

Step response of a first order dynamic system

- Let us consider the following first order dynamic system:

$$G(s) = \frac{1}{1 + \tau s}$$

- The system is characterized only by one parameter: the *time constant* τ .
- If $\tau > 0$ the system pole p has negative real part (the system is stable):

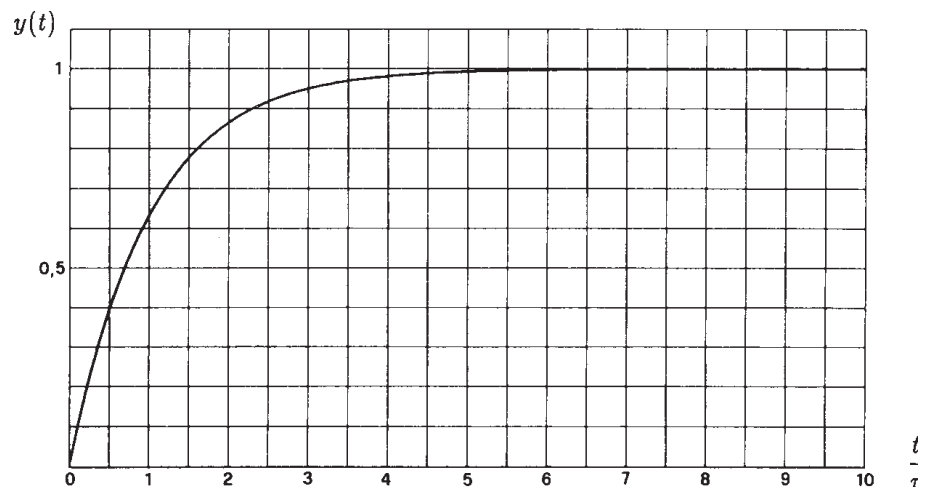
$$p = -\frac{1}{\tau}$$

- The unitary step response of the considered system is the following:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left[\frac{1}{s(1 + \tau s)} \right] = \frac{1}{\tau} \mathcal{L}^{-1} \left[\frac{1}{s(s + \frac{1}{\tau})} \right] \\ &= \frac{1}{\tau} \mathcal{L}^{-1} \left[\frac{\tau}{s} - \frac{\tau}{s + \frac{1}{\tau}} \right] = \mathcal{L}^{-1} \left[\frac{1}{s} - \frac{1}{s + \frac{1}{\tau}} \right] = 1 - e^{-\frac{t}{\tau}} \end{aligned}$$

- The time behavior (the time axis is normalized with respect to τ):

$t = \tau$	→	63,2%
$t = 2\tau$	→	86.5%
$t = 3\tau$	→	95.0%
$t = 5\tau$	→	99.3%
$t = 7\tau$	→	99.9%



- Then step response $y(t)$ of a first order dynamic system is always of *aperiodic* type: the final value is reached without overshoot.
- After three time constants the system has reached the 95% of the final value. The settling time T_a of the system is:

$$T_a = 3\tau = \frac{3}{|p|}$$

Example. Compute the unitary step response $y(t)$ of the following differential equation:

$$a \dot{y}(t) + b y(t) = c x(t)$$

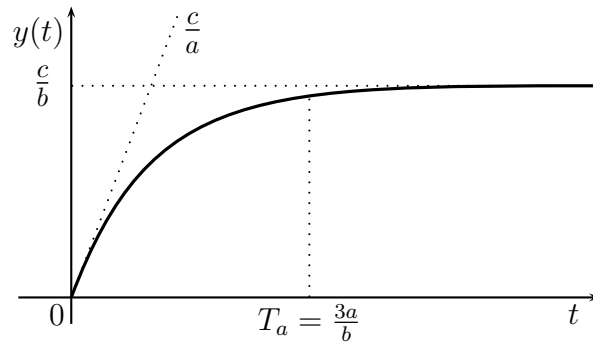
The transfer function $G(s)$ associated to the differential equation is:

$$G(s) = \frac{c}{a s + b} \quad \rightarrow \quad G(s) = \frac{c}{b} \frac{1}{\left(\frac{a}{b} s + 1\right)} \quad \text{where} \quad \tau = \frac{a}{b}$$

The step response $y(t)$ of system $G(s)$ is therefore the following:

$$y(t) = \frac{c}{b} \left(1 - e^{-\frac{b}{a} t}\right)$$

The time behavior of function $y(t)$ is aperiodic and exponential:



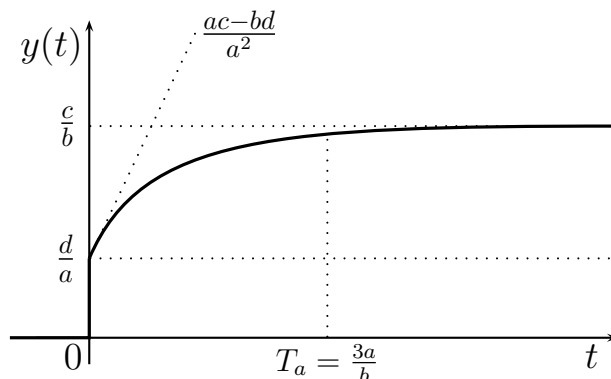
Example. Compute the step response $y(t)$ of the following transfer function $G(s)$:

$$G(s) = \frac{d s + c}{a s + b} \quad \rightarrow \quad G(s) = \frac{d}{a} + \left(\frac{c}{b} - \frac{d}{a}\right) \frac{1}{\left(\frac{a}{b} s + 1\right)}$$

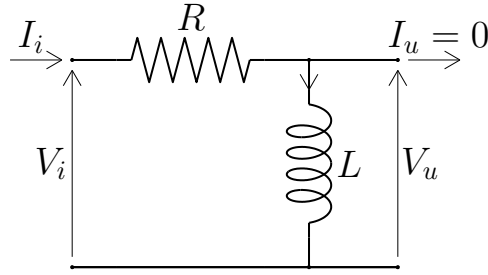
where $\tau = \frac{a}{b}$. The step response $y(t)$ of function $G(s)$ is therefore the following:

$$y(t) = \frac{d}{a} + \frac{ac - bd}{ab} \left(1 - e^{-\frac{b}{a} t}\right)$$

The time behavior of function $y(t)$ is aperiodic and exponential:



Example. Electric system RL:



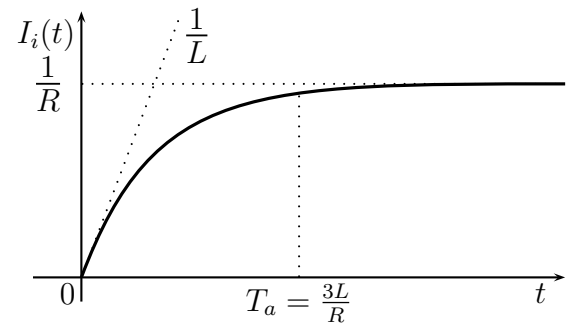
Differential equation:

$$L \dot{I}_i(t) + R I_i(t) = V_i(t)$$

Transfer function:

$$G(s) = \frac{I_i(s)}{V_i(s)} = \frac{1}{Ls + R}$$

Step response when $V_i(t) = 1$:



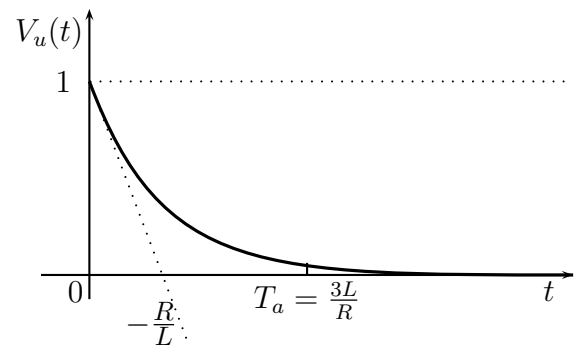
Differential equation:

$$L \dot{V}_u(t) + R V_u(t) = L \dot{V}_i(t)$$

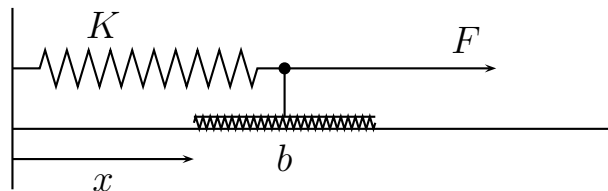
Transfer function:

$$G(s) = \frac{V_u(s)}{V_i(s)} = \frac{Ls}{Ls + R} = 1 - \frac{R}{Ls + R}$$

Step response when $V_i(t) = 1$:



Example. Spring-damper system:



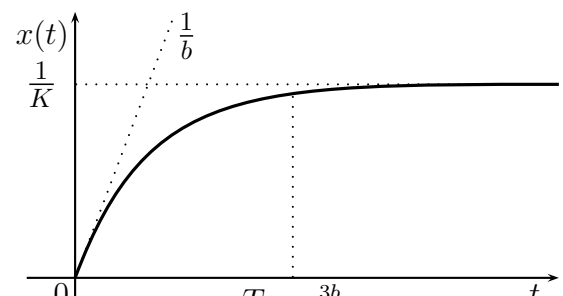
Differential equation:

$$b \dot{x}(t) + K x(t) = F(t)$$

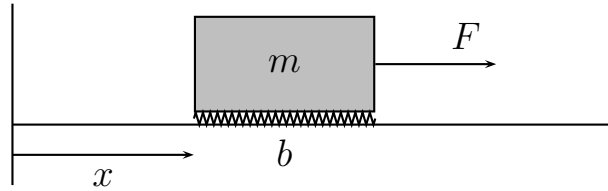
Transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{bs + K}$$

Step response when $F(t) = 1$:



Example. Mass-damper system:



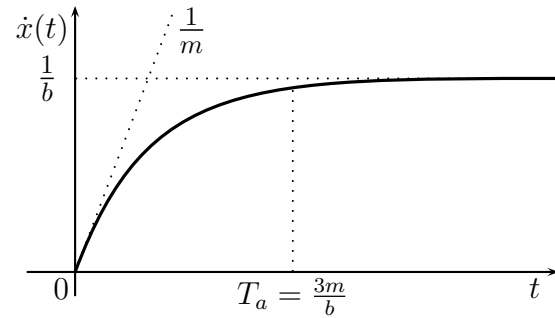
Differential equation:

$$m\ddot{x}(t) + b\dot{x}(t) = F(t)$$

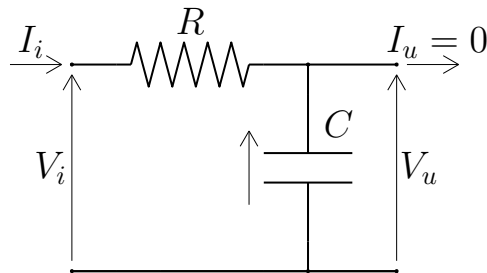
Transfer function:

$$G(s) = \frac{\dot{X}(s)}{F(s)} = \frac{1}{ms + b}$$

Step response when $F(t) = 1$:



Example. Electric system RC:



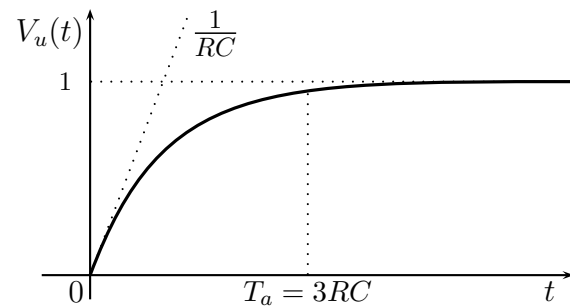
Differential equation:

$$RC\dot{V}_u(t) + V_u(t) = V_i(t)$$

Transfer function:

$$G(s) = \frac{V_u(s)}{V_i(s)} = \frac{1}{RCs + 1}$$

Step response when $V_i(t) = 1$:



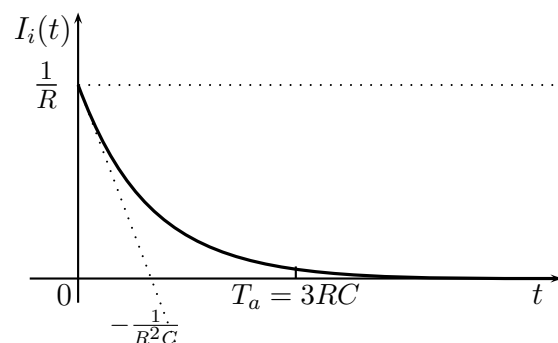
Differential equation:

$$RC\dot{I}_i(t) + I_i(t) = C\dot{V}_i(t)$$

Transfer function:

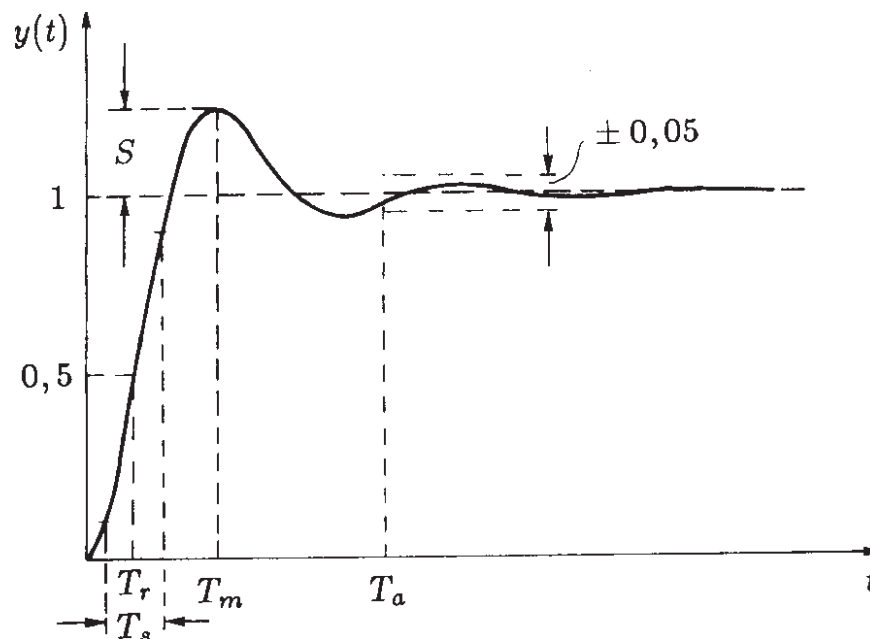
$$G(s) = \frac{I_i(s)}{V_i(s)} = \frac{Cs}{RCs + 1} = \frac{1}{R} - \frac{1}{R(RCs + 1)}$$

Step response when $V_i(t) = 1$:



Second order dynamic system

- Very often the linear dynamic systems show a step response similar to the time step response of a second order system. This happens for systems having two *dominant poles*, i.e. systems characterized by two complex conjugate poles which are near to the imaginary axis.
- The influence of the dominant poles on the time step response of the system is much higher than that of all the other poles.



- Main parameters which describe the time behavior of the step response:
 1. *Maximum overshoot* S : difference between the maximum and the steady-state values of the output signal $y(t)$. This parameter is expressed in % of the final value.
 2. *Delay time* T_r : is the time it takes for the system transient to reach the 50% of the final value.
 3. *Rise time* T_s : is the time it takes for the system transient to move from 10% to 90% of the final value.
 4. *Settling time* T_a : is the time it takes for the system transient to enter and stay within $\pm 5\%$ of the final value.
 5. *Instant of the maximum overshoot* T_m : instant at which the maximum overshoot is reached.

- Typical transfer function of a second order system:

$$G(s) = \frac{1}{1 + 2\delta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

where $\delta = \cos \varphi$ is the *damping coefficient* and ω_n is the *natural frequency* of the system.

- The unitary step response of the given system is:

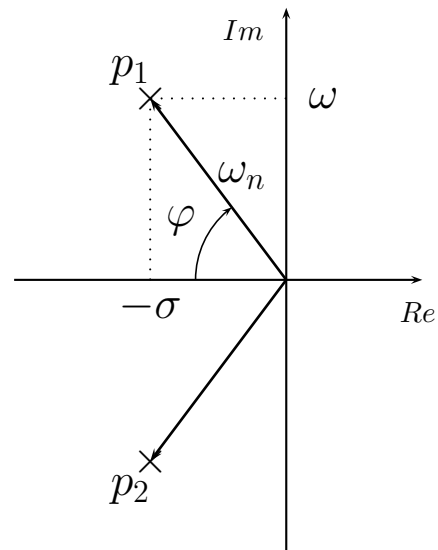
$$y(t) = \mathcal{L}^{-1} \left[\frac{\omega_n^2}{s(s^2 + 2\delta\omega_n s + \omega_n^2)} \right]$$

$$= 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \sin(\omega t + \varphi)$$

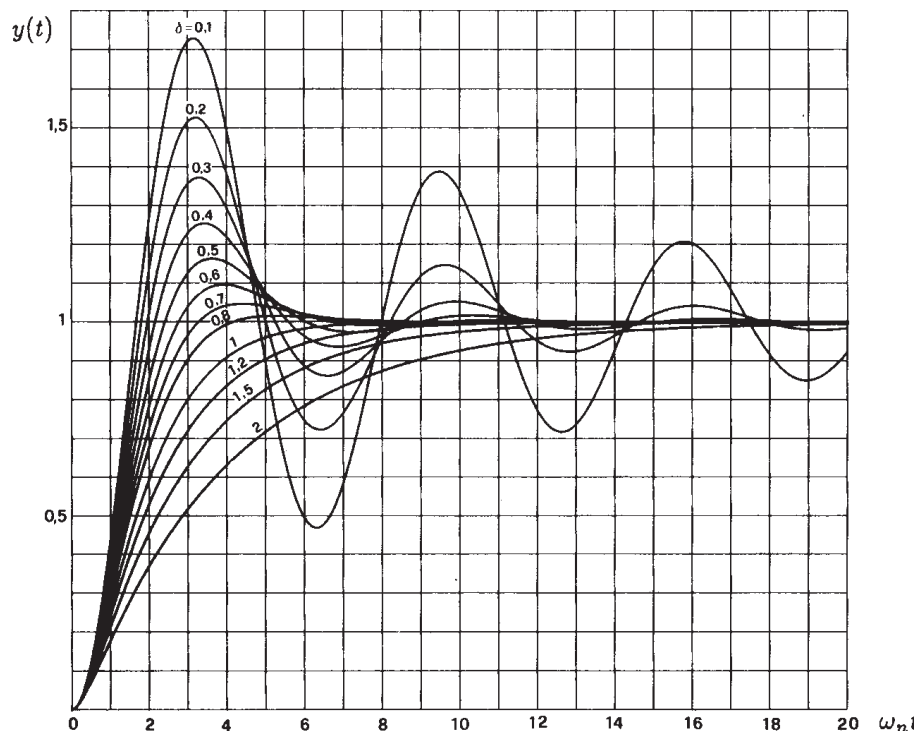
$$\omega := \omega_n \sqrt{1-\delta^2}$$

$$\sigma := \delta\omega_n$$

$$\varphi := \arccos \delta = \arctan \frac{\sqrt{1-\delta^2}}{\delta}$$



- The time behavior of the time response $y(t)$ as a function of parameter δ is the following (the time axis is normalized with respect to ω_n):



- When $\delta = 1$, the overshoot S is zero: $y(t)$ asymptotically tends to the final value without exceeding it.
- Let us compute the maximum and minimum points. Let $A = \frac{1}{\sqrt{1-\delta^2}}$ and then compute the time-derivative of function $y(t)$:

$$\frac{dy(t)}{dt} = -A e^{-\delta\omega_n t} \omega \cos(\omega t + \varphi) + A \delta \omega_n e^{-\delta\omega_n t} \sin(\omega t + \varphi)$$

When the time-derivative is equal to zero, one obtains:

$$-\omega_n \sqrt{1-\delta^2} \cos(\omega t + \varphi) + \delta \omega_n \sin(\omega t + \varphi) = 0$$

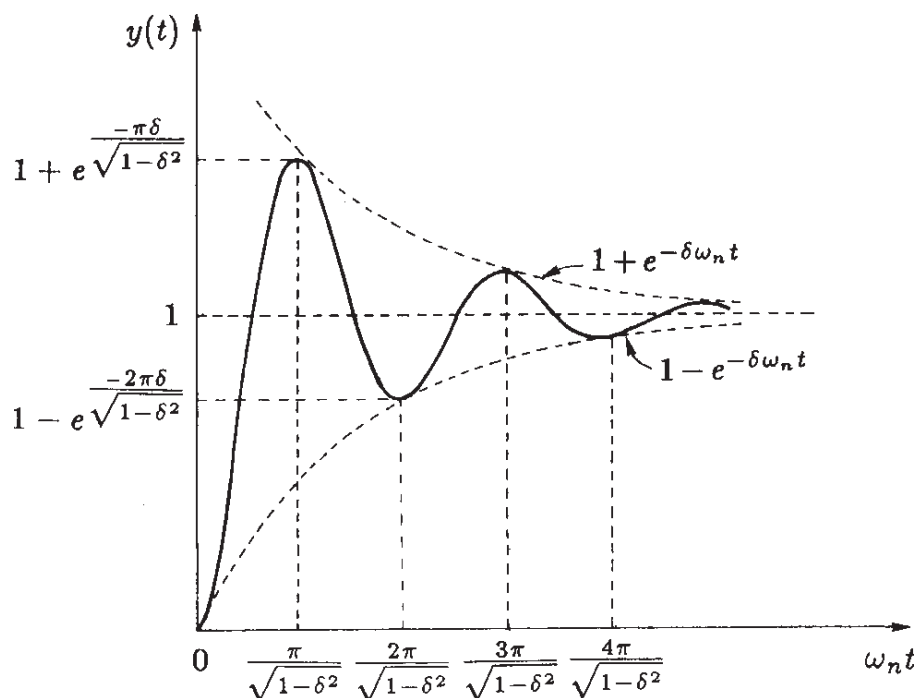
from which it follows that

$$\tan(\omega t + \varphi) = \frac{\sqrt{1-\delta^2}}{\delta} \quad \Leftrightarrow \quad \omega t = n\pi$$

(for $n = 0, 1, \dots$), that is:

$$t = \frac{n\pi}{\omega_n \sqrt{1-\delta^2}} = \frac{n\pi}{\omega}$$

- The time behavior of the maximum and minimum points is the following:



- Output values in correspondence of the maximum and minimum points:

$$y(t) \Big|_{\substack{\text{max} \\ \text{min}}} = 1 - \frac{e^{\frac{-n\pi\delta}{\sqrt{1-\delta^2}}}}{\sqrt{1-\delta^2}} \sin(n\pi + \varphi)$$

from which it follows that

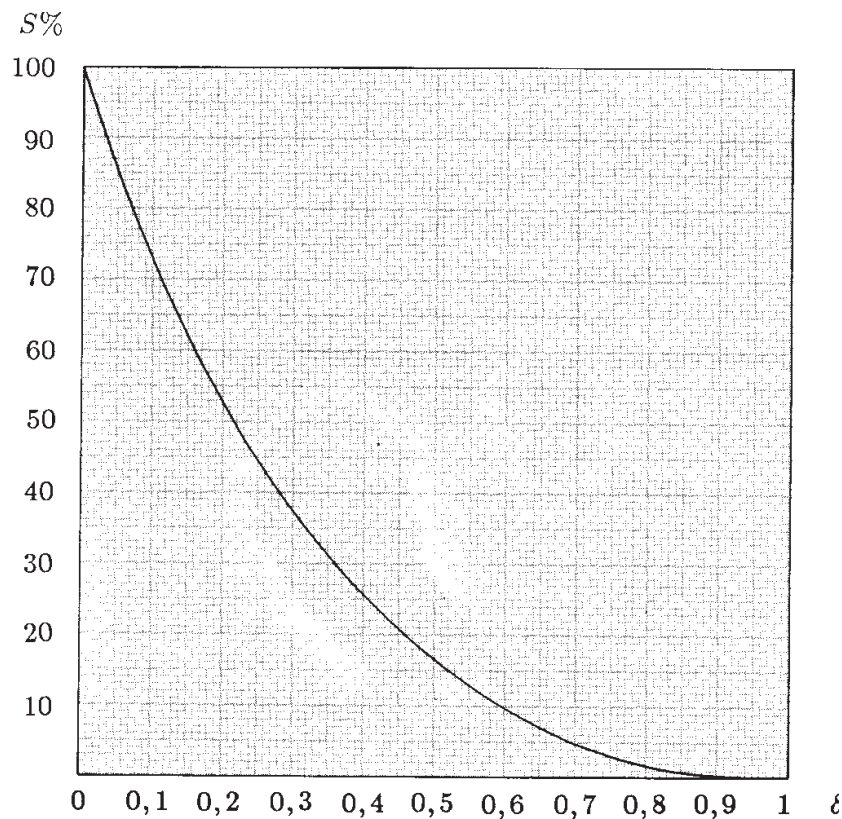
$$y(t) \Big|_{\substack{\text{max} \\ \text{min}}} = 1 - (-1)^n e^{\frac{-n\pi\delta}{\sqrt{1-\delta^2}}}$$

- The maximum overshoot S satisfies the following relation:

$$S = 100 \frac{(y_{\text{max}} - y_{\infty})}{y_{\infty}},$$

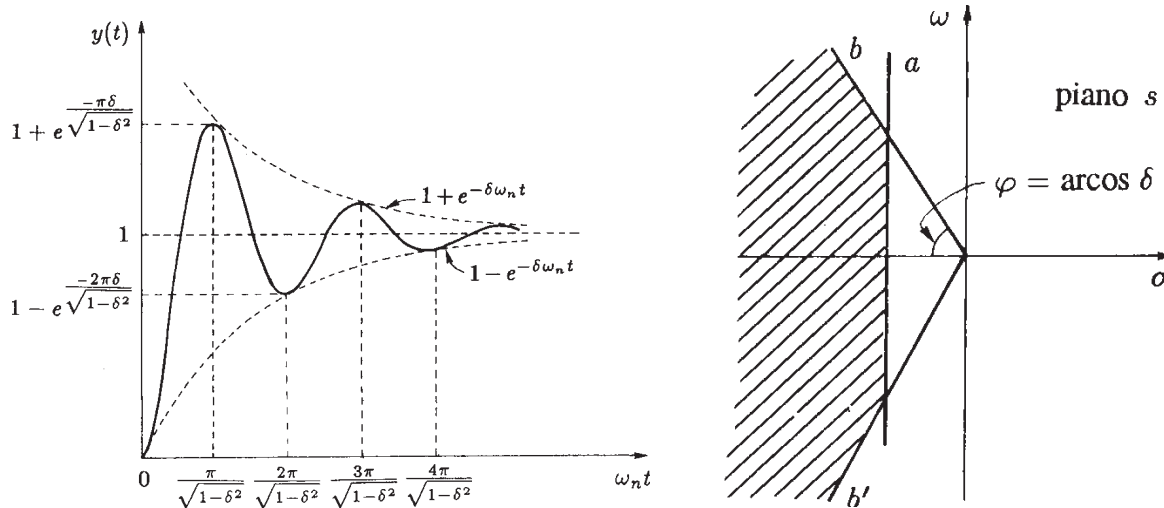
$$S = 100 e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}$$

- The maximum overshoot S depends only on the damping coefficient δ , and it is equal to 100% when $\delta = 0$



- The natural frequency ω_n does not affect the maximum overshoot S .

- The maximum overshoot does not exceed an assigned value if the poles of the system are included within the sector delimited by two lines b and b' uniquely determined by the damping coefficient δ .



- An upper bound for the settling time T_a can be obtained from the relation

$$e^{-\delta\omega_n T_a} = 0.05$$

from which it follows

$$\delta\omega_n T_a = 3, \quad \text{that is}$$

$$T_a = \frac{3}{\delta\omega_n} = \frac{3}{|\sigma|}$$

- The settling time is not higher than the assigned value T_a if

$$\delta\omega_n \geq \frac{3}{T_a}$$

where $\delta\omega_n$ is the modulus of the real part σ of the system's poles.

- The constraint on settling time is satisfied if all the poles of the system are located to the left of a vertical line a .
- Both specifications, settling time and maximum overshoot, are satisfied if all the poles of the system are located within the dashed area.

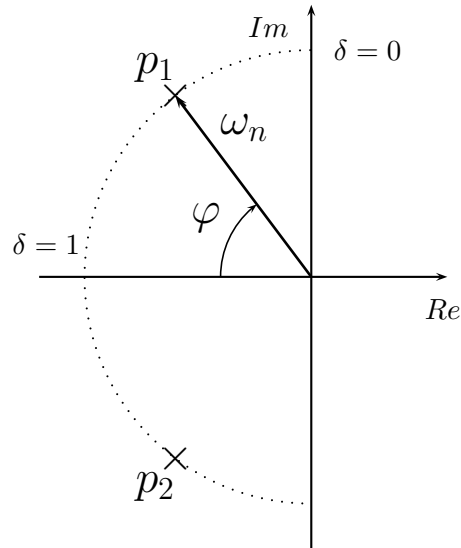
Constant natural frequency ω_n

- Keeping ω_n constant and changing δ means moving the poles of the system along a circumference with radius ω_n :

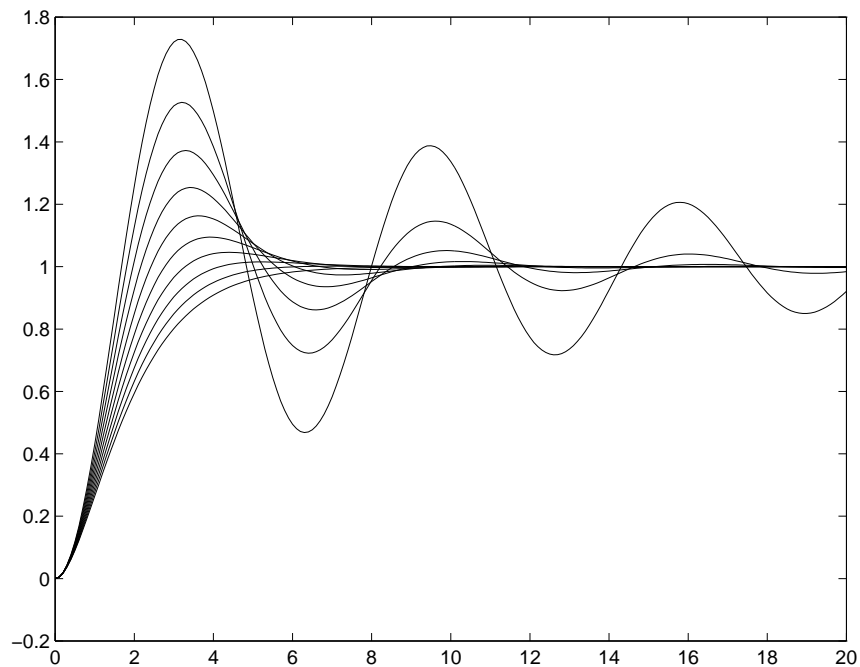
$$0 < \delta < 1$$

$$\frac{\pi}{2} > \varphi > 0$$

ω_n constant



- The unitary step response of system $G(s)$ when $\delta \in [0.1, 0.2, \dots, 1]$ is shown in the following graph:



- The δ coefficient directly affects the maximum overshoot $S\%$:

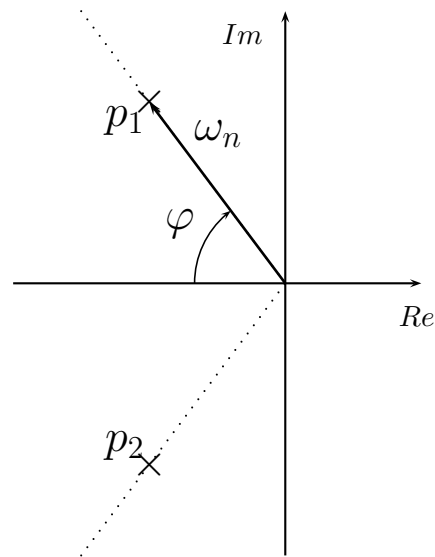
$$S\% = 100 e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}}$$

Constant damping coefficient δ

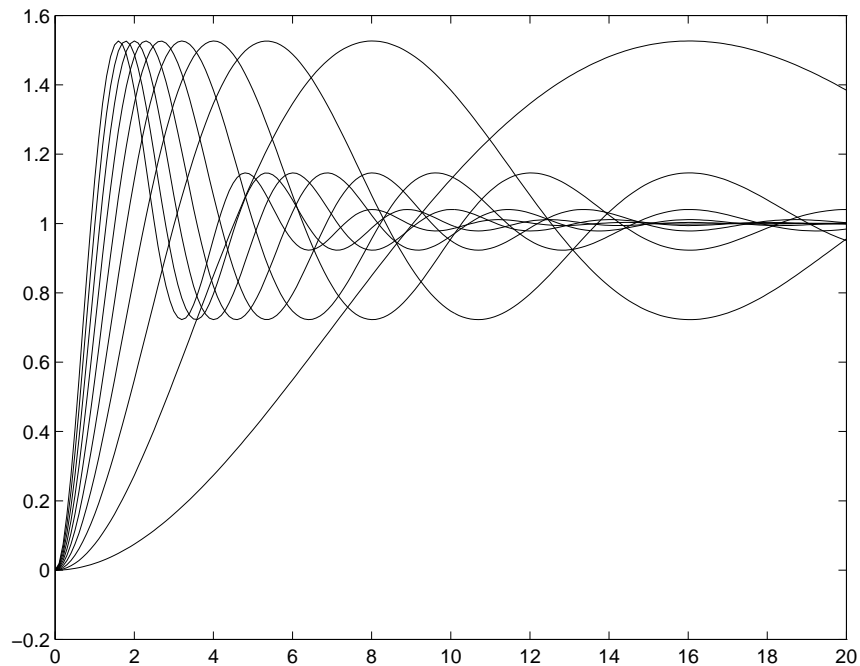
- Keeping coefficient δ constant and varying ω_n means moving the poles of the system along a line that forms an angle $\varphi = \arccos \delta$ with the real negative semiaxis:

δ constant

$\omega_n > 0$

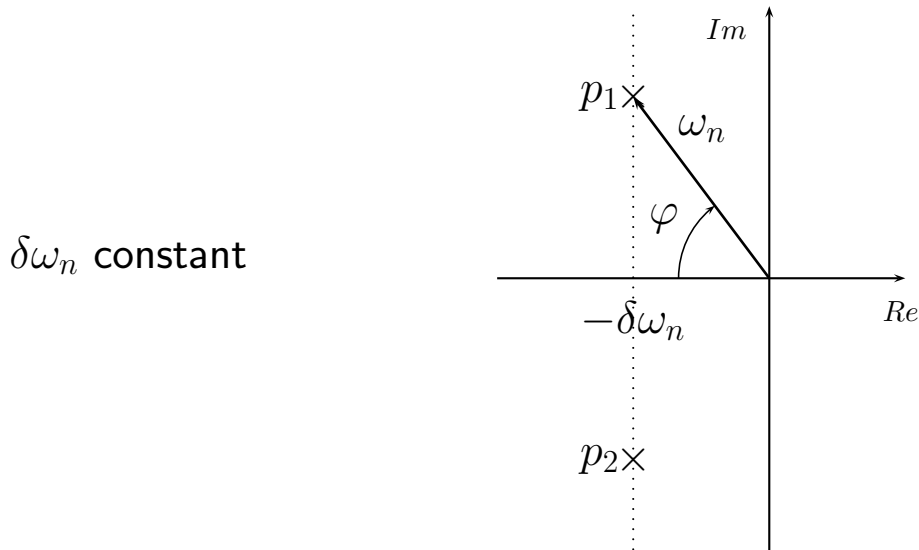


- If you choose $\delta = 0.2$ and change $\omega_n \in [0.2, 0.4, \dots, 2]$ you get the following time behaviors:

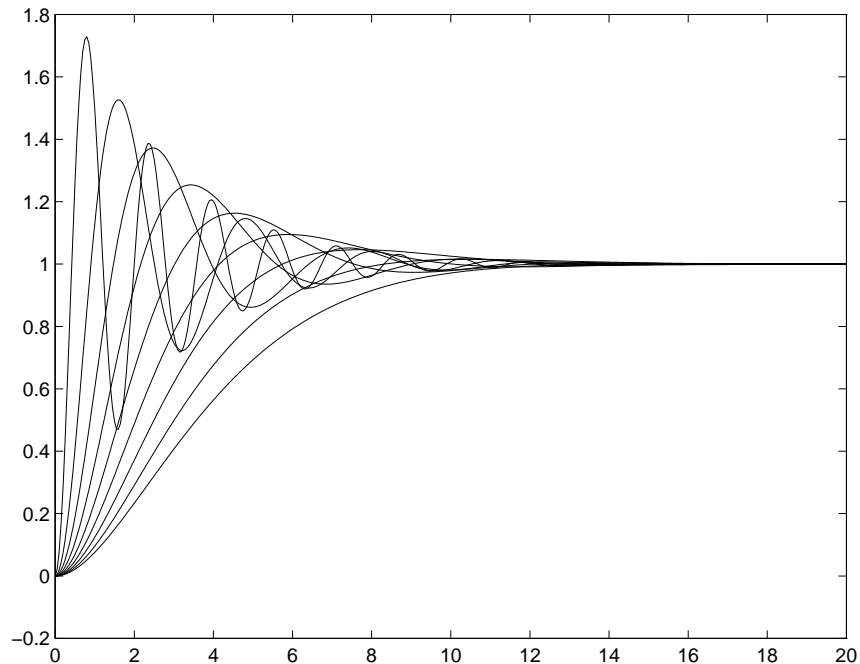


Constant settling time T_a

- Keeping the product $\delta\omega_n$ constant and changing (for example) δ , means moving the poles of the system along a vertical line of abscissa $-\delta\omega_n$:



- The time behaviors obtained by varying $\delta \in [0.1, 0.2, 0.3, \dots, 0.9]$ and keeping constant the product $\delta\omega_n = 0.4$ are the following:



- The settling time T_a (5%) is inversely proportional to $\delta\omega_n$:

$$T_a = \frac{3}{\delta\omega_n}$$

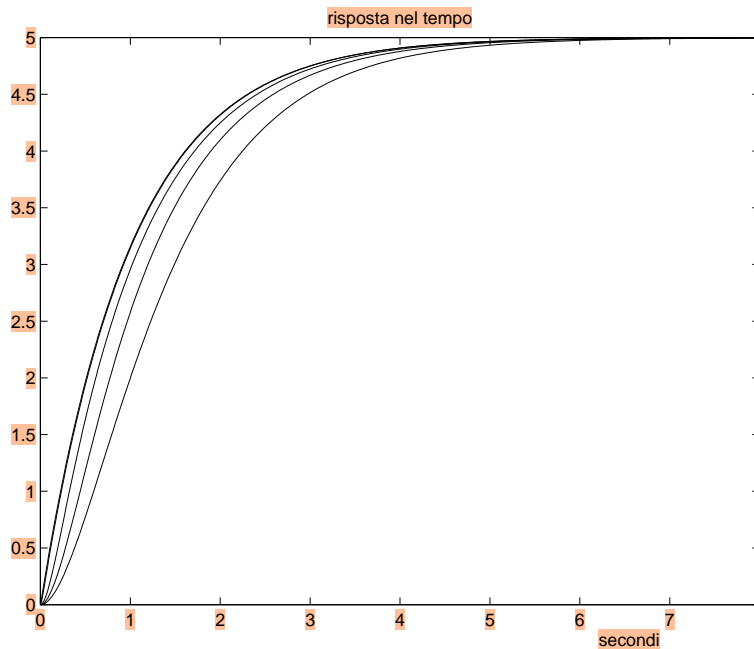
Systems with a dominant pole

- All the following second order systems have static gain $G_i(0) = 5$ and a real pole in -1 . They differ for the position of the second pole which is placed, respectively, in -2 , -4 , -10 , -100 and -1000 :

$$G_1(s) = \frac{10}{(s+1)(s+2)}, \quad G_2(s) = \frac{20}{(s+1)(s+4)}, \quad G_3(s) = \frac{50}{(s+1)(s+10)}$$

$$G_4(s) = \frac{500}{(s+1)(s+100)}, \quad G_5(s) = \frac{5000}{(s+1)(s+1000)}, \quad \left[G(s) = \frac{5}{(s+1)} \right]$$

- The unitary step response of these systems is the following:



- In the graph the slowest time behavior is the one related to system $G_1(s)$, the fastest is the one related to system $G_5(s)$.
- For stable systems, the **dominant pole** is the pole that is closest to the imaginary axis.
- The time response of the system changes “little” when the non-dominant poles have a negative real part bigger than the “dominant” pole.
- The poles that are “10 times farther” from to the dominant pole have little influence on the time response of the system.

Systems with dominant poles

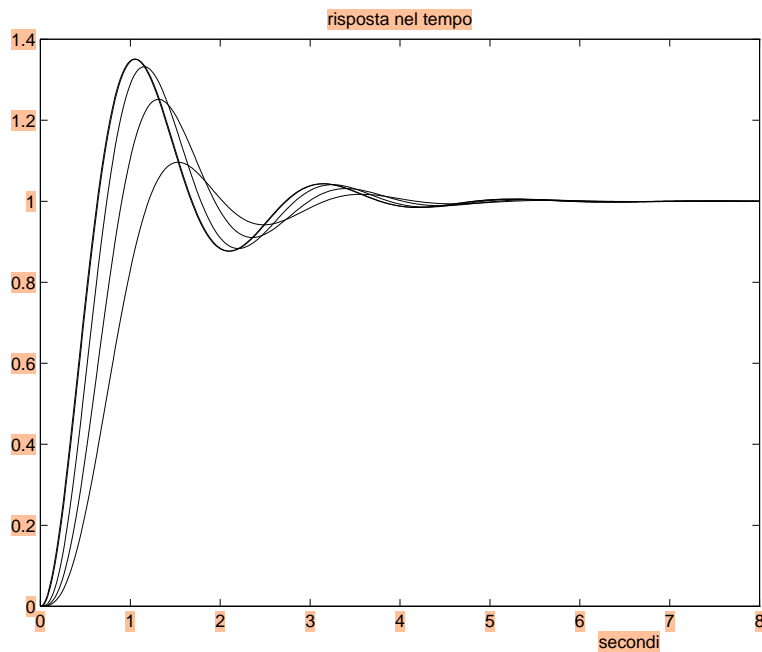
- The same considerations also apply to systems characterized by a couple of complex conjugate dominant poles.
- We define “**dominant poles**” of an asymptotically stable system the two complex conjugate poles that are closer to the imaginary axis than any other pole of the system.
- The unitary step responses of the following systems

$$G_1(s) = \frac{10}{[(s+1)^2 + 3^2](1 + \frac{s}{2})}, \quad G_2(s) = \frac{10}{[(s+1)^2 + 3^2](1 + \frac{s}{4})}$$

$$G_3(s) = \frac{10}{[(s+1)^2 + 3^2](1 + \frac{s}{10})}, \quad G_4(s) = \frac{10}{[(s+1)^2 + 3^2](1 + \frac{s}{100})}$$

$$G_5(s) = \frac{10}{[(s+1)^2 + 3^2](1 + \frac{s}{1000})}, \quad \left[G(s) = \frac{10}{[(s+1)^2 + 3^2]} \right]$$

are shown in the following graph:



- Also in this case, the poles that are “10 times farther” than the pair of “dominating poles” have little influence on the time response of the system.

Second-order dynamic system

- Any second-order dynamic system without zeros:

$$G(s) = \frac{c}{s^2 + a s + b}$$

can be rewritten as follows:

$$G(s) = K \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

- The natural frequency ω_n , the damping coefficient δ and the static gain K can be obtained as follows:

$$\boxed{\omega_n = \sqrt{b}, \quad \delta = \frac{a}{2\sqrt{b}}}, \quad K = \frac{c}{b} = G(s)|_{s \rightarrow 0}$$

- The geometric meaning of these parameters on the complex plane is the following:

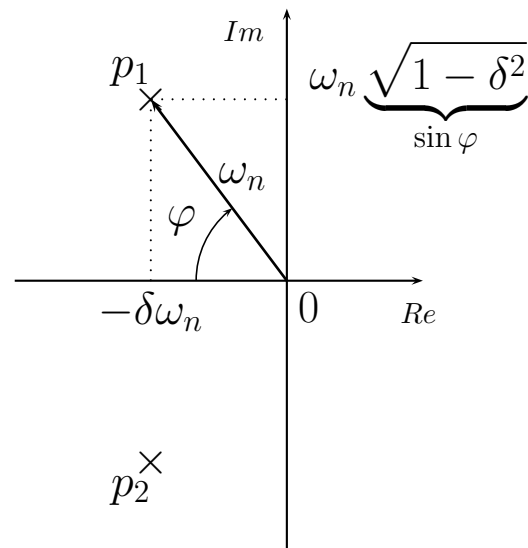
$$\begin{aligned} p_{1,2} &= -\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2} \\ &= -\sigma \pm j\omega \end{aligned}$$

$$\delta = \cos \varphi$$

$$\sigma = \delta \omega_n$$

$$\omega = \omega_n \sqrt{1-\delta^2}$$

$$\omega_n = \sqrt{\sigma^2 + \omega^2} = |p_1| = |p_2|$$



- On the complex plane s the *natural frequency* ω_n is the distance of the conjugated complex poles $p_{1,2}$ from the origin.
- The *damping coefficient* δ is equal to the cosine of the φ angle that the $\overline{p_1 0}$ segment forms with the negative semiaxis.