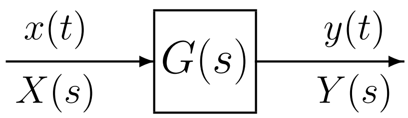
Simple fractions decomposition

- *Forced evolution* of a differential equation: the Laplace transform $Y(s)$ of the output signal $y(t)$ is equal to the product of the Laplace transform $X(s)$ of the input signal $x(t)$ and the transfer function $G(s)$ associated to the differential equation:



$$\leftrightarrow \qquad Y(s) = G(s)\, X(s)$$

- The forced evolution of a differential equation can be exactly determined by computing the inverse Laplace transform of a rational fraction function having the following form:

$$Y(s) = \frac{N(s)}{D(s)} := \frac{b_m\, s^m + b_{m-1}\, s^{m-1} + \ldots + b_1\, s + b_0}{s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0}$$

- **Relative degree** of a rational function $Y(s)$: is the difference $r = n - m$ between the degrees of the $D(s)$ and $N(s)$ polynomials.

- The function $Y(s)$ can be *decomposed in simple fractions* only if it is strictly proper, that is if it has a relative degree $r \geq 1$. If the function $Y(s)$ has relative degree $r = 0$, it can be decomposed as follows:

$$Y(s) = y_0 + Y_1(s)$$

where constant y_0 and function $Y_1(s)$, with relative degree $r \geq 1$, are:

$$y_0 = \lim_{s\to\infty} Y(s), \qquad\qquad Y_1(s) = Y(s) - y_0.$$

- The function $Y(s)$ can always be expressed in factored form:

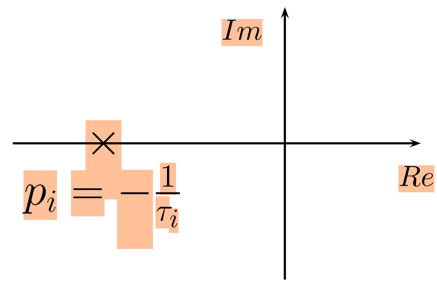
$$Y(s) = K\, \frac{(s - z_1)\,(s - z_2)\ldots(s - z_m)}{(s - p_1)\,(s - p_2)\ldots(s - p_n)}$$

- The complex constants $z_1, \ldots, z_m$ and $p_1, \ldots, p_n$ are called **zeros** and **poles** of function $Y(s)$, respectively.

- Since $Y(s) = G(s)X(s)$, the poles of function $Y(s)$ are the union of the poles of the two functions $G(s)$ and $X(s)$.

- For each *real pole* p_i of function $Y(s)$ it is possible to define a **time constant** τ_i as follows:

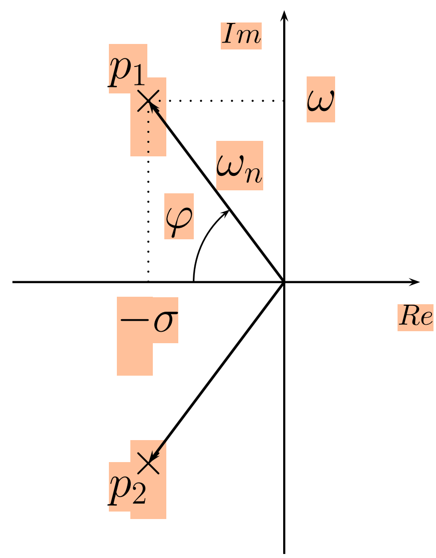
$$\boxed{\tau_i = -\frac{1}{p_i}}$$



The time constant τ_i is positive only if the real pole p_i is negative. Similarly, for the *real zeros* z_j the following relation holds: $\tau_j = -1/z_j$.

- In the case of *conjugated complex poles*, $p_{1,2} = -\sigma \pm j\,\omega$, ($-\sigma$ is the real part and ω is the imaginary part of the complex poles) the following parametrization is also used:

$$\begin{aligned} p_{1,2} &= -\sigma \pm j\,\omega \\ &= -\delta\,\omega_n \pm j\,\omega_n\sqrt{1-\delta^2} \\ &= -\omega_n \cos\varphi \pm j\,\omega_n \sin\varphi \end{aligned}$$



where ω_n is called *natural frequency*:

$$\boxed{\omega_n = |p_1| = |p_2| = \sqrt{\sigma^2 + \omega^2}}$$

and δ is called *damping ratio*:

$$\boxed{\delta = \cos\varphi = \frac{\sigma}{\omega_n} = \frac{\sigma}{\sqrt{\sigma^2+\omega^2}}}$$

The following relations hold:

$$\boxed{\sigma = \delta\,\omega_n, \qquad \omega = \omega_n\sqrt{1-\delta^2}}$$

- The simple fractions decomposition shows two different cases:

  i) all the poles of function $Y(s)$ are simple;

  ii) function $Y(s)$ has multiple poles.

Simple fractions decomposition when the poles are *simple*

- If all the poles are simple, function $Y(s)$ can be decomposed as follows:

$$Y(s) = \frac{N(s)}{(s-p_1)(s-p_2)\dots(s-p_n)} = \sum_{i=1}^{n} \frac{K_i}{s-p_i}$$

- The constants K_i (called *residues*) are real for real poles, and complex conjugate for complex conjugate poles. The constants K_i can be obtained using the following formula:

$$K_i = (s-p_i)Y(s)\Big|_{s=p_i}$$

- Once the $Y(s)$ function has been decomposed into simple fractions, it is immediate to compute its inverse Laplace transform:

$$y(t) = \sum_{i=1}^{n} K_i\, e^{p_i t}$$

- Example:

$$Y(s) := \frac{5\,s+3}{(s+1)\,(s+2)\,(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

The residues are computed as follows:

$$\begin{aligned} K_1 &= \frac{5\,(-1)+3}{(-1+2)\,(-1+3)} = -1 \\ K_2 &= \frac{5\,(-2)+3}{(-2+1)\,(-2+3)} = 7 \\ K_3 &= \frac{5\,(-3)+3}{(-3+1)\,(-3+2)} = -6 \end{aligned}$$

It follows that

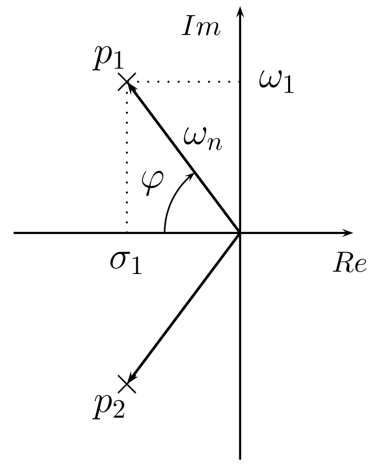
$$Y(s) = -\frac{1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3}$$

and therefore

$$y(t) = -e^{-t} + 7\,e^{-2t} - 6\,e^{-3t}$$

Let us consider two *complex conjugates poles*:

$$p_1 = \sigma_1 + j\,\omega_1, \qquad p_2 = \sigma_1 - j\,\omega_1$$



The corresponding residues are complex conjugates:

$$K_1 = u_1 + j\,v_1, \qquad K_2 = u_1 - j\,v_1$$

The sum of the corresponding simple fractions is

$$\frac{u_1 + j\,v_1}{s - \sigma_1 - j\,\omega_1} + \frac{u_1 - j\,v_1}{s - \sigma_1 + j\,\omega_1}$$

Defining

$$\boxed{M_1 := 2|K_1| = 2\sqrt{u_1^2 + v_1^2}, \qquad \varphi_1 := \arg K_1 = \arg\left(u_1 + j\,v_1\right)},$$

the following relation holds

$$\frac{M_1}{2}\left(\frac{e^{j\varphi_1}}{s - \sigma_1 - j\,\omega_1} + \frac{e^{-j\varphi_1}}{s - \sigma_1 + j\,\omega_1}\right),$$

from which, computing the inverse Laplace transform, one obtains

$$\frac{M_1}{2}\left(e^{\sigma_1 t + j(\omega_1 t + \varphi_1)} + e^{\sigma_1 t - j(\omega_1 t + \varphi_1)}\right),$$

which can be rewritten in the following form

$$\boxed{M_1\, e^{\sigma_1 t} \cos(\omega_1 t + \varphi_1)} \qquad \text{or} \qquad \boxed{M_1\, e^{\sigma_1 t}\, \sin(\omega_1 t + \varphi_1 + \pi/2)}$$

Example. Let us consider the following function:

$$Y(s) := \frac{7s^2 - 8s + 5}{s^3 + 2s^2 + 5s} = \frac{K_1}{s} + \frac{K_2}{s + 1 - j2} + \frac{K_3}{s + 1 + j2}$$

The corresponding residues can be computed as follows:

$$\begin{aligned}
K_1 &= \tfrac{7\cdot 0 - 8\cdot 0 + 5}{(0+1-j2)\,(0+1+j2)} = 1 \\
K_2 &= \tfrac{7\,(-1+j2)^2 - 8\,(-1+j2) + 5}{(-1+j2)\,(-1+j2+1+j2)} = 3 + j4 \\
K_3 &= \tfrac{7\,(-1-j2)^2 - 8\,(-1-j2) + 5}{(-1-j2)\,(-1-j2+1-j2)} = 3 - j4
\end{aligned}$$

and therefore

$$Y(s) = \frac{1}{s} + \frac{3 + j4}{s + 1 - j2} + \frac{3 - j4}{s + 1 + j2},$$

from which, computing the inverse Laplace transform, one obtains

$$y(t) = 1 + 10\, e^{-t} \cos\left(2t + \varphi\right),$$

where $10 = 2|K_2| = 2\sqrt{3^2 + 4^2}$ and $\varphi = \arctan(4/3) = 53.13°$.

Simple fractions decomposition when $Y(s)$ has *multiple* poles

- If the rational function $Y(s)$ has h distinct poles p_i $(i=1,\dots,h)$, each characterized by a multiplicity order $r_i \geq 1$, the following relation holds:

$$Y(s) = \frac{N(s)}{(s-p_1)^{r_1}(s-p_2)^{r_2}\dots(s-p_h)^{r_h}} = \sum_{i=1}^{h}\sum_{\ell=1}^{r_i} \frac{K_{i\ell}}{(s-p_i)^{r_i-\ell+1}}$$

- The constants $K_{i\ell}$ can be computed using the following formula

$$K_{i\ell} = \frac{1}{(\ell-1)!}\frac{d^{\,\ell-1}}{ds^{\,\ell-1}}\Big[(s-p_i)^{r_i}\,Y(s)\Big]\Big|_{s=p_i}$$

where $(i = 1,\dots,h;\ell = 1,\dots,r_i)$. Computing the inverse Laplace transform of function $Y(s)$ one obtains:

$$y(t) = \sum_{i=1}^{h}\sum_{\ell=1}^{r_i}\frac{K_{i\ell}}{(r_i-\ell)!}\,t^{r_i-\ell}\,e^{p_i t}$$

- Example:

$$Y(s) = \frac{1}{(s+2)\,(s+1)^2} = \frac{K_{11}}{s+2} + \frac{K_{22}}{s+1} + \frac{K_{21}}{(s+1)^2}$$

where

$$\begin{aligned}
K_{11} &= [(s+2)\,Y(s)]\,\big|_{s=-2} = 1\\
K_{22} &= \frac{d}{ds}\left[\,(s+1)^2\,Y(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{1}{s+2}\right]\Big|_{s=-1} = -1\\
K_{21} &= \left[(s+1)^2\,Y(s)\right]\big|_{s=-1} = 1
\end{aligned}$$

Computing the inverse Laplace transform:

$$y(t) = e^{-2t} - e^{-t} + t\,e^{-t}$$

Properties of the simple fractions decomposition.

- Let us consider the following rational function

$$Y(s) = \frac{b_m s^m + b_{m-1}\, s^{m-1} + \ldots + b_1\, s + b_0}{a_n s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0}$$

- The following properties hold:

  i) if $n = m+1$, then the sum of the residues of function $Y(s)$ is $\frac{b_m}{a_n}$;

  ii) if $n > m+1$, then the sum of residues of function $Y(s)$ is zero.

  Note: the residues of a function $Y(s)$ decomposed in simple fractions are the coefficients of all the terms having a denominator of order one.

- Example 1:

$$Y(s) = \frac{s - z_1}{(s-p_1)\,(s-p_2)\,(s-p_3)} = \frac{A}{s-p_1} + \frac{B}{s-p_2} + \frac{C}{s-p_3}$$

Computed A and B, the residue C can be easily computed as follows: $C = -(A+B)$.

- Example 2:

$$Y(s) = \frac{1}{(s-p_1)^2\,(s-p_2)} = \frac{A}{(s-p_1)^2} + \frac{B}{s-p_1} + \frac{C}{s-p_2}$$

The coefficient A and the residue C can be easily computed as follows:

$$A = \frac{1}{p_1 - p_2}\,, \qquad C = \frac{1}{(p_2-p_1)^2}$$

Applying the property *ii)* it follows that: $B = -C$.

- Example 3:

$$Y(s) = \frac{s - z_1}{(s-p_1)^3\,(s-p_2)} = \frac{A}{(s-p_1)^3} + \frac{B}{(s-p_1)^2} + \frac{C}{s-p_1} + \frac{D}{s-p_2}$$

The coefficient A and the residues D and C can be computed directly:

$$A = \frac{p_1 - z_1}{p_1 - p_2} \qquad D = \frac{p_2 - z_1}{(p_2-p_1)^3} \qquad C = -D = \frac{z_1 - p_2}{(p_2-p_1)^3}$$

The coefficient B can be computer as follows:

$$B = \frac{d}{ds}\left(\frac{s-z_1}{s-p_2}\right)\bigg|_{s=p_1} = \frac{z_1 - p_2}{(s-p_2)^2}\bigg|_{s=p_1} = \frac{z_1 - p_2}{(p_1-p_2)^2}$$

Modes of the time response

- The most difficult point in computing the inverse Laplace transform of a rational functions $Y(s)$ is the factorization of the denominator polynomial.
- The time behavior of the inverse transformed function $y(t)$ is essentially determined by the position of the poles p_i of function $Y(s)$. Simple poles $s_i = -\sigma$ and complex conjugate poles $s_j = -\sigma \pm j\,\omega$ are associated to continuous-time terms $y_i(t)$ and $y_j(t)$, called *modes*, of the following form:

$$(s+\sigma) \quad \Rightarrow \quad y_i(t) = K\,e^{-\sigma t}$$

$$(s+\sigma)^2+\omega^2 \quad \Rightarrow \quad y_j(t) = M\,e^{-\sigma t}\sin(\omega\,t+\varphi)$$

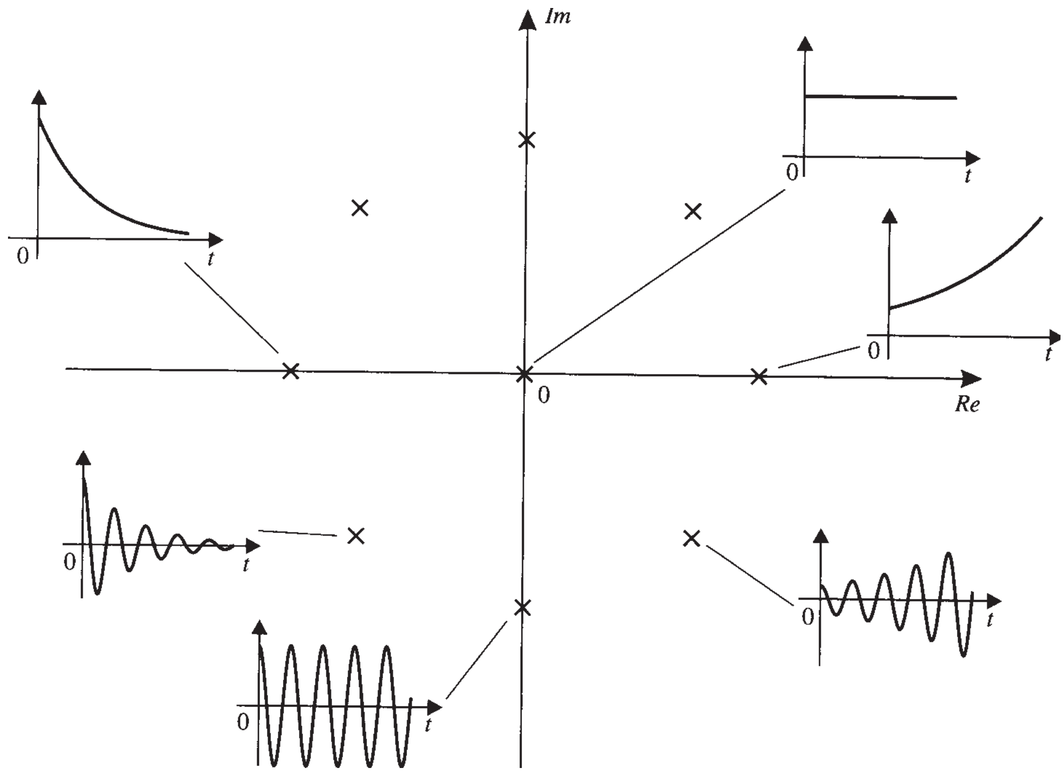
- In the case of poles with multiplicity degree $r > 1$, the *modes* are:

$$(s+\sigma)^r \quad \Rightarrow \quad y_i(t) = \sum_{n=0}^{r-1} K_n\,t^n\,e^{-\sigma t}$$

$$[(s+\sigma)^2+\omega^2]^r \quad \Rightarrow \quad y_j(t) = \sum_{n=0}^{r-1} M_n\,t^n\,e^{-\sigma t}\sin(\omega\,t+\varphi)$$

- In the case of *simple poles*, the modes $y_i(t)$ and $y_j(t)$ tend to zero for $t \to \infty$ if the real part of the poles is negative ($-\sigma < 0$), remain limited if it is zero ($\sigma = 0$) and diverge if it is positive ($-\sigma > 0$).
- In the case of *multiple poles*, the modes tend to zero if the real part of the poles is negative ($-\sigma < 0$) and diverge if it is positive or zero ($-\sigma \geq 0$).
- The inverse transformed function $y(t)$ of a rational function $Y(s)$ remains limited if and only if $Y(s)$ has no poles in the positive real half plane and the poles on the imaginary axis are simple. Otherwise function $y(t)$ diverges.
- The poles of the time response $Y(s) = G(s)X(s)$ are the sum of the poles of the transfer function $G(s)$ plus the poles of the input function $X(s)$.
- A system $G(s)$ is *asymptotically stable* if all its poles have negative real part: in this case all the modes $y_i(t)$ and $y_j(t)$ associated with the poles of function $G(s)$ tend to zero when t tends to infinity.

- Modes of the time response in the case of simple poles ($r = 1$):



- Modes of the time response in the case of poles with multiplicity $r = 2$: