

Simple fractions decomposition

- *Forced evolution* of a differential equation: the Laplace transform $Y(s)$ of the output signal $y(t)$ is equal to the product of the Laplace transform $X(s)$ of the input signal $x(t)$ and the transfer function $G(s)$ associated to the differential equation:

$$\begin{array}{ccc} \begin{array}{c} x(t) \\ \longrightarrow \\ X(s) \end{array} & \begin{array}{c} \boxed{G(s)} \\ \longleftarrow \quad \longrightarrow \end{array} & \begin{array}{c} y(t) \\ \longrightarrow \\ Y(s) \end{array} \end{array} \quad \Leftrightarrow \quad Y(s) = G(s) X(s)$$

- The forced evolution of a differential equation can be exactly determined by computing the inverse Laplace transform of a rational fraction function having the following form:

$$Y(s) = \frac{N(s)}{D(s)} := \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

- **Relative degree** of a rational function $Y(s)$: is the difference $r = n - m$ between the degrees of the $D(s)$ and $N(s)$ polynomials.
- The function $Y(s)$ can be *decomposed in simple fractions* only if it is strictly proper, that is if it has a relative degree $r \geq 1$. If the function $Y(s)$ has relative degree $r = 0$, it can be decomposed as follows:

$$Y(s) = y_0 + Y_1(s)$$

where constant y_0 and function $Y_1(s)$, with relative degree $r \geq 1$, are:

$$y_0 = \lim_{s \rightarrow \infty} Y(s), \quad Y_1(s) = Y(s) - y_0.$$

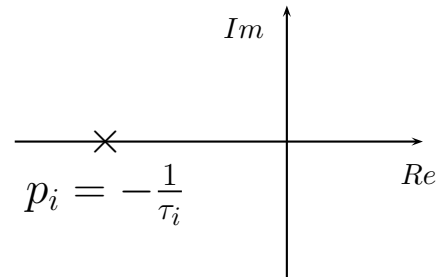
- The function $Y(s)$ can always be expressed in factored form:

$$Y(s) = K \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

- The complex constants z_1, \dots, z_m and p_1, \dots, p_n are called **zeros** and **poles** of function $Y(s)$, respectively.
- Since $Y(s) = G(s)X(s)$, the poles of function $Y(s)$ are the union of the poles of the two functions $G(s)$ and $X(s)$.

- For each *real pole* p_i of function $Y(s)$ it is possible to define a **time constant** τ_i as follows:

$$\tau_i = -\frac{1}{p_i}$$



The time constant τ_i is positive only if the real pole p_i is negative. Similarly, for the *real zeros* z_j the following relation holds: $\tau_j = -1/z_j$.

- In the case of *conjugated complex poles*, $p_{1,2} = -\sigma \pm j\omega$, ($-\sigma$ is the real part and ω is the imaginary part of the complex poles) the following parametrization is also used:

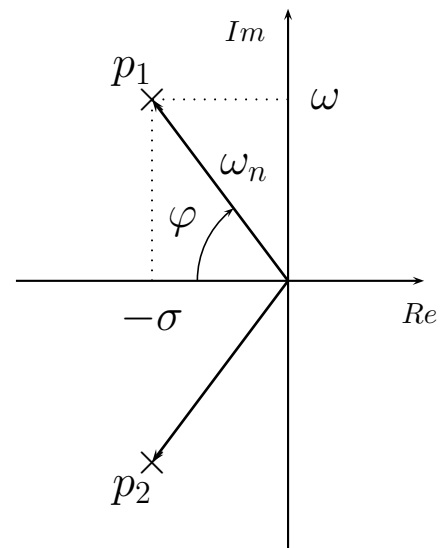
$$\begin{aligned} p_{1,2} &= -\sigma \pm j\omega \\ &= -\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2} \\ &= -\omega_n\cos\varphi \pm j\omega_n\sin\varphi \end{aligned}$$

where ω_n is called natural frequency:

$$\omega_n = |p_1| = |p_2| = \sqrt{\sigma^2 + \omega^2}$$

and δ is called damping ratio:

$$\delta = \cos\varphi = \frac{\sigma}{\omega_n} = \frac{\sigma}{\sqrt{\sigma^2 + \omega^2}}$$



The following relations hold:

$$\sigma = \delta\omega_n, \quad \omega = \omega_n\sqrt{1-\delta^2}$$

- The simple fractions decomposition shows two different cases:
 - all the poles of function $Y(s)$ are simple;*
 - function $Y(s)$ has multiple poles.*

Simple fractions decomposition when the poles are simple

- If all the poles are simple, function $Y(s)$ can be decomposed as follows:

$$Y(s) = \frac{N(s)}{(s - p_1)(s - p_2) \dots (s - p_n)} = \sum_{i=1}^n \frac{K_i}{s - p_i}$$

- The constants K_i (called *residues*) are real for real poles, and complex conjugate for complex conjugate poles. The constants K_i can be obtained using the following formula:

$$K_i = (s - p_i)Y(s) \Big|_{s=p_i}$$

- Once the $Y(s)$ function has been decomposed into simple fractions, it is immediate to compute its inverse Laplace transform:

$$y(t) = \sum_{i=1}^n K_i e^{p_i t}$$

- Example:

$$Y(s) := \frac{5s + 3}{(s + 1)(s + 2)(s + 3)} = \frac{K_1}{s + 1} + \frac{K_2}{s + 2} + \frac{K_3}{s + 3}$$

The residues are computed as follows:

$$K_1 = \frac{5(-1) + 3}{(-1 + 2)(-1 + 3)} = -1$$

$$K_2 = \frac{5(-2) + 3}{(-2 + 1)(-2 + 3)} = 7$$

$$K_3 = \frac{5(-3) + 3}{(-3 + 1)(-3 + 2)} = -6$$

It follows that

$$Y(s) = -\frac{1}{s + 1} + \frac{7}{s + 2} - \frac{6}{s + 3}$$

and therefore

$$y(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}$$

Let us consider two *complex conjugates poles*:

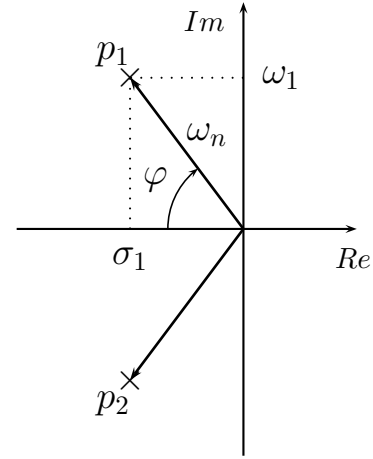
$$p_1 = \sigma_1 + j\omega_1, \quad p_2 = \sigma_1 - j\omega_1$$

The corresponding residues are complex conjugates:

$$K_1 = u_1 + jv_1, \quad K_2 = u_1 - jv_1$$

The sum of the corresponding simple fractions is

$$\frac{u_1 + jv_1}{s - \sigma_1 - j\omega_1} + \frac{u_1 - jv_1}{s - \sigma_1 + j\omega_1}$$



Defining

$$M_1 := 2|K_1| = 2\sqrt{u_1^2 + v_1^2}, \quad \varphi_1 := \arg K_1 = \arg(u_1 + jv_1),$$

the following relation holds

$$\frac{M_1}{2} \left(\frac{e^{j\varphi_1}}{s - \sigma_1 - j\omega_1} + \frac{e^{-j\varphi_1}}{s - \sigma_1 + j\omega_1} \right),$$

from which, computing the inverse Laplace transform, one obtains

$$\frac{M_1}{2} (e^{\sigma_1 t + j(\omega_1 t + \varphi_1)} + e^{\sigma_1 t - j(\omega_1 t + \varphi_1)}),$$

which can be rewritten in the following form

$$M_1 e^{\sigma_1 t} \cos(\omega_1 t + \varphi_1)$$

or

$$M_1 e^{\sigma_1 t} \sin(\omega_1 t + \varphi_1 + \pi/2)$$

Example. Let us consider the following function:

$$Y(s) := \frac{7s^2 - 8s + 5}{s^3 + 2s^2 + 5s} = \frac{K_1}{s} + \frac{K_2}{s + 1 - j2} + \frac{K_3}{s + 1 + j2}$$

The corresponding residues can be computed as follows:

$$K_1 = \frac{7 \cdot 0 - 8 \cdot 0 + 5}{(0+1-j2)(0+1+j2)} = 1$$

$$K_2 = \frac{7(-1+j2)^2 - 8(-1+j2) + 5}{(-1+j2)(-1+j2+1+j2)} = 3 + j4$$

$$K_3 = \frac{7(-1-j2)^2 - 8(-1-j2) + 5}{(-1-j2)(-1-j2+1-j2)} = 3 - j4$$

and therefore

$$Y(s) = \frac{1}{s} + \frac{3 + j4}{s + 1 - j2} + \frac{3 - j4}{s + 1 + j2},$$

from which, computing the inverse Laplace transform, one obtains

$$y(t) = 1 + 10 e^{-t} \cos(2t + \varphi),$$

where $10 = 2|K_2| = 2\sqrt{3^2 + 4^2}$ and $\varphi = \arctan(4/3) = 53.13^\circ$.

Simple fractions decomposition when $Y(s)$ has multiple poles

- If the rational function $Y(s)$ has h distinct poles p_i ($i = 1, \dots, h$), each characterized by a multiplicity order $r_i \geq 1$, the following relation holds:

$$Y(s) = \frac{N(s)}{(s - p_1)^{r_1} (s - p_2)^{r_2} \dots (s - p_h)^{r_h}} = \sum_{i=1}^h \sum_{\ell=1}^{r_i} \frac{K_{i\ell}}{(s - p_i)^{r_i - \ell + 1}}$$

- The constants $K_{i\ell}$ can be computed using the following formula

$$K_{i\ell} = \frac{1}{(\ell - 1)!} \left. \frac{d^{\ell-1}}{ds^{\ell-1}} \left[(s - p_i)^{r_i} Y(s) \right] \right|_{s=p_i}$$

where ($i = 1, \dots, h; \ell = 1, \dots, r_i$). Computing the inverse Laplace transform of function $Y(s)$ one obtains:

$$y(t) = \sum_{i=1}^h \sum_{\ell=1}^{r_i} \frac{K_{i\ell}}{(r_i - \ell)!} t^{r_i - \ell} e^{p_i t}$$

- Example:

$$Y(s) = \frac{1}{(s + 2)(s + 1)^2} = \frac{K_{11}}{s + 2} + \frac{K_{22}}{s + 1} + \frac{K_{21}}{(s + 1)^2}$$

where

$$K_{11} = [(s + 2)Y(s)] \Big|_{s=-2} = 1$$

$$K_{22} = \frac{d}{ds} [(s + 1)^2 Y(s)] \Big|_{s=-1} = \frac{d}{ds} \left[\frac{1}{s + 2} \right] \Big|_{s=-1} = -1$$

$$K_{21} = [(s + 1)^2 Y(s)] \Big|_{s=-1} = 1$$

Computing the inverse Laplace transform:

$$y(t) = e^{-2t} - e^{-t} + t e^{-t}$$

Properties of the simple fractions decomposition.

- Let us consider the following rational function

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

- The following properties hold:

- i)* if $n = m + 1$, then the sum of the residues of function $Y(s)$ is $\frac{b_m}{a_n}$;
- ii)* if $n > m + 1$, then the sum of residues of function $Y(s)$ is zero.

Note: the residues of a function $Y(s)$ decomposed in simple fractions are the coefficients of all the terms having a denominator of order one.

- Example 1:

$$Y(s) = \frac{s - z_1}{(s - p_1)(s - p_2)(s - p_3)} = \frac{A}{s - p_1} + \frac{B}{s - p_2} + \frac{C}{s - p_3}$$

Computed A and B , the residue C can be easily computed as follows: $C = -(A + B)$.

- Example 2:

$$Y(s) = \frac{1}{(s - p_1)^2(s - p_2)} = \frac{A}{(s - p_1)^2} + \frac{B}{s - p_1} + \frac{C}{s - p_2}$$

The coefficient A and the residue C can be easily computed as follows:

$$A = \frac{1}{p_1 - p_2}, \quad C = \frac{1}{(p_2 - p_1)^2}$$

Applying the property *ii)* it follows that: $B = -C$.

- Example 3:

$$Y(s) = \frac{s - z_1}{(s - p_1)^3(s - p_2)} = \frac{A}{(s - p_1)^3} + \frac{B}{(s - p_1)^2} + \frac{C}{s - p_1} + \frac{D}{s - p_2}$$

The coefficient A and the residues D and C can be computed directly:

$$A = \frac{p_1 - z_1}{p_1 - p_2} \quad D = \frac{p_2 - z_1}{(p_2 - p_1)^3} \quad C = -D = \frac{z_1 - p_2}{(p_2 - p_1)^3}$$

The coefficient B can be computer as follows:

$$B = \frac{d}{ds} \left(\frac{s - z_1}{s - p_2} \right) \Big|_{s=p_1} = \frac{z_1 - p_2}{(s - p_2)^2} \Big|_{s=p_1} = \frac{z_1 - p_2}{(p_1 - p_2)^2}$$

Modes of the time response

- The most difficult point in computing the inverse Laplace transform of a rational function $Y(s)$ is the factorization of the denominator polynomial.
- The time behavior of the inverse transformed function $y(t)$ is essentially determined by the position of the poles p_i of function $Y(s)$. Simple poles $s_i = -\sigma$ and complex conjugate poles $s_j = -\sigma \pm j\omega$ are associated to continuous-time terms $y_i(t)$ and $y_j(t)$, called *modes*, of the following form:

$$(s + \sigma) \quad \Rightarrow \quad y_i(t) = K e^{-\sigma t}$$

$$(s + \sigma)^2 + \omega^2 \quad \Rightarrow \quad y_j(t) = M e^{-\sigma t} \sin(\omega t + \varphi)$$

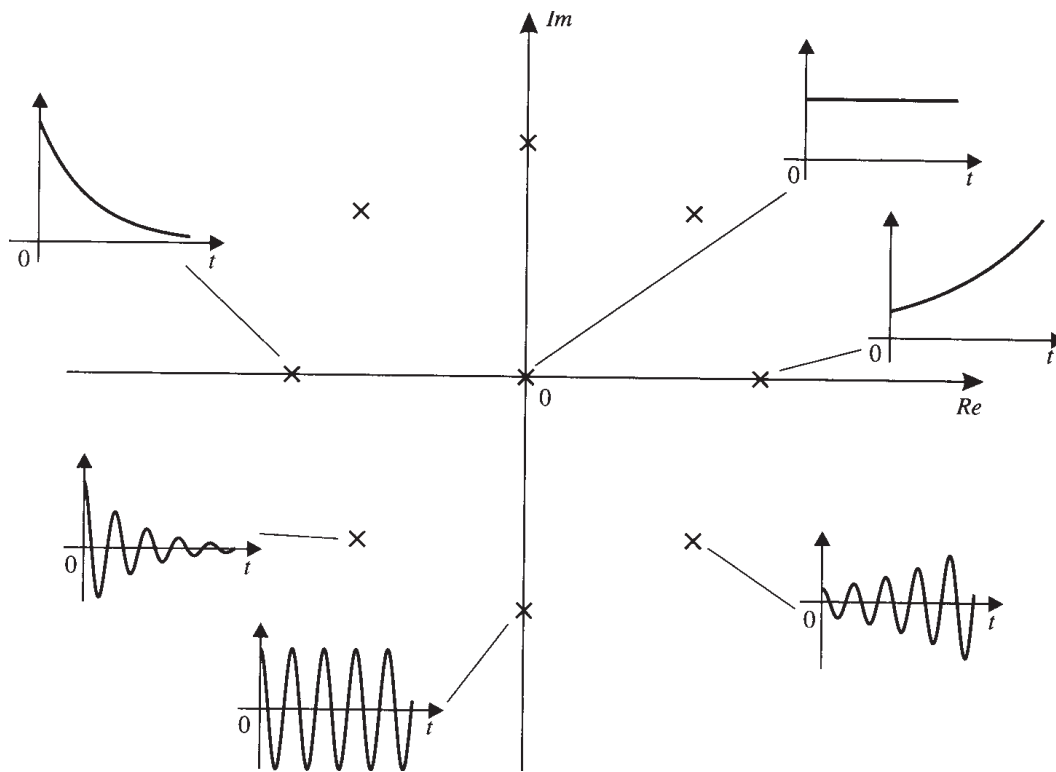
- In the case of poles with multiplicity degree $r > 1$, the *modes* are:

$$(s + \sigma)^r \quad \Rightarrow \quad y_i(t) = \sum_{n=0}^{r-1} K_n t^n e^{-\sigma t}$$

$$[(s + \sigma)^2 + \omega^2]^r \quad \Rightarrow \quad y_j(t) = \sum_{n=0}^{r-1} M_n t^n e^{-\sigma t} \sin(\omega t + \varphi)$$

- In the case of *simple poles*, the modes $y_i(t)$ and $y_j(t)$ tend to zero for $t \rightarrow \infty$ if the real part of the poles is negative ($-\sigma < 0$), remain limited if it is zero ($\sigma = 0$) and diverge if it is positive ($-\sigma > 0$).
- In the case of *multiple poles*, the modes tend to zero if the real part of the poles is negative ($-\sigma < 0$) and diverge if it is positive or zero ($-\sigma \geq 0$).
- The inverse transformed function $y(t)$ of a rational function $Y(s)$ remains limited if and only if $Y(s)$ has no poles in the positive real half plane and the poles on the imaginary axis are simple. Otherwise function $y(t)$ diverges.
- The poles of the time response $Y(s) = G(s)X(s)$ are the sum of the poles of the transfer function $G(s)$ plus the poles of the input function $X(s)$.
- A system $G(s)$ is *asymptotically stable* if all its poles have negative real part: in this case all the modes $y_i(t)$ and $y_j(t)$ associated with the poles of function $G(s)$ tend to zero when t tends to infinity.

- Modes of the time response in the case of simple poles ($r = 1$):



- Modes of the time response in the case of poles with multiplicity $r = 2$:

