Exercises on the simple fractions decomposition

Compute the impulsive time response $g(t)$ of the following functions $G(s)$:

1. **Sum of known functions.**

$$G(s) = 3 + \frac{10}{(s+5)^3} \qquad \rightarrow \qquad g(t) = 3\,\delta(t) + 5\,t^2 e^{-5\,t}$$

2. **A rational function $G(s)$ having zero relative degree:** $r = 0$.

$$G(s) = \frac{s+10}{s+a} = g_0 + G_1(s)$$

In this case function $G(s)$ can be rewritten as the sum of a constant g_0 and a function $G_1(s)$ having relative degree $r \geq 1$:

$$g_0 = \lim_{s\to\infty} G(s) = 1, \qquad G_1(s) = G(s) - g_0 = \frac{(10-a)}{s+a}$$

Then the inverse Laplace transform of each element can be easily computed:

$$G(s) = 1 + \frac{(10-a)}{s+a} \qquad \rightarrow \qquad g(t) = \delta(t) + (10-a)\,e^{-a\,t}$$

3. **A function $G(s)$ with time delay.**

$$G(s) = \frac{e^{-t_0\,s}}{s^2} \qquad \rightarrow \qquad g(t) = \begin{cases} 0 & t < t_0 \\ t - t_0 & t \geq t_0 \end{cases}$$

In this case, the first step is to compute the inverse Laplace transform $g_1(t) = t$ of function $G_1(s) = \frac{1}{s^2}$ obtained from $G(s)$ eliminating the time delay. Then, the desired function $g(t)$ is obtained shifting in time (t_0) the function $g_1(t)$ remembering that function $g(t)$ must be zero for $t \leq t_0$.

4. **Function $G(s)$ given in the poles-zeros factorized form.**

$$G(s) = \frac{2}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

In this case the function can be decomposed in simple fractions using the residues formula:

$$K_1 = s\,G(s)\Big|_{s=0} = 1, \qquad K_2 = (s+1)\,G(s)\Big|_{s=-1} = -2, \qquad K_3 = (s+2)\,G(s)\Big|_{s=-2} = 1$$

In this case the sum of the residues is zero because the relative degree of function $G(s)$ is $r = 3$. Computing the inverse Laplace transform one obtains:

$$G(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \qquad \rightarrow \qquad g(t) = 1 - 2\,e^{-t} + e^{-2\,t}$$

5. **Function $G(s)$ given in a mixed factorized form.**

$$G(s) = \frac{7}{(4\,s+1)(s+2)^2}$$

In this case, to avoid errors, before computing the inverse Laplace transform it is better to put function $G(s)$ in the poles-zeros factorized form:

$$G(s) = \frac{7}{4(s+0.25)(s+2)^2} = \frac{K_1}{s+0.25} + \frac{K_2}{s+2} + \frac{K_3}{(s+2)^2}$$

In this case the parameters K_1 e K_3 can be easily computed:

$$K_1 = (s+0.25)\,G(s)\Big|_{s=-0.25} = \frac{7}{4(-0.25+2)^2} = \frac{4}{7}, \qquad K_3 = (s+2)^2\,G(s)\Big|_{s=-2} = \frac{7}{4(-2+0.25)} = -1$$

The term K_2 can be easily calculated remembering that, in this case, the sum of residues is zero $K_1 + K_2 = 0$, and therefore $K_2 = -K_1 = -\frac{4}{7}$. Computing the inverse Laplace transform one obtains:

$$G(s) = \frac{4}{7(s+0.25)} - \frac{4}{7(s+2)} - \frac{1}{(s+2)^2} \qquad \rightarrow \qquad g(t) = \frac{4}{7}\,e^{-0.25\,t} - \frac{4}{7}\,e^{-2\,t} - t\,e^{-2\,t}$$

6. **Function $G(s)$ characterized by only two complex conjugate poles.**

$$G(s) = \frac{a\,s+b}{(s+\sigma)^2+\omega^2}$$

In this case it is useful to write function $G(s)$ as the sum of two terms, one proportional to the $\sin(\cdot)$ function and the other proportional to the $\cos(\cdot)$ function:

$$\begin{aligned} G(s) &= \frac{a\,(s+\sigma-\sigma)+b}{(s+\sigma)^2+\omega^2} = \frac{a\,(s+\sigma)+(b-a\sigma)}{(s+\sigma)^2+\omega^2} \\ &= a\,\frac{(s+\sigma)}{(s+\sigma)^2+\omega^2} + \frac{(b-a\sigma)}{\omega}\,\frac{\omega}{(s+\sigma)^2+\omega^2} \end{aligned}$$

The fact that each term in s is associated with the additive term $+\sigma$ indicates that the corresponding functions $\sin(\omega t)$ and $\cos(\omega t)$ must be multiplied by the exponential term $e^{-\sigma t}$. Computing the inverse Laplace transform one obtains:

$$g(t) = a\,e^{-\sigma t}\cos(\omega t) + \frac{(b-a\sigma)}{\omega}\,e^{-\sigma t}\sin(\omega t)$$

7. **Function $G(s)$ given as sum and product of factors.**

$$G(s) = \left[\frac{3\,b}{(s+a)^2} + 4\right]\frac{1}{(s+a)^2}$$

In this case the function $G(s)$ mast be rewritten as sum of simple terms before computing the inverse Laplace transform:

$$G(s) = \frac{3\,b}{(s+a)^4} + \frac{4}{(s+a)^2} \qquad \rightarrow \qquad g(t) = \frac{3b}{6}\,t^3\,e^{-a\,t} + 4\,t\,e^{-a\,t}$$