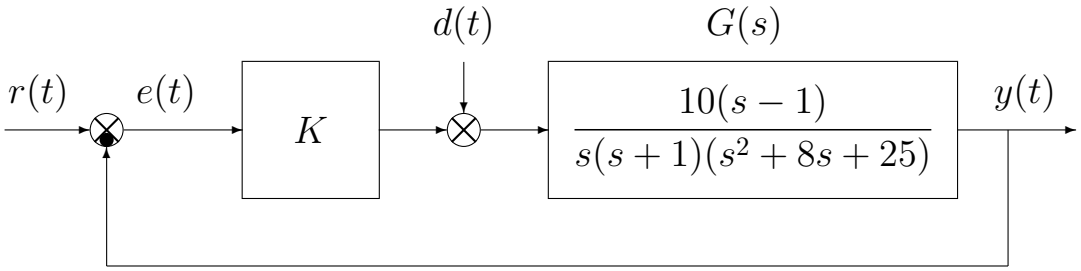
Root locus: qualitative drawing

Example. Let us consider the following feedback system:



For a qualitative and rapid plotting of the root locus, it is useful to apply the following rules in sequence:

1) Write the characteristic equation of the feedback system in the following form:

$$1 + K_1 G_1(s) = 0 \qquad \rightarrow \qquad 1 + \frac{10K(s-1)}{s(s+1)(s^2+8s+25)} = 0$$

where $G_1(s)$ must be in the "poles-zeros" factorized form in order to easily determine if the parameter K_1 has the same sign of the parameter K. In the considered case the system is already in the factorized form and the parameters K and K_1 have the same sign: $K_1 = 10\,K$.

2) Draw on the complex plane the position of the poles and zeros of function $G_1(s)$: use a "cross" for the poles and a "small circle" for the zeros. In the case of multiple poles (or zeros) it is useful to draw on the complex plane a number of crosses (or circles) equal to the multiplicity of the pole (or zero). This helps make no mistakes in applying the following rules.

3) Highlight the points of the real axis that belong to the root locus. When $K_1 > 0$, *the points of the real axis that belong to the root locus leave to their right a total "odd" number of poles and zeros*.

4) Draw arrows on the real axis in the vicinity of the poles and zeros, remembering that *the branches of the root locus start from the poles for* $K_1 = 0^+$ *and reach the zeros for* $K_1 \rightarrow \infty$. For each real pole (zero) draw an arrow exiting the pole (entering the zero) in direction of the points of the real axis that belong to the root locus. In case of multiple poles (zeros) draw the arrows remembering that *the branches exiting the multiple pole (entering the multiple zero) "locally" divide the plane in equal parts*.

5) The number of asymptotes is equal to the relative degree $n - m$. The asymptotes are half-lines exiting a point σ_a of the real axis:

$$\sigma_a = \frac{1}{n-m}\left(\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i\right) \quad \rightarrow \quad \sigma_a = \frac{1}{3}(-1-8-1) = -\frac{10}{3} = -3.333$$

The center of the asymptotes must be computed only if the relative degree is greater or equal 2, $r = n - m \geq 2$, that is when the root locus has at least two asymptotes.

The asymptotes *exit from point σ_a and divide the plane in equal parts which are symmetric with respect to the real axis*. When the relative degree $r = n - m$ is odd, at least one asymptote belongs to the real axis. When $K_1 > 0$ and r is odd, the negative real axis belongs to the root locus and it coincides with one of the asymptotes. When $K_1 < 0$ and r is odd the positive real axis belongs to the root locus and it coincides with one of the asymptotes.

The center of the asymptotes σ_a is the same both for $K_1 > 0$ and $K_1 < 0$. The position of the asymptotes for $K_1 < 0$ can be obtained from the position of the asymptotes when $K_1 > 0$ simply "rotating" (clockwise or counterclockwise) by an angle $\varphi = \frac{\pi}{n-m}$.

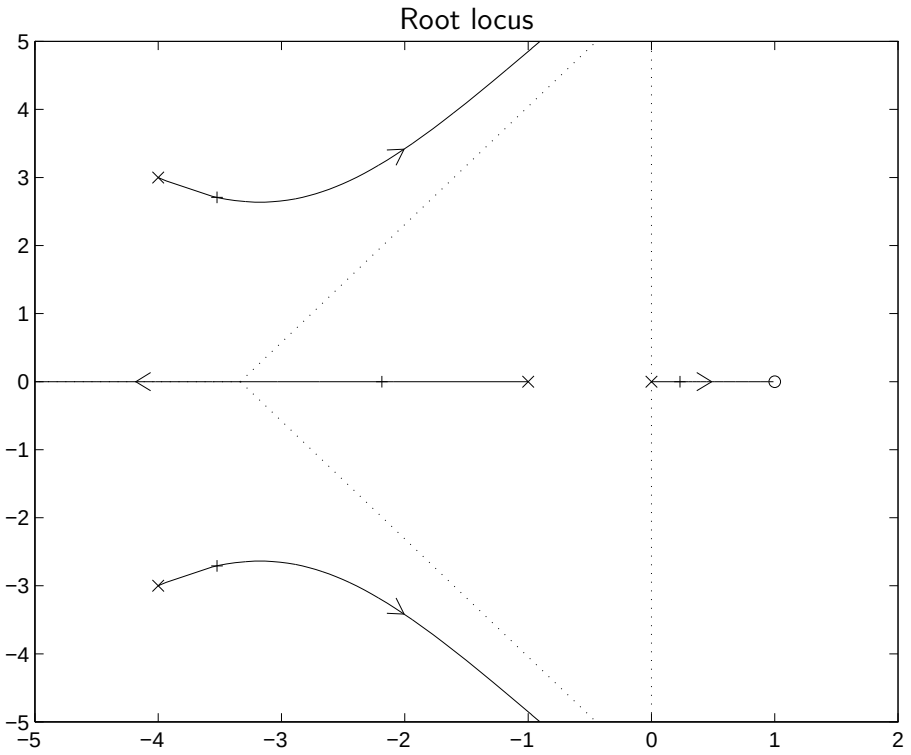
For each asymptote there is one branch of the root locus that goes towards the infinity when $K_1 \to \infty$: an arrow that goes towards the infinity can also be added to each branch that tends to an asymptote.

6) The previously plotted arrows help to identify if there are branching points in the parts of the real axis that belong to the real axis and if they are "entering" or "exiting" branching points. The branching points of order 2 on the real axis are "double" solutionis of the characteristic equation $1 + K_1 G_1(s) = 0$, so they are also solutions of the "derivative" of the characteristic equation:

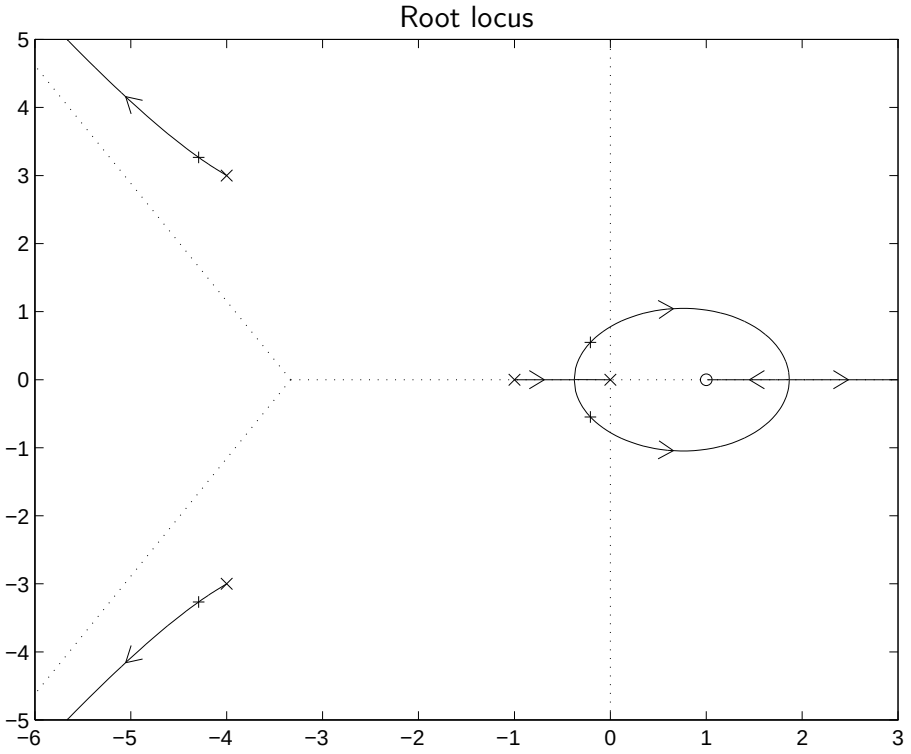
$$\frac{d\,G_1(s)}{d\,s} = 0 \qquad \rightarrow \qquad N(s)D'(s) - N'(s)D(s) = 0$$

where $N(s)$ and $D(s)$ are, respectively, the numerator and the denominator of the function $G(s) = N(s)/D(s)$. Without the aid of a computer, the "exact" determination of the branching points is not an easy task.

- Root locus of function $G(s)$ when $K > 0$:



- Root locus of function $G(s)$ when $K < 0$:



7) The intersections with the imaginary axis can be determined using the Routh criterion. The limit value K^* and the frequency ω^* obtained from the Routh criterion indicate that when $K = K^*$ the root locus intersects the imaginary axis at point $s = \pm j\,\omega^*$.

   For the considered case the intersection with the imaginary axis is obtained for $K^* = -1.956$ at point $\pm j\omega^* = \pm j0.7771$.

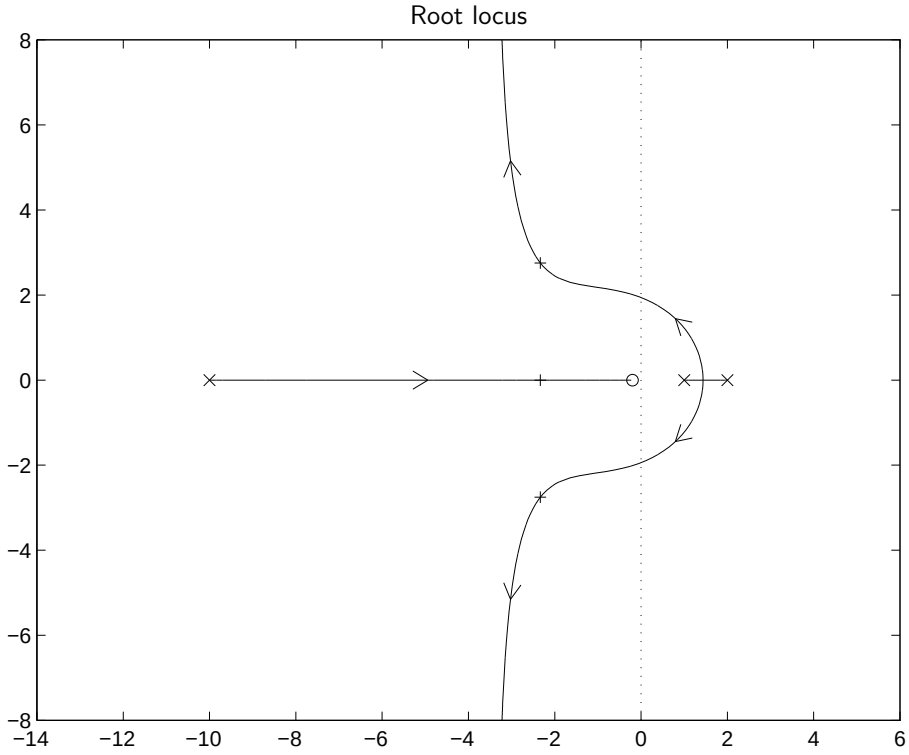
The root locus shows on the complex plane how the poles of the feedback system move when parameter K ranges from 0 to ∞. his method clearly shows the condition of *minimum settling time*: this condition is obtained when the poles of the feedback system are at the "maximum" distance from the imaginary axis.

Often the condition of minimum settling time coincides with one of the branching points present on the real axis. In the considered case, for example, the minimum settling time is obtained at the branching point located within the segment $[-1,\ 0]$ of the real axis.

In other cases the minimum settling time is obtained in correspondence of certain alignment condition for the poles. Consider, for example, the following characteristic equation:

$$1 + \frac{K_2(s+0.2)}{(s-1)(s-2)(s+10)} = 0 \qquad \rightarrow \qquad 1 + K_2 G_2(s) = 0$$

Root locus of function $G_2(s)$ when $K_2 > 0$:



The center of the asymptotes is:

$$\sigma_a = \frac{1}{2}\left(-10 + 1 + 2 + 0.2\right) = -3.4$$

In this case, the minimum settling time T_a is obtained when the three poles of the feedback system are aligned. In this case the ascissa σ_0 of the three aligned poles can be determined using the **Centroid theorem**: *referring to the characteristic equation* $1 + K_2 G_2(s) = 0$, *if the relative degree of function* $G_2(s)$ *is greater than or equal to 2, the sum of the roots of the characteristic equation, i.e. the poles* $\bar{p}_i$ *of the feedback system* $G_0(s)$, *remains constant when parameter* K_2 *changes*. Applying the centroid theorem to the considered system we have that:

$$\sum_{i=1}^{3} \bar{p}_i = \sum_{i=1}^{3} p_i \qquad \leftrightarrow \qquad 3\sigma_0 = -10 + 1 + 2 \qquad \leftrightarrow \qquad \sigma_0 = -\frac{7}{3}$$

The value $\bar{K}_2$ for which the alignment condition is obtained can be determined as follows:

$$\bar{K}_2 = -\left.\frac{1}{G_2(s)}\right|_{s=\sigma_0} = -\left.\frac{(s-1)(s-2)(s+10)}{(s+0.2)}\right|_{s=\sigma_0} = 51.9$$

Unfortunately, not always it is possibile to use the centroid theorem to determine the condition of alignment of the poles. In these the problem cab be solve numerically with the aid of a computer.