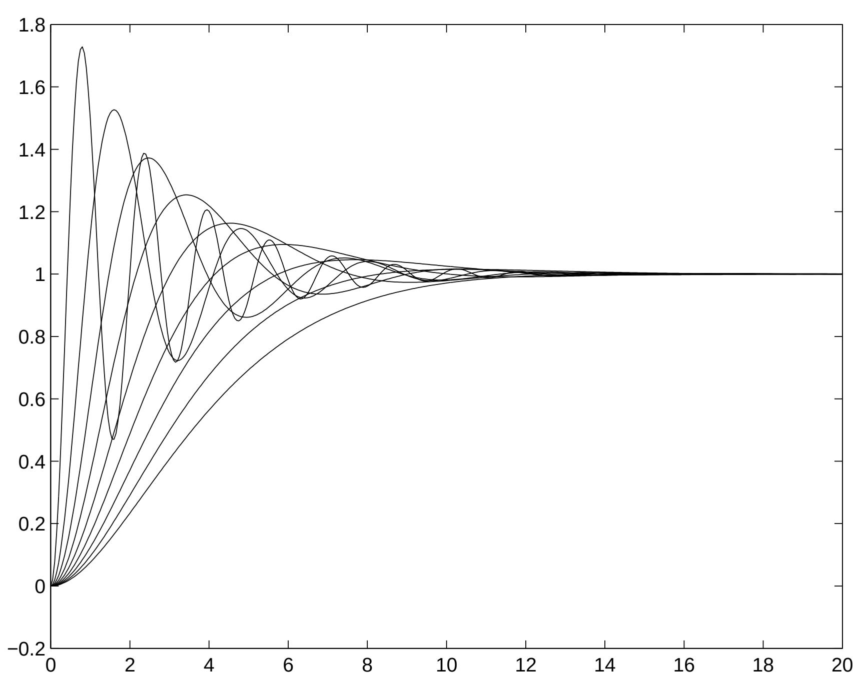
## Constant settling time $T_a$

- Keeping the product $\delta\omega_n$ constant and changing (for example) $\delta$, means moving the poles of the system along a vertical line of abscissa $-\delta\omega_n$:

$\delta\omega_n$ constant




- The time behaviors obtained by varying $\delta \in [0.1, 0.2, 0.3, \ldots, 0.9]$ and keeping constant the product $\delta\omega_n = 0.4$ are the following:




- The settling time $T_a$ (5%) is inversely proportional to $\delta\omega_n$:

$$\boxed{T_a = \frac{3}{\delta\omega_n}}$$

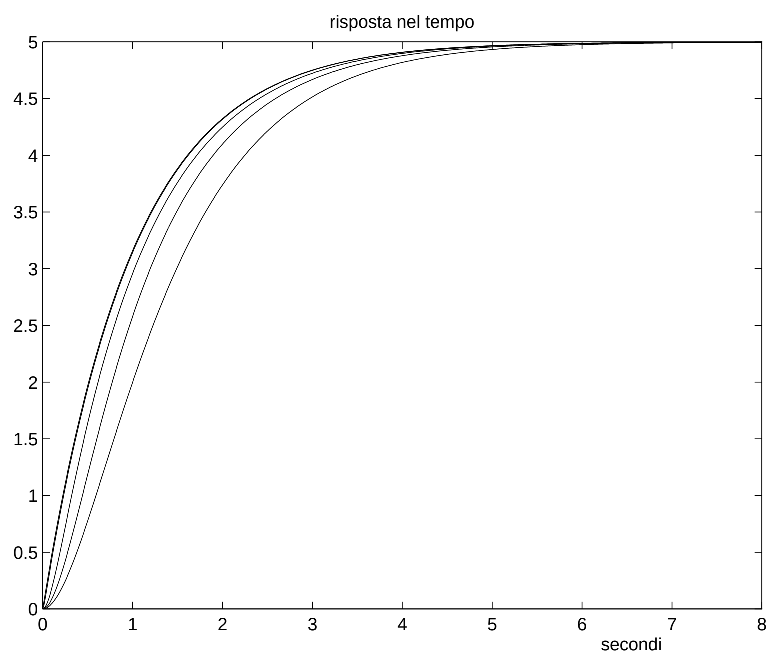
## Systems with a dominant pole

- All the following second order systems have static gain $G_i(0) = 5$ and a real pole in $-1$. They differ for the position of the second pole which is placed, respectively, in $-2$, $-4$, $-10$, $-100$ and $-1000$:

$$G_1(s) = \frac{10}{(s+1)(s+2)}, \quad G_2(s) = \frac{20}{(s+1)(s+4)}, \quad G_3(s) = \frac{50}{(s+1)(s+10)}$$

$$G_4(s) = \frac{500}{(s+1)(s+100)}, \quad G_5(s) = \frac{5000}{(s+1)(s+1000)}, \quad \left[G(s) = \frac{5}{(s+1)}\right]$$

- The unitary step response of these systems is the following:




- In the graph the slowest time behavior is the one related to system $G_1(s)$, the fastest is the one related to system $G_5(s)$.
- For stable systems, the **dominant pole** is the pole that is closest to the imaginary axis.
- The time response of the system changes "little" when the non-dominant poles have a negative real part bigger than the "dominant" pole.
- The poles that are "10 times farer" from to the dominant pole have little influence on the time response of the system.

## Systems with dominant poles

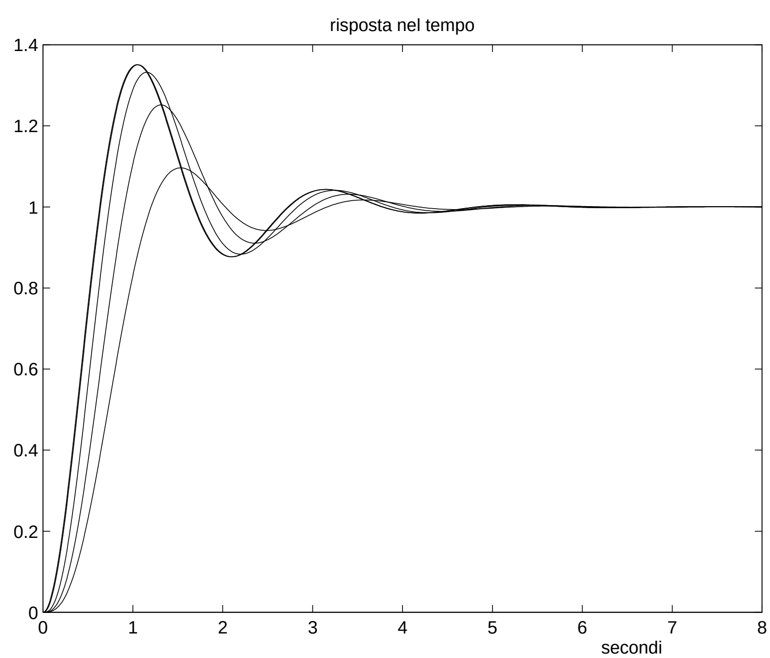
- The same considerations also apply to systems characterized by a couple of complex conjugate dominant poles.
- We define "**dominant poles**" of an asymptotically stable system the two complex conjugate poles that are closer to the imaginary axis than any other pole of the system.
- The unitary step responses of the following systems

$$G_1(s) = \frac{10}{[(s+1)^2+3^2](1+\frac{s}{2})}, \quad G_2(s) = \frac{10}{[(s+1)^2+3^2](1+\frac{s}{4})}$$

$$G_3(s) = \frac{10}{[(s+1)^2+3^2](1+\frac{s}{10})}, \quad G_4(s) = \frac{10}{[(s+1)^2+3^2](1+\frac{s}{100})}$$

$$G_5(s) = \frac{10}{[(s+1)^2+3^2](1+\frac{s}{1000})}, \qquad \left[G(s) = \frac{10}{[(s+1)^2+3^2]}\right]$$

are shown in the following graph:




- Also in this case, the poles that are "10 times farer" than the pair of "dominating poles" have little influence on the time response of the system.

## Second-order dynamic system

- Any second-order dynamic system without zeros:

$$G(s) = \frac{c}{s^2 + a\,s + b}$$

can be rewritten as follows:

$$G(s) = K\,\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

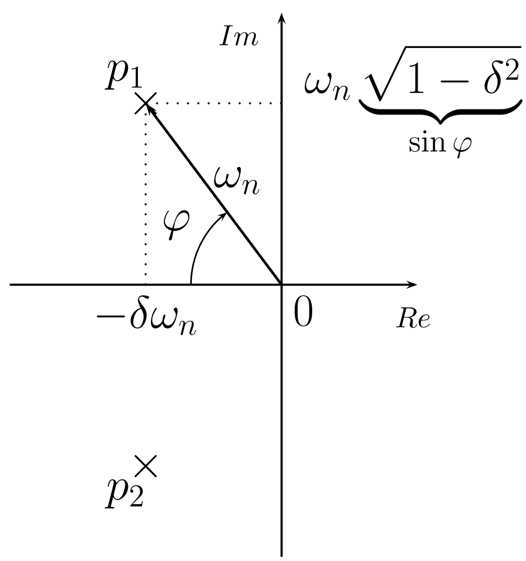
- The natural frequency $\omega_n$, the damping coefficient $\delta$ and the static gain $K$ can be obtainedL as follows:

$$\boxed{\omega_n = \sqrt{b}, \qquad \delta = \frac{a}{2\sqrt{b}}}, \qquad K = \frac{c}{b} = G(s)|_{s\to 0}$$

- The geometric meaning of these parameters on the complex plane is the following:

$$\begin{aligned} p_{1,2} &= -\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2} \\ &= -\sigma \pm j\omega \end{aligned}$$

$$\begin{aligned} \delta &= \cos\varphi \\ \sigma &= \delta\,\omega_n \\ \omega &= \omega_n\sqrt{1-\delta^2} \\ \omega_n &= \sqrt{\sigma^2+\omega^2} = |p_1| = |p_2| \end{aligned}$$




- On the complex plane $s$ the *natural frequency* $\omega_n$ is the distance of the conjugated complex poles $p_{1,2}$ from the origin.

- The *damping coefficient* $\delta$ is equal to the cosine of the $\varphi$ angle that the $\overline{p_1 0}$ segment forms with the negative semiaxis.