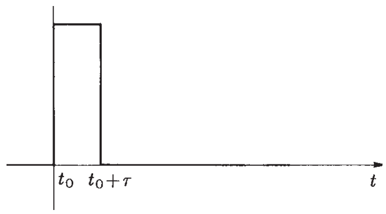
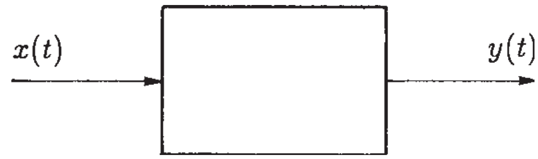
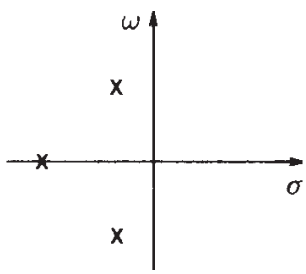
Stability and feedback

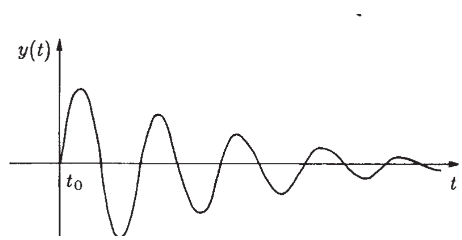
- Stability of linear dynamic systems:



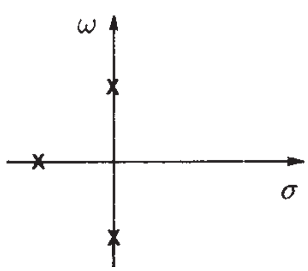


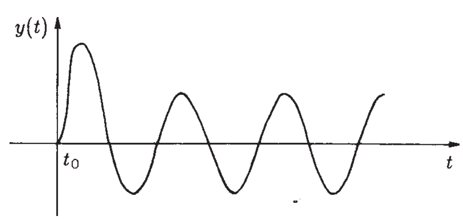
- A system $G(s)$ is **asymptotically stable** if all its poles have negative real part:



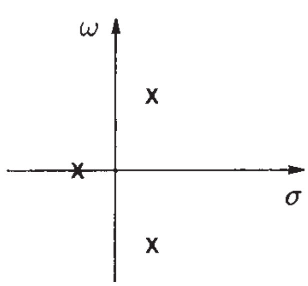


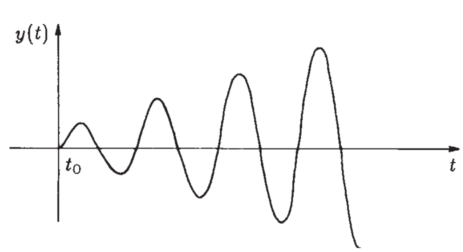
- A system $G(s)$ is **stable** if all its poles have negative real part o zero real part with multiplicity equal to 1.





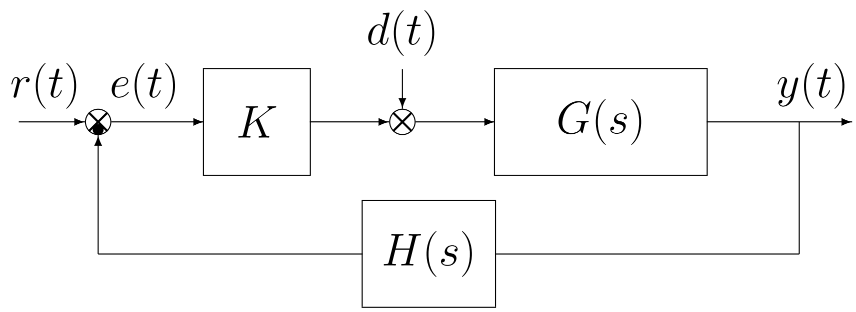
- A system $G(s)$ is **unstable** if it has at least one pole with positive real part, or poles with zero real part and multiplicity ≥ 2.





Routh Criterion

- Let us refer to the following feedback system:



- The transfer function $G_0(s)$ of the feedback system is:

$$G_0(s) = \frac{Y(s)}{R(s)} = \frac{K\,G(s)}{1 + K\,G(s)H(s)}$$

- The **characteristic equation** of the feedback system is:

$$1 + K\,G(s)H(s) = 0$$

- The stability of the feedback system $G_0(s)$ is uniquely determined by the position on the complex plane of the poles of the function $G_0(s)$, that is the zeros of the characteristic equation.

- To determine if a system $G_0(s)$ is stable, it is not strictly necessary to know the exact position of its poles, it is sufficient to know if they are on the left or on the right of the imaginary axis.

- The Routh criterion allows to determine if system $G_0(s)$ is stable without calculating the exact position of its poles.

- The characteristic equation must be rewritten in the polynomial form:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

with $a_n > 0$ and $a_0 \neq 0$

- *A necessary condition: all the roots of the characteristic equation have negative real part only if all its coefficients a_i are positive:*

$$a_n > 0, \quad a_{n-1} > 0, \quad \ldots, \quad a_1 > 0, \quad a_0 > 0$$

- Given the following characteristic equation:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

build the Routh table as follows:

$$\begin{array}{c|cccc} n & a_n & a_{n-2} & a_{n-4} & a_{n-6} \\ n-1 & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} \\ n-2 & b_{n-2} & b_{n-4} & b_{n-6} & \ldots \\ n-3 & c_{n-3} & c_{n-5} & \ldots & \end{array}$$

where

$$b_{n-2} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \qquad b_{n-4} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \quad \ldots$$

$$c_{n-3} = \frac{b_{n-2}a_{n-3} - a_{n-1}b_{n-4}}{b_{n-2}}, \qquad c_{n-5} = \frac{b_{n-2}a_{n-5} - a_{n-1}b_{n-6}}{b_{n-2}}, \quad \ldots$$

- **Routh criterion**: *each change of sign between the coefficients of the first column of the Routh table corresponds to a root with a positive real part, each permanence of sign corresponds to a root with a negative real part. (C.N.S.)*

- Example:

$$s^3 - 4s^2 + s + 6 = 0 \qquad \rightarrow \qquad (s+1)(s-2)(s-3) = 0$$

Routh table:

$$\begin{array}{c|ccc} 3 & 1 & 1 & 0 \\ 2 & -4 & 6 & 0 \\ 1 & \frac{-4-6}{-4} = 2.5 & 0 & \\ 0 & \frac{2.5 \cdot 6}{2.5} = 6 & & \end{array}$$

There are two variations of sign in the first column of the Routh table, and therefore the equation has two roots with positive real part.

- Example:

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

Routh table:

4	2	3	10
3	1	5	0
2	-7	10	
1	$\frac{45}{7}$	0	
0	10		

2 variations $\rightarrow$

$\rightarrow$ 2 roots with positive real part

- The Routh criterion remains valid also if, during the construction of the Routh table, **all** the coefficients of the same row are multiplied by a **positive** constant.

$$4s^4 + 3s^3 + 5s^2 + 2s + 1 = 0$$

Routh table:

4	4	5	1	
3	3	2	0	
2	7	3		$\leftarrow$ line not devided by 3
1	5	0		$\leftarrow$ line not devided by 7
0	3			

All the roots of the given equation have negative real part.

- The Routh criterion can be used also in presence of symbolic variables:

$$s^3 + 3s^2 + 2s + K = 0$$

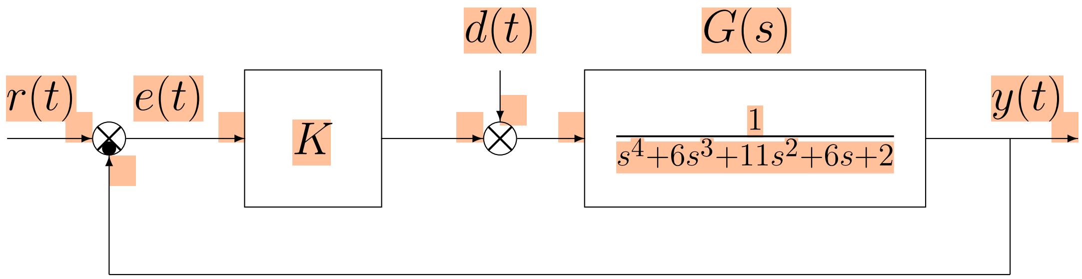
Routh table:

3	1	2
2	3	K
1	$6 - K$	
0	K	

All the roots of the given equation have negative real part if and only if:

$$0 < K < 6.$$

- Example:



- The Routh criterion can also be used to check the stability of system $G(s)$:

$$G(s) = \frac{1}{s^4 + 6s^3 + 11s^2 + 6s + 2}.$$

In this case the Routh criterion must be applied to the denominator of function $G(s)$. The Routh table is:

4	1	11	2
3	6	6	0
2	10	2	
1	48	0	
0	2		

From Routh criterion it follows that function $G(s)$ is asymptotically stable.

- The transfer function $G_0(s)$ of the feedback system is:

$$G_0(s) = \frac{K\,G(s)}{1 + K\,G(s)}$$

In this case the characteristic equation is function of parameter K:

$$s^4 + 6s^3 + 11s^2 + 6s + K + 2 = 0$$

Routh table:

4	1	11	$K+2$
3	6	6	0
2	10	$K+2$	
1	$48-6K$	0	
0	$K+2$		

Conditions for the asymptotic stability:

$$48 - 6K > 0, \qquad K + 2 > 0$$

The system is asymptotically stable if and only if:

$$-2 < K < 8$$

- For $\bar{K} = -2$ the feedback system has a simple pole in the origin. In fact, for $K = -2$, in the Routh table there is only one change of sign: the change of sign of the coefficient of line 0. In this case, for $\bar{K} = -2$, the feedback system has one pole that crosses the imaginary axis in the origin.
- For $K^* = 8$ the feedback system has two complex conjugate poles which cross the imaginary axis in correspondence of frequency:

$$\omega = \sqrt{\frac{K^* + 2}{10}} = 1$$

In fact in the Routh table the first coefficient of line 1 chances sign for $K = 8$. It follows that for $K = 8$ there are two changes of sign between the coefficients of the first column of the Routh table, and therefore there are two poles of the feedback system that are crossing the imaginary axis.

- The crossing frequency ω^* on the imaginary axis can be determined writing the following "*auxiliary equation*":

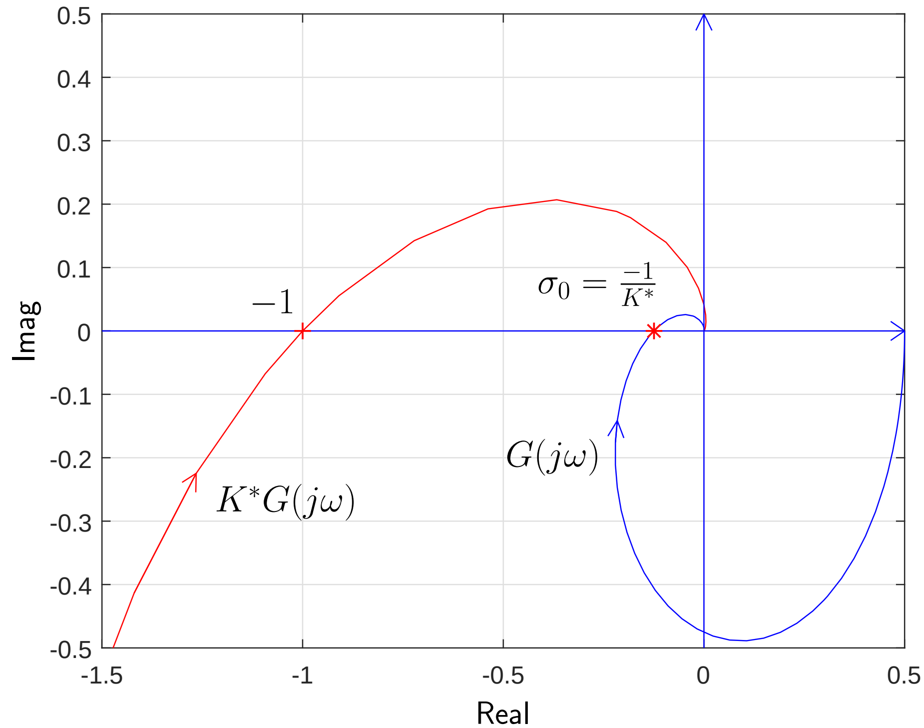
$$a_2\, s^2 + a_0 = 0 \qquad \Leftrightarrow \qquad 10\, s^2 + K^* + 2 = 0$$

substituting $s = j\,\omega^*$, and then solving the auxiliary equation for ω^*:

$$s = \pm j\sqrt{\frac{a_0}{a_2}} \qquad \Rightarrow \qquad \omega^* = \sqrt{\frac{a_0}{a_2}} \qquad \Leftrightarrow \qquad \omega^* = \sqrt{\frac{K^* + 2}{10}} = 1.$$

- This method can be applied only is line 1 of the Routh table is zero for $K = K^*$.

- Often the Routh criterion can also be used to determine the point σ_0 where the Nyquist diagram of function $G(s)$ crosses the negative real axis.
- For the considered example the Nyquist diagram of function $G(s)$ has the following form:



- Known K^* from the Routh criterion, the point σ_0 where the Nyquist diagram $G(j\omega)$ intersects the negative real axis is:

$$\sigma_0 = -\frac{1}{K^*}$$

Routh criterion: symbolic examples

1) Let us consider the following dynamic system:

$$G_1(s) = \frac{\alpha}{s(s+a)(s+b)}$$

The characteristic equation of the corresponding feedback system is:

$$1 + K\,G(s) = 0 \qquad \Leftrightarrow \qquad s^3 + (a+b)s^2 + a\,b\,s + \alpha K = 0.$$

Routh table:

$$\begin{array}{c|ll} 3 & 1 & ab \\ 2 & a+b & \alpha K \\ 1 & (a+b)ab - \alpha K & \\ 0 & \alpha K & \end{array}$$

The feedback system is asymptotically stable for $0 < K < K^*$ where:

$$K^* = \frac{(a+b)ab}{\alpha}, \qquad \omega^* = \sqrt{ab}.$$

Frequency ω^* is the position of the two poles on the imaginary axis when $K = K^*$.

2) Let us consider the following dynamic system:

$$G_2(s) = \frac{(s+c)}{s(s+a)(s+b)}$$

The characteristic equation of the corresponding feedback system is:

$$s^3 + (a+b)s^2 + (a\,b + K)\,s + K\,c = 0$$

Routh table:

$$\begin{array}{c|ll} 3 & 1 & ab + K \\ 2 & a+b & K\,c \\ 1 & (a+b)(ab+K) - K\,c & \\ 0 & K\,c & \end{array}$$

The feedback system is stable for $0 < K < K^*$ if $c > a+b$, and for $K > 0$ if $c < a+b$. The values of parameters K^* and ω^* are:

$$K^* = \frac{(a+b)ab}{c-a-b}, \qquad \omega^* = \sqrt{\frac{abc}{c-a-b}}$$

Calculation of the limit frequency ω^*

- *Let us consider the following third order characteristic equation:*

$$a_3(K)s^3 + a_2(K)s^2 + a_1(K)s + a_0(K) = 0$$

where the coefficients $a_i(K)$, for $i = 0, \ldots, 3$, are functions of parameter K. The frequency ω^* corresponding to the limit value K^* can be computed as follows:

$$\boxed{\omega^* = \sqrt{\frac{a_0(K^*)}{a_2(K^*)}} = \sqrt{\frac{a_1(K^*)}{a_3(K^*)}}} \qquad (1)$$

if the coefficient of the first line is zero for $K = K^*$.
The Routh table is:

$$\begin{array}{c|cc} 3 & a_3(K) & a_1(K) \\ 2 & a_2(K) & a_0(K) \\ 1 & a_2(K)a_1(K) - a_0(K)a_3(K) & \\ 0 & a_0(K) & \end{array}$$

The limit value K^* is usually obtained when the coefficient of line 1 is zero:

$$a_2(K*)a_1(K^*) - a_0(K^*)a_3(K^*) = 0 \qquad \rightarrow \qquad \frac{a_0(K^*)}{a_2(K*)} = \frac{a_1(K^*)}{a_3(K^*)} \qquad (2)$$

The frequency ω^* can be obtained using the auxiliary equation of line 2:

$$a_2(K^*)s^2 + a_0(K^*) = 0 \qquad \rightarrow \qquad \omega^* = \sqrt{\frac{a_0(K^*)}{a_2(K^*)}} \qquad (3)$$

Relation (1) is obtained by combining the two relations (2) and (3).

- If a fourth order characteristic equation is considered:

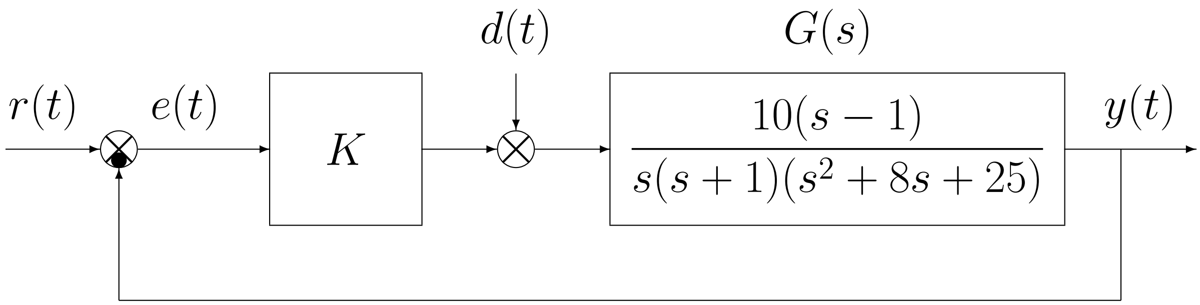
$$a_4(K)s^4 + a_3(K)s^3 + a_2(K)s^2 + a_1(K)s + a_0(K) = 0$$

the frequency ω^* can be determined as follows:

$$\boxed{\omega^* = \sqrt{\frac{a_1(K^*)}{a_3(K^*)}}}$$

Example

Consider the following feedback system:



Determine the values of parameter K for which the feedback system is asymptotically stable. The characteristic equation of the feedback system is:

$$1 + K\,G(s) = 0 \qquad \Leftrightarrow \qquad 1 + \frac{10K(s-1)}{s(s+1)(s^2+8s+25)} = 0$$

which can be rewritten in polynomial form as follows:

$$s^4 + 9s^3 + 33s^2 + (25 + 10K)s - 10K = 0$$

The corresponding Routh table has the following form:

$$\begin{array}{c|ccc} 4 & 1 & 33 & -10K \\ 3 & 9 & 25+10K & \\ 2 & 272-10K & -90K & \\ 1 & (272-10K)(25+10K)+810K & & \\ 0 & -90K & & \end{array} \tag{4}$$

Note: the Routh table (4) has been calculated without dividing each row by the first coefficient of the previous line (for example line 2 was not divided by 9 and line 1 was not divided by $272 - 10K$). This can be done because all the coefficients of the first column will be required to be positive in order to guarantee the asymptotic stability of the feedback system.

The feedback system is asymptotically stable if and only if all the elements of the first column are positive:

$$2)\ 272 - 10\,K > 0 \qquad 1)\ 340 + 164K - 5K^2 > 0 \qquad 0)\ \ -90\,K > 0$$

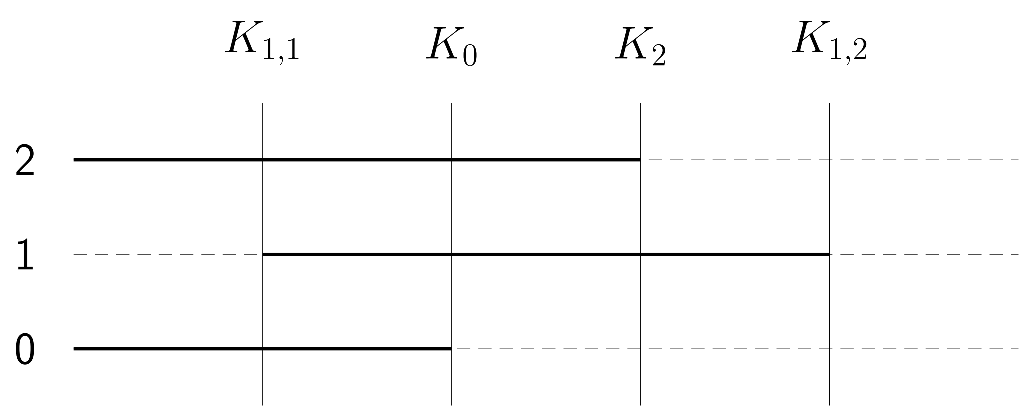
These three inequalities are zero for the following values of parameter K:

$$2)\ K_2 = 27.2, \quad 1)\ K_{1,*} = \begin{cases} K_{1,1} = \frac{164-\sqrt{33696}}{10} = -1.956 \\ K_{1,2} = \frac{164+\sqrt{33696}}{10} = 34.7565 \end{cases} \quad 0)\ K_0 = 0$$

The feedback system is asymptotically stable if and only if:

$$2)\ K < K_2 \qquad 1)\ K_{1,1} < K < K_{1,2} \qquad 0)\ K < K_0$$

Graphical representation of these three inequalities:



From the graphical representation it follows that the feedback system is asymptotically stable if and only if:

$$K_{1,1} < K < K_0 \qquad \leftrightarrow \qquad -1.956 < K < 0$$

The frequency ω^* corresponding to the limit value $K_{1,1}$ is:

$$\omega^* = \sqrt{\frac{a_1(K_{1,1})}{a_3}} = \sqrt{\frac{25 + 10K_{1,1}}{9}} = 0.7771.$$