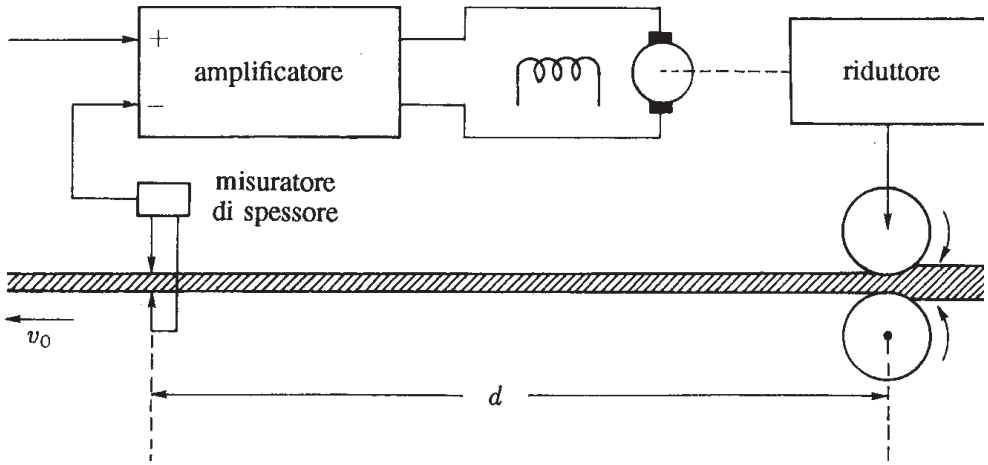
Systems with finite delay

In many controlled systems (i.e. pneumatic, hydraulic or mechanical transmissions) a finite delay $e^{-t_0 s}$ is often present. Example: a system for controlling the thickness of a steel laminate:



This system can be properly described by using the following transcendent transfer function:

$$F(s) = \frac{P(s)}{Q(s)}\, e^{-t_0 s}$$

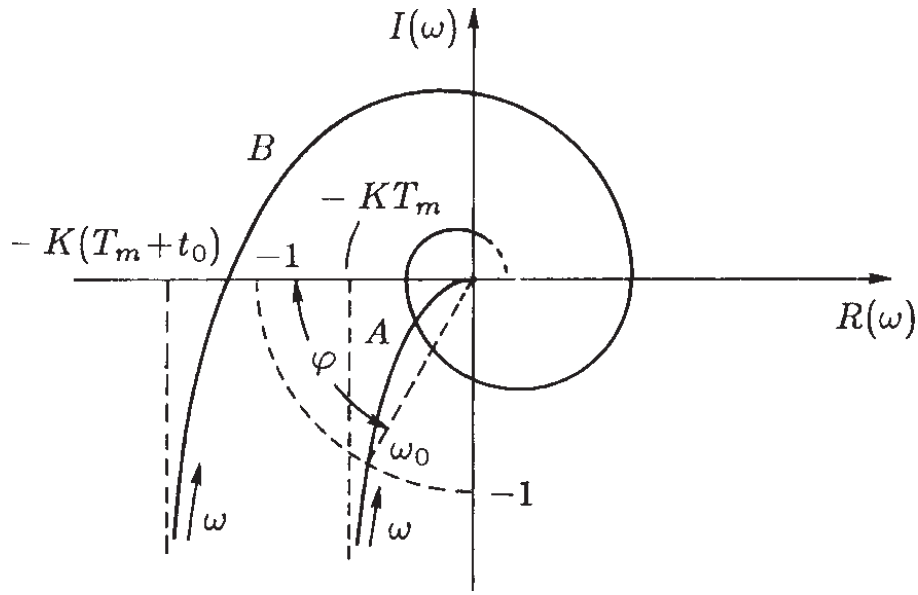
where $P(s)$ and $Q(s)$ are polynomials in s and $e^{-t_0 s}$ is the transfer function of a finite delay t_0.

The Routh criterion and the root locus method cannot be applied to system $F(s)$ with finite delays. On the contrary, the Nyquist criterion can be used, without modification, also for systems $F(s)$ with finite delays.

Example. In the considered example, the thickness of the laminate is measured at a certain distance d from the cylinders: if the speed v_0 of the laminate is constant, the finite delay within the system is $t_0 = d/v_0$. A simplified transfer function $F(s)$ for the considered system is the following:

$$G(s) = \frac{K}{s\,(1 + T_m\, s)}, \qquad F(s) = \frac{K\, e^{-t_0 s}}{s\,(1 + T_m\, s)}$$

The Nyquist diagram of function $F(s)$ without delay, when $t_0 = 0$, is denoted with A in the following figure:



When the finite delay $e^{-t_0 s}$ is present, each point $F(j\omega)$ of the nyquist diagram is delayed by an angle $\omega\, t_0$ which is proportional to the frequency ω, and the Nyquist diagram assumes the spiral shape indicated with B in the figure.

Note: if ω_0 the frequency for which the Nyquist diagram A intersects the unitary circle, then the feedback system is stable if and only if the finite delay t_0 satisfies the relation $\omega_0\, t_0 < \varphi$, that is iff:

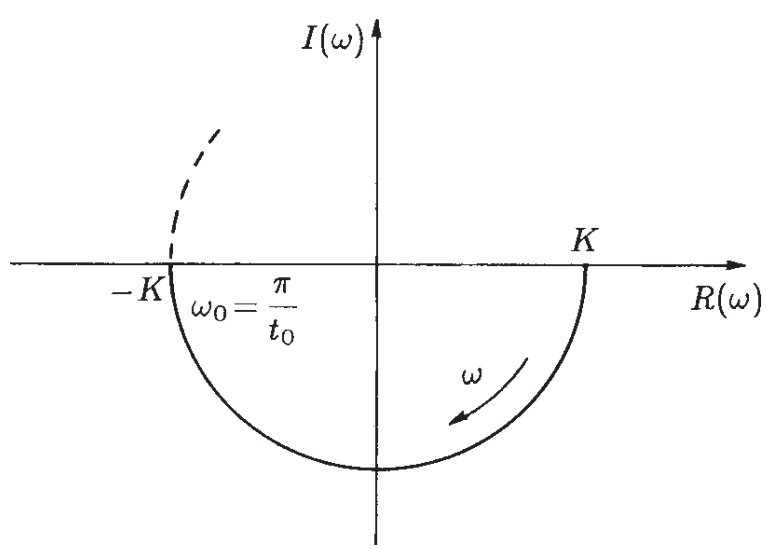
$$t_0 < \frac{\varphi}{\omega_0}$$

where φ is the phase margin of the system when $t_0 = 0$.

Example. Let us consider a type 0 system $F(s)$ having the following structure

$$F(s) = \frac{K\, e^{-t_0 s}}{(1+\tau_1 s)\,(1+\tau_2 s)\,\ldots}$$

with $t_0 \gg \tau_1, \tau_2 \ldots$ The corresponding Nyquist diagram can be of the following type:



The Nyquist diagram has a spiral form, but the initial part of the diagram can be approximated with a circumference: this happens when the phase shift due to the delay occurs before the attenuation and the phase shift due to the other terms.

Using the Nyquist criterion, one can state that the feedback system $F(s)$ is asymptotically stable if and only if $K < 1$. The value $K < 1$ for the static gain K is unacceptable because the corresponding position steady-state error $e_p = \frac{R_0}{1+K}$ is always bigger than $\frac{R_0}{2}$.

In This problem can be solved by using a controller of *integral type*: $\frac{K}{s}$. The new transfer function becomes:

$$F(s) = \frac{K\, e^{-t_0 s}}{s\,(1+\tau_1\, s)\,(1+\tau_2\, s)\,\ldots}$$

Assuming $t_0 \gg \tau_1, \tau_2\, \ldots$, the function $F(s)$ can be approximated as follows:

$$F(s) \simeq \frac{K\, e^{-t_0 s}}{s}$$

The frequency ω_0 at which function $F(j\omega)$ intersects the unitary circle at a point $B = e^{j(-\pi+M_F)}$ corresponding to a margin phase M_F, is determined as follows:

$$\text{Arg}\left[\frac{K\, e^{-t_0 s}}{s}\right]_{s=j\omega_0} = -\pi + M_F \qquad \Rightarrow \qquad -t_0\,\omega_0 - \frac{\pi}{2} = -\pi + M_F$$

from which one obtains:

$$\omega_0 = \frac{1}{t_0}\left(\frac{\pi}{2} - M_F\right)$$

The value of the gain K corresponding to phase margin M_F is determined by imposing that the module of the function $F(j\omega)$ at frequency ω_0 is unitary:

$$\left|\frac{K\, e^{-t_0 s}}{s}\right|_{s=j\omega_0} = 1 \qquad \Rightarrow \qquad K = \omega_0$$

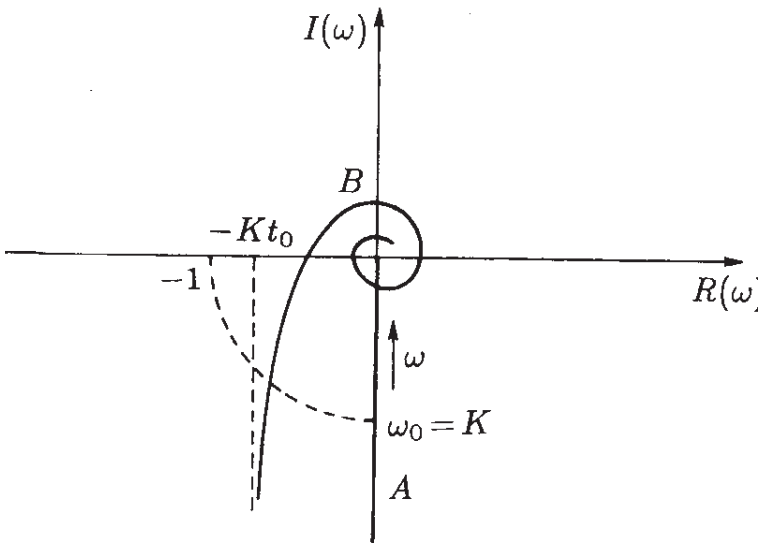
A phase margin M_F can be imposed to the feedback system $F(s)$ by choosing K as follows:

$$\boxed{K = \omega_0 = \frac{1}{t_0}\left(\frac{\pi}{2} - M_F\right)}$$

The maximum value K^* of parameter K that can guarantee the stability of the feedback system can determined by imposing $M_F = 0$ in the previous equation:

$$\boxed{K^* = \omega_0 = \frac{\pi}{2t_0}}$$

$$K = \frac{1}{t_0}\left(\frac{\pi}{2} - M_F\right) .$$



The presence of the pole in the origin makes it possible to obtain a zero the steady-state error for step input signals. This solution is frequently adopted for the stabilization of feedback systems having big finite delays.