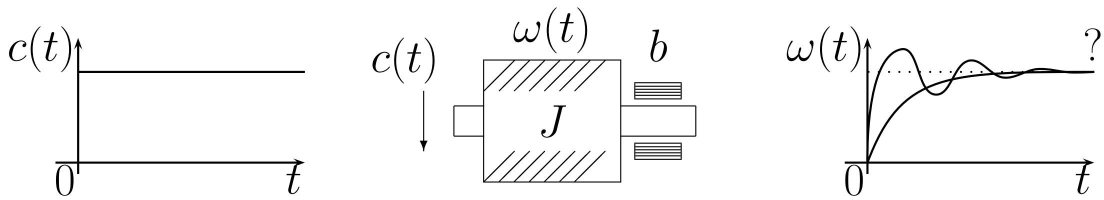
Examples of physical models

1) Dynamics of the rotor of an electric motor.

Consider a mechanical element with inertia J, linear friction coefficient b that rotates at the angular velocity ω to which an external torque $c(t)$ is applied.



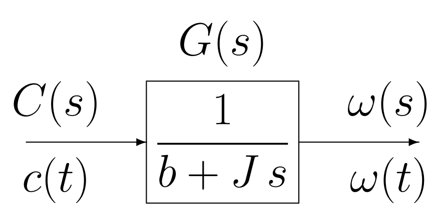
The differential equation of the system can be obtained from the law of conservation of the angular momentum:

$$\frac{d[J\omega(t)]}{dt} = c(t) - b\,\omega(t) \qquad \leftrightarrow \qquad J\dot{\omega}(t) + b\,\omega(t) = c(t)$$

Starting from zero initial conditions and applying the Laplace transform one obtains:

$$J\,s\,\omega(s) + b\,\omega(s) = C(s) \qquad \leftrightarrow \qquad \omega(s) = \frac{1}{b + J\,s}\,C(s)$$

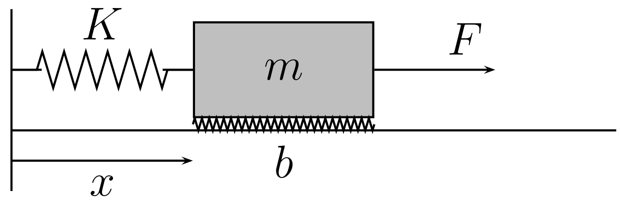
So, the physical system can be described as follows:



where $G(s)$ is the transfer function that describes the physical system:

$$G(s) = \frac{1}{b + J\,s}$$

2) **Mass-spring-damper system.**



- Variables and parameters of the physical system:

$x(t)$	: position	m	: mass
$\dot{x}(t)$	: velocity	K	: stiffness of the spring
$\ddot{x}(t)$	: acceleration	b	: linear friction coefficient
$F(t)$	: applied force	$P(t)$	: momentum

- Conservation law of the momentum $P(t) = m\,\dot{x}(t)$:

$$\frac{d}{dt}[P(t)] = \sum_i F_i(t) \qquad \rightarrow \qquad m\,\frac{d}{dt}[\dot{x}(t)] = \sum_i F_i(t)$$

- The following differential equation is then obtained:

$$\frac{d}{dt}[m\,\dot{x}(t)] = F - b\,\dot{x}(t) - K\,x(t)$$

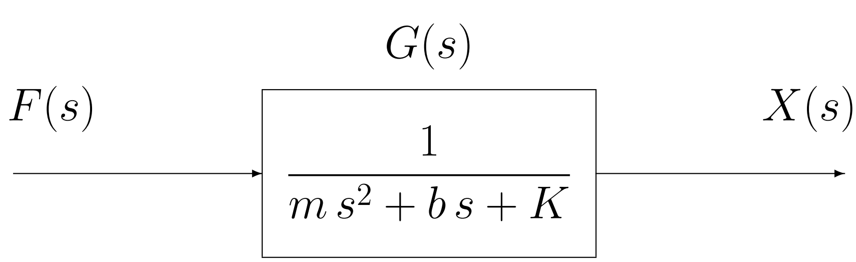
which can be rewritten as follows:

$$m\ddot{x}(t) + b\,\dot{x}(t) + K\,x(t) = F(t)$$

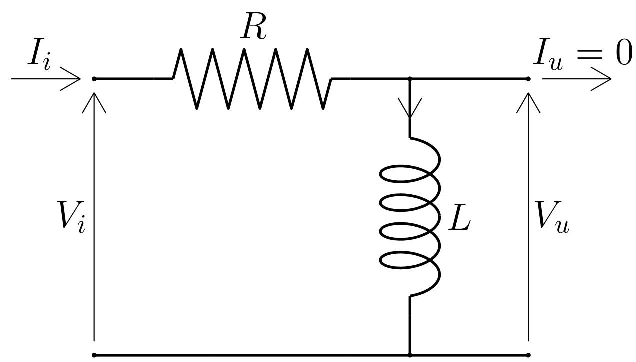
- Using the Laplace transform ($x(0) = \dot{x}(0) = 0$) one obtains:

$$m\,s^2 X(s) + b\,s\,X(s) + K\,X(s) = F(s) \qquad \leftrightarrow \qquad X(s) = \frac{F(s)}{m\,s^2 + b\,s + K}$$

The system can therefore be represented as follows:



3) **RL electrical system.**



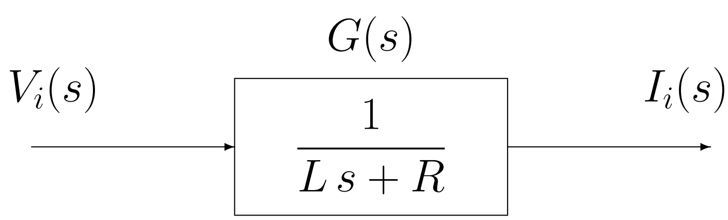
- Physical law: the variation of the concatenated flux $\phi_c(t) = LI_i(t)$ is equal to the voltage $V_u(t)$ applied to the inductance.

$$\frac{d}{dt}[\phi_c(t)] = V_u(t) \qquad \rightarrow \qquad L\,\frac{d}{dt}[I_i(t)] = V_i(t) - R\,I_i(t)$$

- Applying the Laplace transform, with zero initial conditions, we have:

$$L\,s\,I_i(s) + R\,I_i(s) = V_i(s) \qquad \leftrightarrow \qquad I_i(s) = \underbrace{\frac{1}{L\,s + R}}_{G(s)}\,V_i(s)$$

- So, the system can be represented as follows:



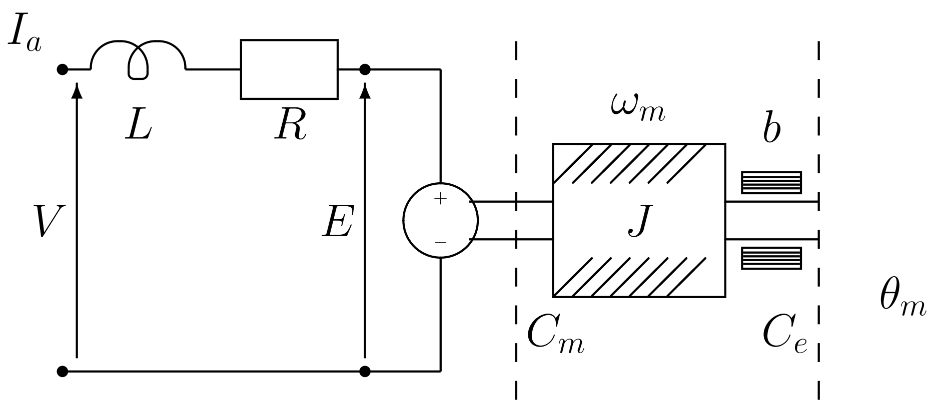
- The mathematical relation between the input voltage $V_i(t)$ and the output voltage $V_u(t)$ is described by the following transfer function:

$$G_1(s) = \frac{V_u(s)}{V_i(s)} = \frac{L\,s}{L\,s + R}$$

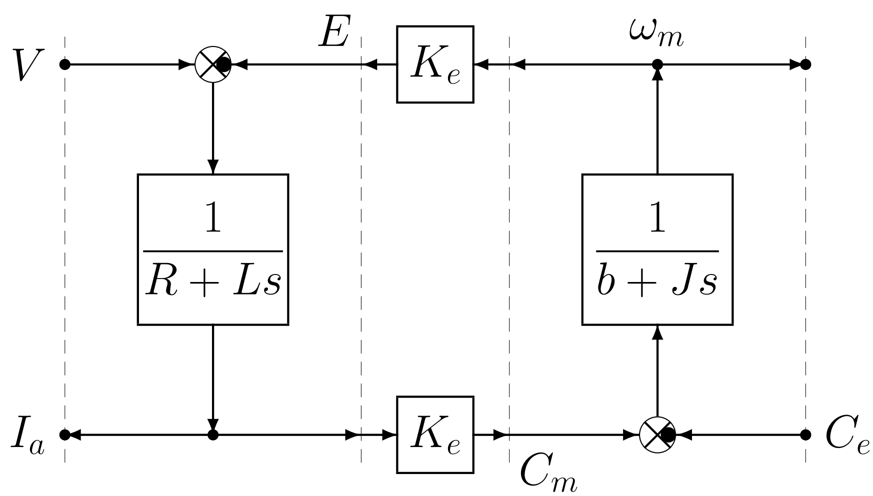
The corresponding differential equation is:

$$L\,\dot{V}_u(t) + R\,V_u(t) = L\,\dot{V}_i(t)$$

4) Direct Current Electric Motor.



- POG block diagram of the DC electric motor:



The system is described by the following two differential equations:

$$\begin{cases} L\dot{I}_a = -RI_a - K_e\,\omega_m + V \\ J\dot{\omega}_m = K_e I_a - b\,\omega_m - C_e \end{cases}$$

- For the principle of energy conservation, the constant K_e links both the armature current I_a to the motor torque C_m and links the counter motor force E to the angular velocity ω_m:

$$C_m = K_e\,I_a, \qquad\qquad E = K_e\,\omega_m.$$

- Using the "Mason formula" one obtains the following relation between the output variable $\omega_m(t)$ and the input variables $V(t)$ and $C_e(t)$:

$$\omega_m(s) = G_1(s)\,V(s) + G_2(s)\,C_e(s)$$

where $G_1(s)$ is the transfer function between input $V(t)$ and output $\omega_m(t)$

$$G_1(s) = \frac{K_e}{(R + L\,s)(b + J\,s) + K_e^2}$$

and $G_2(s)$ is the transfer function between input $C_e(t)$ and output $\omega_m(t)$:

$$G_2(s) = \frac{-(R + L\,s)}{(R + L\,s)(b + J\,s) + K_e^2}$$

- The previous relationship can also be rewritten as follows:

$$\left[L\,J\,s^2 + (R\,J + L\,b)s + R\,b + K_e^2\right]\omega_m(s) = K_e V(s) - (R + L\,s)C_e(s)$$

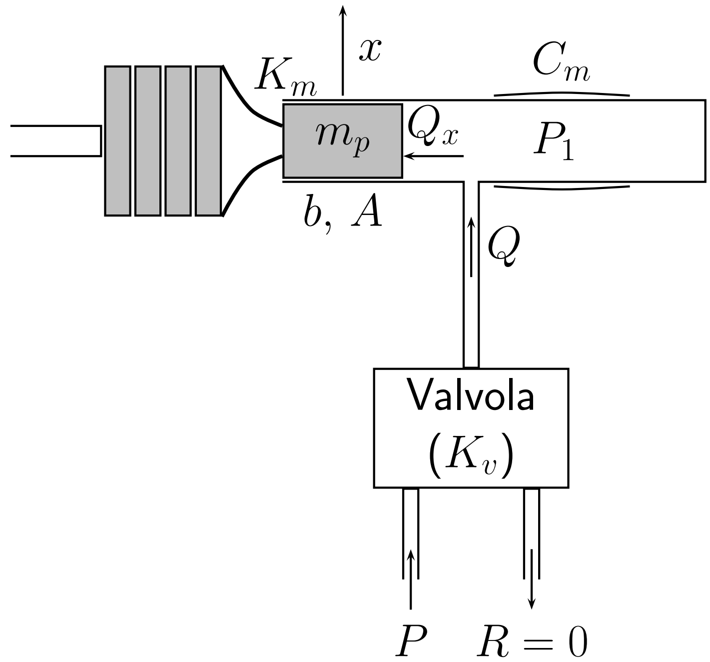
which corresponds to the following second order differential equation:

$$L\,J\,\ddot{\omega}_m + (R\,J + L\,b)\dot{\omega}_m + (R\,b + K_e^2)\omega_m = K_e V - R\,C_e - L\,\dot{C}_e$$

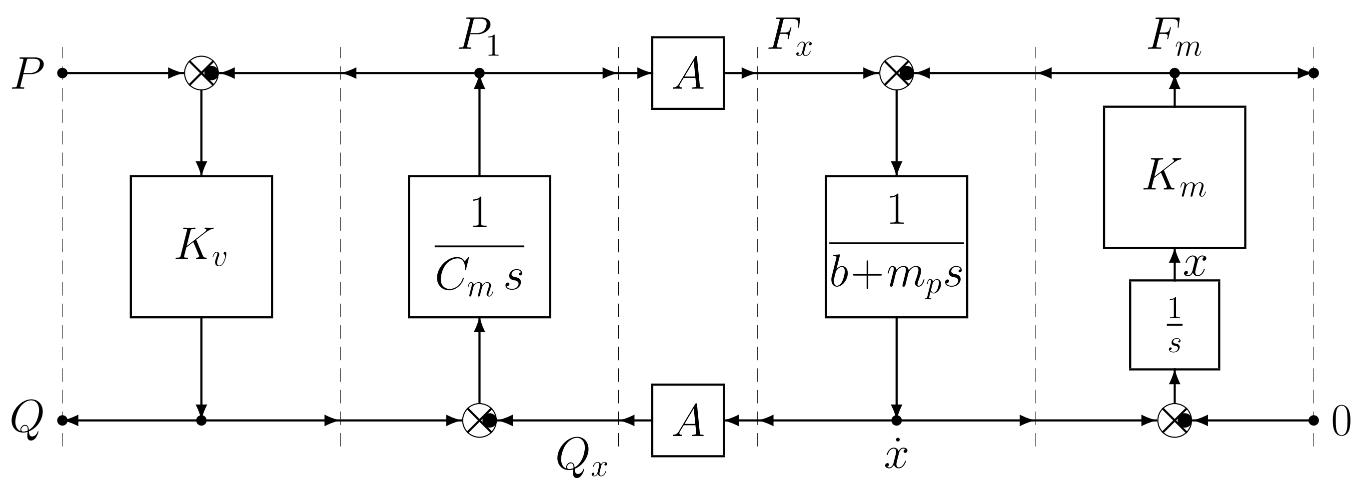
5) **Hydraulic clutch.**

Consider the following simplified hydraulic model of a clutch:

P	Supply pressure
Q	Volumetric flow rate in the valve
K_v	Constant of prop. of the valve
C_m	Cylinder hydraulic capacity
P_1	Pressure inside the cylinder
A	Piston section
x	Piston position
$\dot{x}$	Speed of the piston
m_p	Piston mass
b	Linear friction of the piston
K_m	Spring Stiffness
F_m	Spring force on the piston



- A block diagram describing the dynamics of the system is the following:



- The transfer function $G(s)$ which links the input P to the output F_m is:

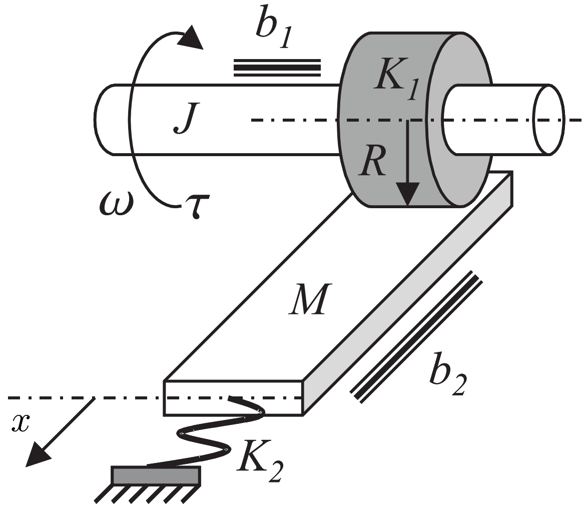
$$G(s) = \frac{F_m(s)}{P(s)} = \frac{A K_m K_v}{C_m m_p s^3 + (C_m b + K_v m_p)s^2 + (A^2 + C_m K_m + K_v b)s + K_m K_v}$$

which corresponds to the following third order differential equation:

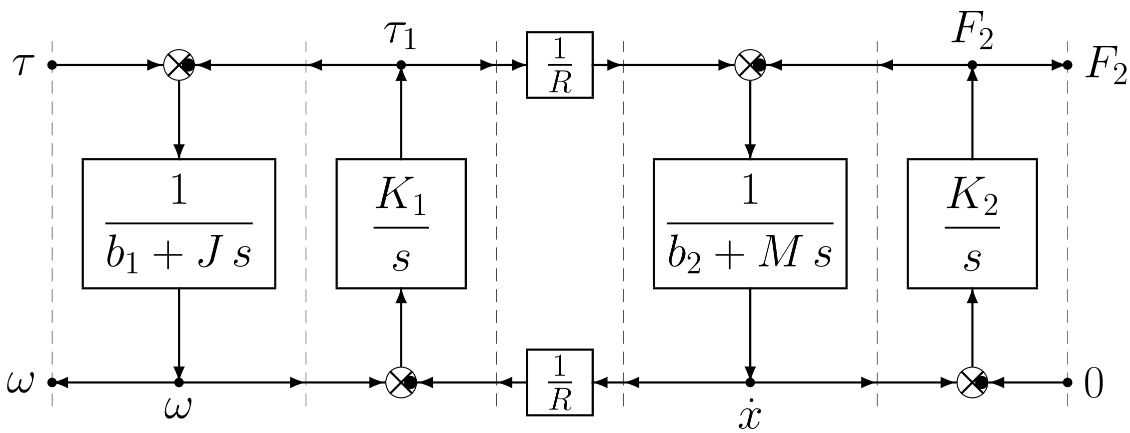
$$C_m m_p \dddot{F}_m + (C_m b + K_v m_p)\ddot{F}_m + (A^2 + C_m K_m + K_v b)\dot{F}_m + K_m K_v F_m = A K_m K_v P(t)$$

6) Mechanical transmission system.

Consider the mechanical system shown in figure: an external torque τ is applied to a shaft with inertia J which rotates at speed ω; the shaft moves an elastic gear characterized by a torsional stiffness K_1 and a radius R; the elastic gear moves a mass M which compresses a linear spring having stiffness coefficient K_2.



A block diagram describing the dynamics of the system is the following:



The transfer function $G(s)$ that links the input torque τ to the output force F_2 can be easily obtained using the Mason formula:

$$G(s) = \frac{K_1 K_2 R}{a_4\, s^4 + a_3\, s^3 + a_2 s^2 + a_1 s + a_0}$$

where

$$\begin{aligned} a_4 &= J\, M\, R^2 \\ a_3 &= (b_2 J + b_1 M) R^2 \\ a_2 &= J\, K_1 + b_1\, b_2\, R^2 + J\, K_2\, R^2 + K_1\, M\, R^2 \\ a_1 &= b_1\, K_1 + b_2\, K_1\, R^2 + b_1\, K_2\, R^2 \\ a_0 &= K_1\, K_2\, R^2 \end{aligned}$$