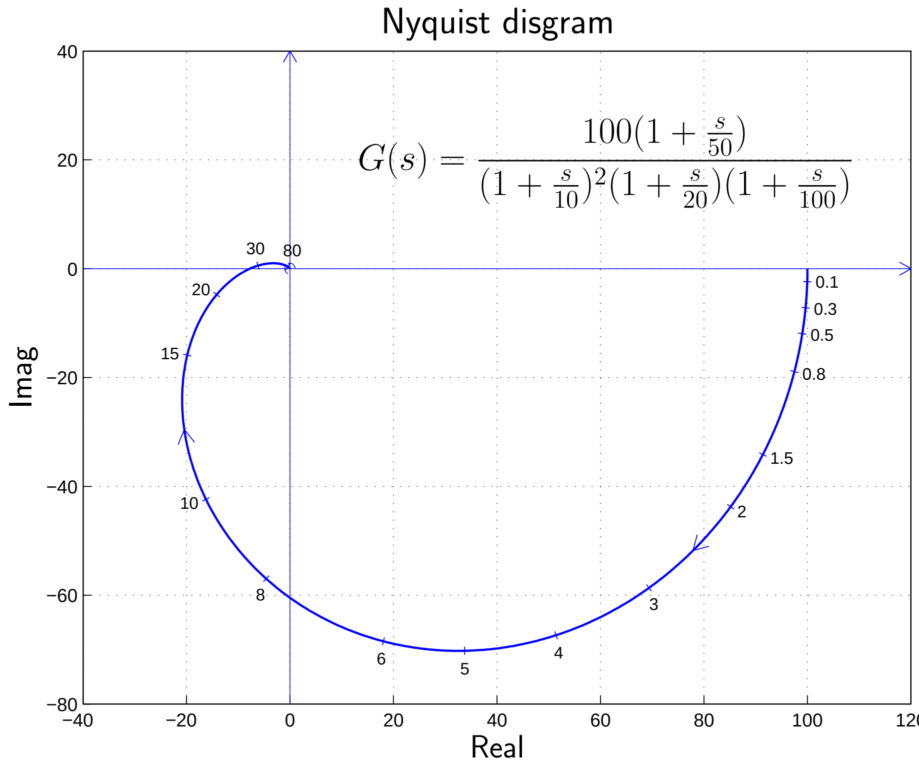
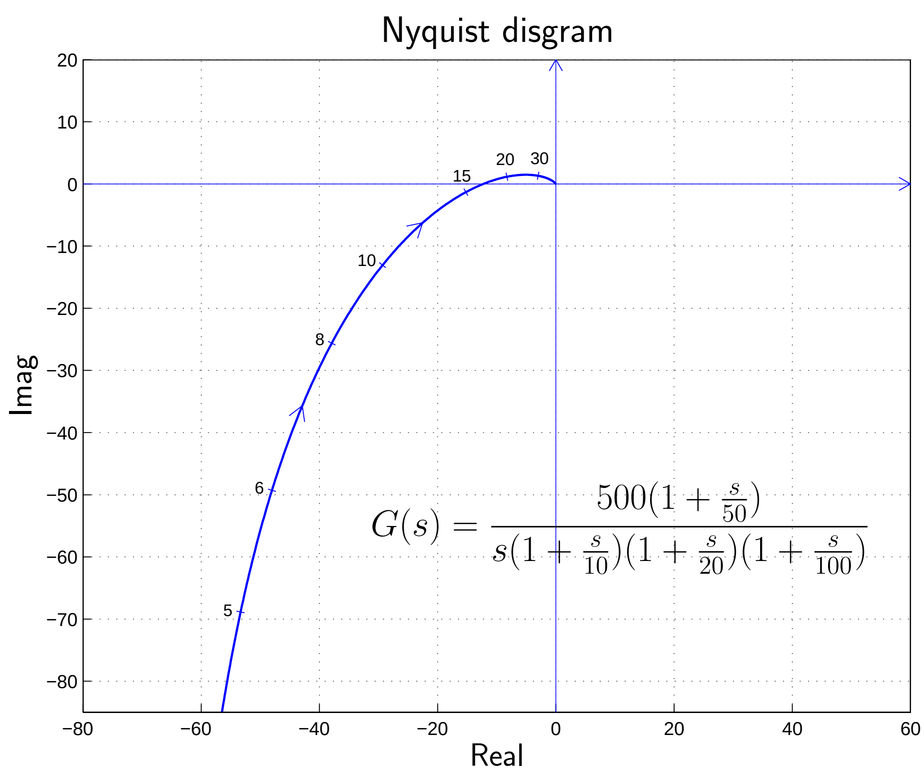
Nyquist (or polar) diagram

- Example of a Nyquist diagram of a system $G(s)$ without poles in the origin:



- Example of a Nyquist diagram of a system $G(s)$ with a pole in the origin:



- The Nyquist diagrams show, on the complex plane, how the complex number $G(j\omega)$ varies as a function of frequency ω.

Function $G(s)$: canonical forms

Canonical forms of a given function $G(s)$:

- *Polynomial form*:

$$G(s) = K_1 \frac{s^m + b_{m-1}\, s^{m-1} + \ldots + b_1\, s + b_0}{s^h\, (s^{n-h} + a_{n-1}\, s^{n-h-1} + \ldots + a_{h+1}\, s + a_h)}$$

- *Poles and zeros factorized form*:

$$G(s) = K_1 \frac{(s - z_1)\,(s - z_2)\, \ldots\, (s - z_m)}{s^h\, (s - p_1)\,(s - p_2)\, \ldots\, (s - p_{n-h})}$$

- *Time constants factorized form*:

$$G(s) = K \frac{\left(1+\tau_1' s\right)\left(1+\tau_2' s\right) \ldots \left(1+2\delta_1' \frac{s}{\omega_{n1}'} + \frac{s^2}{\omega_{n1}'^2}\right) \ldots}{s^h \left(1+\tau_1 s\right)\left(1+\tau_2 s\right) \ldots \left(1+2\delta_1 \frac{s}{\omega_{n1}} + \frac{s^2}{\omega_{n1}^2}\right) \ldots}$$

- The following properties hold:

1. Static gains:

$$b_0 = \prod_{i=1}^{m} (-z_i), \qquad a_h = \prod_{i=1}^{n-h} (-p_i), \qquad K = K_1 \frac{b_0}{a_h}.$$

2. Time constants:

$$\frac{b_1}{b_0} = -\sum_{i=1}^{m} \frac{1}{z_i} = \sum_{i=1}^{m} \tau_i' = \tau_1' + \tau_2' + \ldots + \frac{2\delta_1'}{\omega_{n1}'} + \frac{2\delta_2'}{\omega_{n2}'} + \ldots$$

$$\frac{a_{h+1}}{a_h} = -\sum_{i=1}^{n-h} \frac{1}{p_i} = \sum_{i=1}^{n-h} \tau_i = \tau_1 + \tau_2 + \ldots + \frac{2\delta_1}{\omega_{n1}} + \frac{2\delta_2}{\omega_{n2}} + \ldots$$

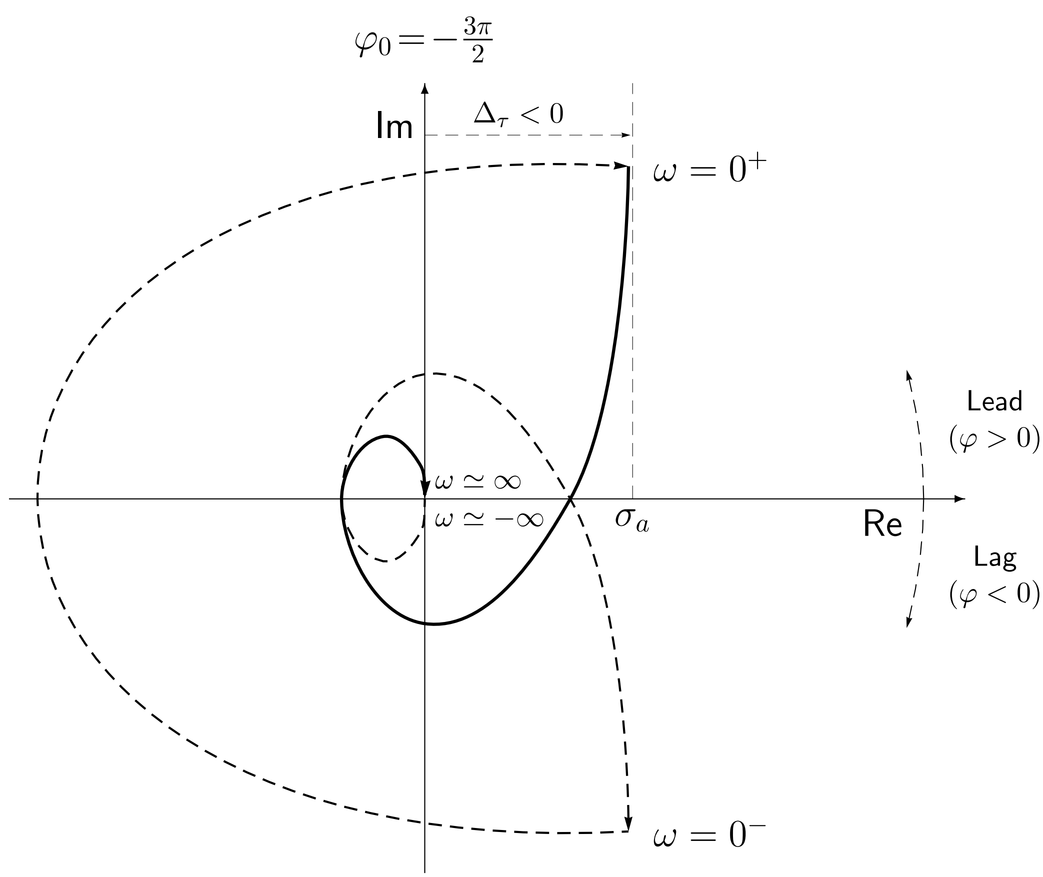
$$\Delta_\tau = \sum_{i=1}^{m} \tau_i' - \sum_{i=1}^{n-h} \tau_i = \sum_{i=1}^{n-h} \frac{1}{p_i} - \sum_{i=1}^{m} \frac{1}{z_i} = \frac{b_1}{b_0} - \frac{a_{h+1}}{a_h}$$

Qualitative plotting of the Nyquist diagram

Let us refer to the following transfer function $G(s)$:

$$G(s) = \frac{10(s-1)}{s(s+1)(s^2+8s+25)}.$$

Qualitative drawing of the Nyquist diagram of function $G(s)$:



1. *Starting point of the Nyquist diagram*. The starting point can be determined using the approximate function $G_0(s)$ when $s \simeq 0$:

$$G_0(s) \simeq G(s)|_{s\simeq 0} = \frac{K}{s^h} = \frac{-10}{25s} \qquad \Rightarrow \qquad \begin{cases} M_0 = \infty \\ \varphi_0 = -\frac{3}{2}\pi \end{cases}$$

The symbols M_0 and φ_0 denote the module and the phase of function $G(s)$ when $s = j\omega \simeq 0$. Constant K is equal to the multiplicative constant of function $G(s)$ given in the "time constants factorized form". Number h is equal to the "type" of system $G(s)$, that is the number of poles in the origin of function $G(s)$. Module M_0 is a function of parameter h:

$$M_0 = \begin{cases} G(0) = G_0(0) & \text{se} \quad h = 0 \\ \infty & \text{se} \quad h \geq 1 \end{cases}$$

2. *Direction of the initial phase φ*. For $\omega \simeq 0^+$ the Nyquist diagram has an initial phase φ which, for increasing ω, moves clockwise or counterclockwise in the complex plane. The sign of the initial phase φ is equal to the sign of the following parameter:

$$\Delta_\tau = \sum_{i=1}^{m} \tau_i' - \sum_{j=1}^{n-h} \tau_j$$

where τ_i' and τ_j are the time constants of the zeros and the poles of function $G(s)$, respectively. Parameter Δ_τ can be determined as follows:

$$\Delta_\tau = \begin{cases} \dfrac{b_1}{b_0} - \dfrac{a_{h+1}}{a_h} & \text{if } G(s) \text{ is given in the ``polinomial form''} \\ \left( \begin{array}{l} +\tau_1' + \tau_2' + \ldots + \frac{2\delta_1'}{\omega_{n1}'} + \ldots \\ -\tau_1 - \tau_2 - \ldots - \frac{2\delta_1}{\omega_{n1}} - \ldots \end{array} \right) & \text{if } G(s) \text{ is given in the ``time constant factorized form''} \end{cases}$$

If $\Delta_\tau > 0$, the initial phase is positive: $\varphi > 0$. If $\Delta_\tau < 0$ the initial phase is negative: $\varphi > 0$. For the considered system we have:

$$\Delta_\tau = -1 - \left(1 + \frac{8}{25}\right) = -\frac{58}{25} < 0.$$

Note: the time constant $\tau = 2\delta/\omega_n = 8/25$ that characterizes the two complex conjugate poles $(s^2 + 8s + 25)$ is obtained by neglecting the quadratic term in s. The constant Δ_τ is negative, so the diagram moves in clockwise direction with respect to the initial phase $\varphi_0 = -\frac{3}{2}\pi$.

3. *Presence of an asymptote*. The Nyquist diagram has an asymptote only if $h = 1$. The asymptote, if it exists, is always vertical. The abscissa σ_a of the vertical asymptote can be determined as follows:

$$\sigma_a = K\Delta_\tau$$

where K is the multiplicative constant of function $G(s)$ given in the *time constants factorized form*. For the considered system we have:

$$\sigma_a = \frac{-10}{25}\left(-1-1-\frac{8}{25}\right) = \frac{116}{125} = 0.928.$$

Note that the position $\sigma_a > 0$ of the asymptote is consistent with the result obtained in point 2 which stated that the Nyquist diagram moves in the clockwise direction with respect to the phase initial φ_0.

4. *Final point of the Nyquist diagram*. The final point φ_∞ can be determined by using the approximate function $G_\infty(s)$ when $s \simeq \infty$:

$$G_\infty(s) \simeq G(s)\big|_{s\simeq\infty} = \frac{K_1}{s^r} = \frac{10}{s^3} \qquad \Rightarrow \qquad \begin{cases} M_\infty = 0 \\ \varphi_\infty = -\frac{3}{2}\pi \end{cases}$$

The symbols M_∞ and φ_∞ denote the module and the phase of function $G(s)$ when $s = j\omega \simeq \infty$. The approximate function $G_\infty(s)$ is always the product of a constant K_1 (the multiplicative constant of function $G(s)$ in the "poles and zeros factorized form") and r integrators, where $r = n - m$ is the the relative degree of function $G(s)$. When $r = 0$ the final point is on the real axis. When $r > 0$ the final point is in the origin.

When $\omega \simeq \infty$, the phase of the Nyquist diagram is higher o smaller than the final phase φ_∞ depending on the sign of the following parameter:

$$\Delta_p = \sum_{i=1}^{m} z_i - \sum_{j=1}^{n} p_j$$

where z_i and p_j denote the zeros and poles, respectively, of function $G(s)$. For the considered system we have:

$$\Delta_p = 1 - (-1-8) = 10 > 0.$$

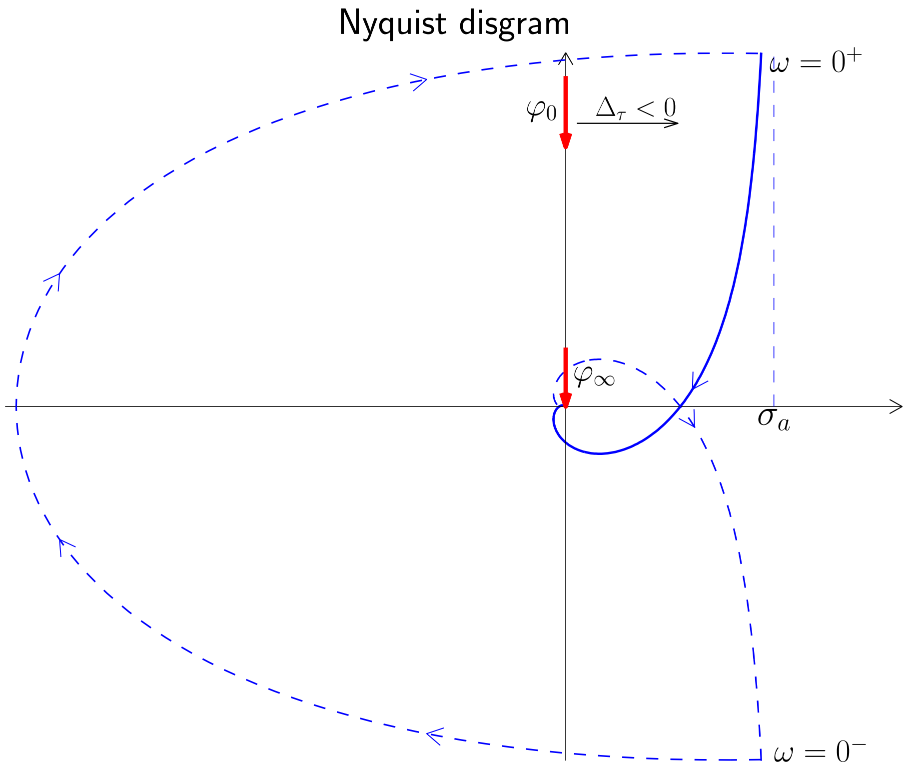
Since $\Delta_p > 0$, the phase of Nyquist diagram when $\omega \simeq \infty$ is higher than the final phase $\varphi_\infty = -\frac{3}{2}\pi$.

5. *Nyquist diagram: qualitative plotting for* $\omega \in]0,\ \infty[$. The starting point $G(j0^+)$ and the final point $G(j\infty)$ of the Nyquist diagram are connected by a curve on the complex plane that can be easily plotted by using the following parameter $\Delta\varphi$:

$$\Delta\varphi = (Z_s + P_i - Z_i - P_s)\frac{\pi}{2}$$

which is the total phase shift provided by the poles and zeros (stable and unstable) of function $G(s)$ for $\omega \in]0,\ \infty[$. Parameters Z_s, P_i, Z_i and P_s denote, respectively, the number of stable zeros, unstable poles, unstable zeros and stable poles of function $G(s)$ without considering the poles or the zeros in the origin. A qualitative plotting of the Nyquist diagram can be easily obtained connecting the starting point $G(j0^+)$ to the final point $G(j\infty)$ with a curve which rotates with respect to the origin of an angular quantity equal to $\Delta\varphi$. In this case we have $\Delta\varphi = -2\pi$.

6. *The "complete" Nyquist diagram*. The Nyquist diagram for $\omega < 0$ is obtained by flipping upsidedown with respect to the real axis the Nyquist diagram plotted for $\omega > 0$. *For $h \geq 1$, the "complete" Nyquist diagram is obtained by closing the diagram at the infinity: point $G(j0^-)$ must be connected to point $G(j0^+)$ by plotting in the clockwise direction as many infinite semi-circumferences as many poles function $G(s)$ has in the origin.*



Qualitative plotting of Nyquist diagrams: examples.

Example. Draw the Nyquist diagram of the following function:

$$G(s) = \frac{10(s+3)}{s(s+0.2)(s^2+15s+100)}$$

1. *Starting point*. Approximate function $G_0(s)$:

$$\lim_{s\to 0} G(s) \simeq G_0(s) = \frac{3}{2s} = \frac{K}{s} \qquad \Rightarrow \qquad \begin{cases} M_0 = \infty \\ \varphi_0 = -\frac{\pi}{2} \end{cases}$$

   The initial phase is $\varphi_0 = -\frac{\pi}{2}$. The initial gain: $K = \frac{3}{2}$.

2. *Direction of the initial phase φ*. The parameter Δ_τ is positive:

$$\Delta_\tau = \sum_{i=1}^{m} \tau_i' - \sum_{j=1}^{n-h} \tau_j = \frac{1}{3} - \frac{1}{0.2} - \frac{15}{100} = -4.82 < 0$$

   The phase φ of the Nyquist diagram for $\omega \simeq 0$ is higher than $\varphi_0 = -\frac{\pi}{2}$.

3. *Eventual asymptote*. The diagram has a vertical asymptote. The abscissa σ_a of the vertical asymptote is:

$$\sigma_a = K\Delta_\tau = \frac{3}{2}(-4.82) = -7.23$$

4. *Final point*. Approximate function $G_\infty(s)$:

$$\lim_{s\to\infty} G(s) \simeq G_\infty(s) = \frac{10}{s^3} = \frac{K_1}{s^3} \qquad \Rightarrow \qquad \begin{cases} M_\infty = 0 \\ \varphi_\infty = -\frac{3}{2}\pi \end{cases}$$

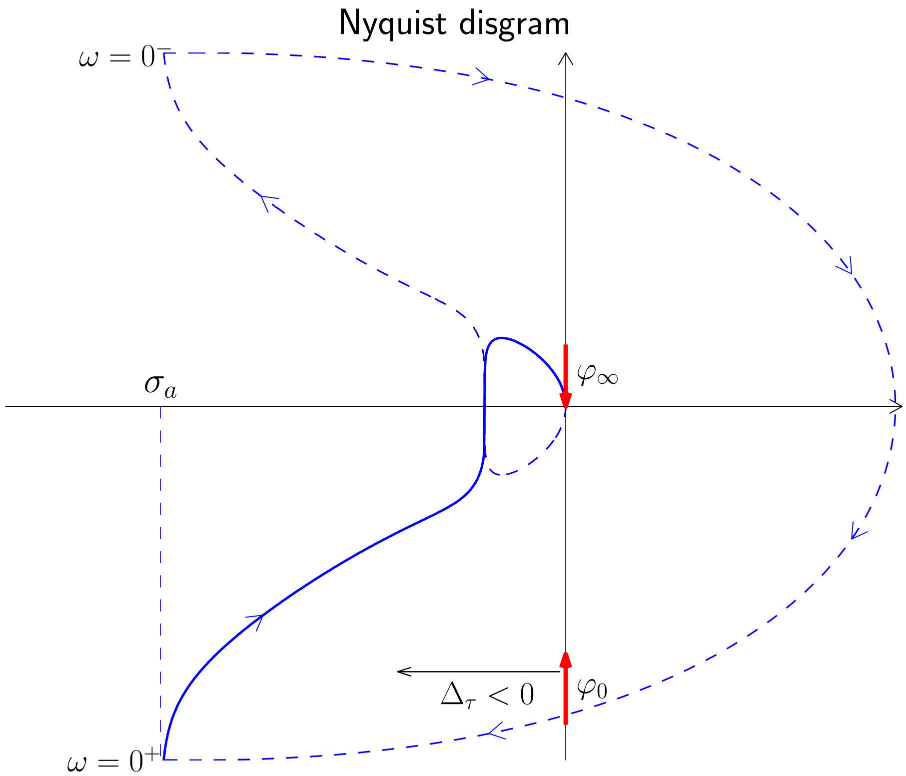
   The phase of the final point is $\varphi_\infty = -\frac{3}{2}\pi$. Since $\Delta_p = -3 + 0.2 + 15 = 12.2 > 0$, the phase φ of the Nyquist diagram for $\omega \simeq \infty$ is higher that the final phase φ_∞.

5. *Phase shift for $\omega \in]0,\ \infty[$*. The phase shift when $\omega \in]0,\ \infty[$ is:

$$\Delta\varphi = \frac{\pi}{2} - \left(\frac{\pi}{2} + \pi\right) = -\pi$$

   Moving from $G(j0^+)$ to $G(j\infty)$ the diagram rotates $-\pi$ counterclockwise.

6. *The "complete" Nyquist diagram*. Being $h = 1$, the Nyquist diagram must be closed with a clockwise infinitely large semi-circumference.



Example. Draw the Nyquist diagram of the following function:

$$G(s) = \frac{10(1+2s)(s-5)}{s^2(s^2+3s+100)}$$

1. *Starting point*. Approximate function $G_0(s)$:

$$\lim_{s\to 0} G(s) \simeq G_0(s) = -\frac{1}{2s^2} = \frac{K}{s^2} \qquad \Rightarrow \qquad \begin{cases} M_0 = \infty \\ \varphi_0 = -2\pi \end{cases}$$

The initial phase of the diagram is $\varphi_0 = -2\pi$.

2. *Direction of the initial phase φ*. The parameter Δ_τ is positive:

$$\Delta_\tau = 2 - \frac{1}{5} - \frac{3}{100} = 1.77 > 0$$

The phase φ of the Nyquist diagram for $\omega \simeq 0$ is higher than $\varphi_0 = -2\pi$.

3. *Eventual asymptote*. The system is of type 2 and therefore the system has no asymptotes.

4. *Final point*. Approximate function $G_\infty(s)$:

$$\lim_{s\to\infty} G(s) \simeq G_\infty(s) = \frac{20}{s^2} = \frac{K_1}{s^2} \qquad \Rightarrow \qquad \begin{cases} M_\infty = 0 \\ \varphi_\infty = -\pi \end{cases}$$

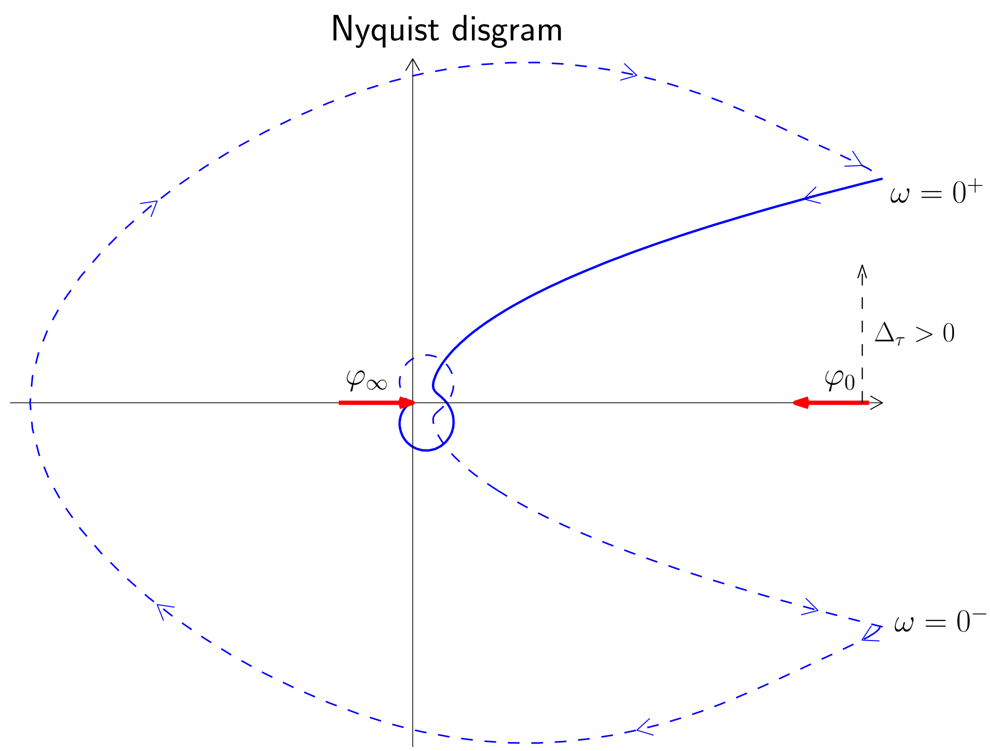
The phase of the final point is $\varphi_\infty = -\pi$. Since $\Delta_p = -0.5 + 5 + 3 = 7.5 > 0$, the phase φ of the Nyquist diagram for $\omega \simeq \infty$ is higher that the final phase φ_∞.

5. *Phase shift for $\omega \in]0,\ \infty[$*. The phase shift when $\omega \in]0,\ \infty[$ is:

$$\Delta\varphi = \frac{\pi}{2} - \frac{\pi}{2} - \pi = -\pi$$

Moving from $G(j0^+)$ to $G(j\infty)$ the diagram rotates π clockwise.

6. *The "complete" Nyquist diagram*. Being $h = 2$, the Nyquist diagram must be closed with a clockwise infinitely large circumference.



Example. Draw the Nyquist diagram of the following function:

$$G(s) = \frac{(1+5s)}{s(s+5)(s^2+s+1)}$$

1. *Starting point*. Approximate function $G_0(s)$:

$$\lim_{s\to 0} G(s) \simeq G_0(s) = \frac{1}{5s} = \frac{K}{s} \qquad \Rightarrow \qquad \begin{cases} M_0 = \infty \\ \varphi_0 = -\dfrac{\pi}{2} \end{cases}$$

   The initial phase of the diagram is $\varphi_0 = -\frac{\pi}{2}$. Initial profit: $K = 0.2$.

2. *Direction of the initial phase φ*. The parameter Δ_τ is positive:

$$\Delta_\tau = 5 - \frac{1}{5} - 1 = 3.8 > 0$$

   The phase φ of the Nyquist diagram for $\omega \simeq 0$ is higher than $\varphi_0 - \frac{\pi}{2}$.

3. *Eventual asymptote*. The abscissa σ_a of the vertical asymptote $(h = 1)$ is:

$$\sigma_a = K\Delta_\tau = 0.2 \cdot 3.8 = 0.76$$

4. *Final point*. Approximate function$G_\infty(s)$:

$$\lim_{s\to\infty} G(s) \simeq G_\infty(s) = \frac{5}{s^3} \qquad \Rightarrow \qquad \begin{cases} M_\infty = 0 \\ \varphi_\infty = -\dfrac{3}{2}\pi \end{cases}$$

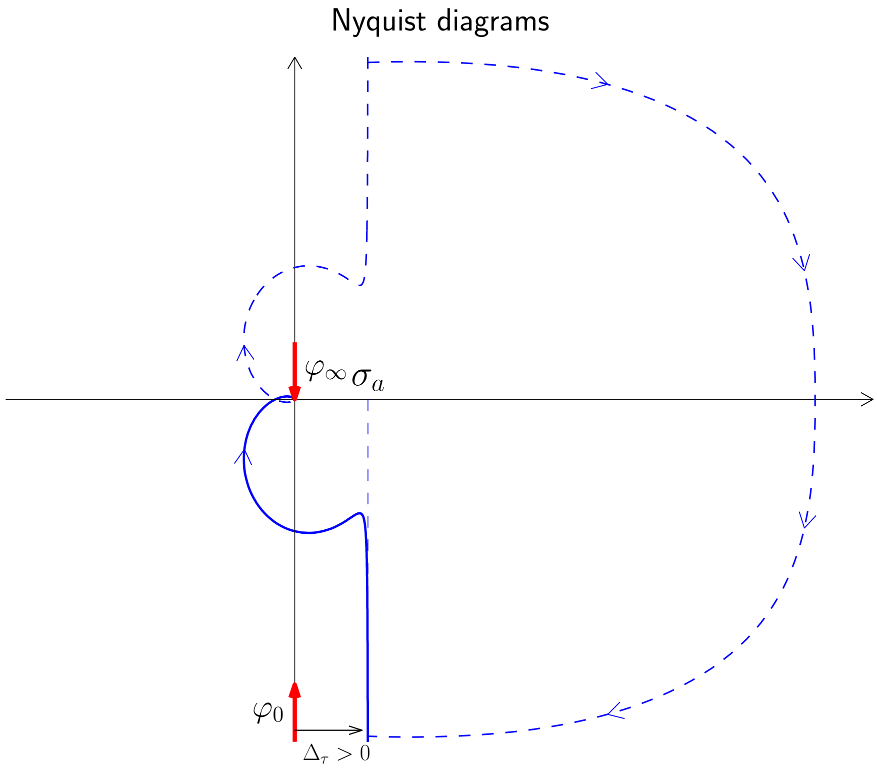
   The phase of the final point is $\varphi_\infty = -\frac{3}{2}\pi$. Since $\Delta_p = -0.2 + 5 + 1 = 5.8 > 0$, the phase φ of the Nyquist diagram for $\omega \simeq \infty$ is higher that the final phase φ_∞.

5. *Phase shift for $\omega \in]0,\ \infty[$*. The phase shift when $\omega \in]0,\ \infty[$ is:

$$\Delta\varphi = \frac{\pi}{2} - \frac{\pi}{2} - \pi = -\pi$$

   Moving from $G(j0^+)$ to $G(j\infty)$ the diagram rotates π clockwise.

6. *The "complete" Nyquist diagram*. Being $h = 1$, the Nyquist diagram must be closed with a clockwise infinitely large semi-circumference.



Example. Draw the Nyquist diagram of the following function:

$$G(s) = \frac{(s-1)(s-3000)}{s(s^2+12s+144)}$$

1. *Starting point*. Approximate function $G_0(s)$:

$$\lim_{s\to 0} G(s) \simeq G_0(s) = \frac{3000}{144\,s} = \frac{K}{s} \qquad \Rightarrow \qquad \begin{cases} M_0 = \infty \\ \varphi_0 = -\dfrac{\pi}{2} \end{cases}$$

   The initial phase of the diagram is $\varphi_0 = -\frac{\pi}{2}$. Initial earning: $K =$20.83.

2. *Direction of the initial phase φ*. The parameter Δ_τ is negative:

$$\Delta_\tau = -1 - \frac{1}{3000} - \frac{12}{144} = -1.084 < 0$$

   The phase φ of the Nyquist diagram for $\omega \simeq 0$ is lower than $\varphi_0 - \frac{\pi}{2}$.

3. *Eventual asymptote*. The abscissa σ_a of the vertical asymptote ($h = 1$) is:

$$\sigma_a = K\Delta_\tau = 20.83\,(-1.084) = -22.58$$

4. *Final point*. Approximate function $G_\infty(s)$:

$$\lim_{s\to\infty} G(s) \simeq G_\infty(s) = \frac{1}{s} \qquad \Rightarrow \qquad \begin{cases} M_\infty = 0 \\ \varphi_\infty = -\dfrac{\pi}{2} \end{cases}$$

The phase of the final point is $\varphi_\infty = -\frac{\pi}{2}$. Since $\Delta_p = 1 + 3000 + 12 = 3013 > 0$, the phase φ of the Nyquist diagram for $\omega \simeq \infty$ is higher that the final phase φ_∞.

5. *Phase shift for $\omega \in]0, \infty[$*. The phase shift when $\omega \in]0, \infty[$ is:

$$\Delta\varphi = -\frac{\pi}{2} - \frac{\pi}{2} - \pi = -2\pi$$

Moving from $G(j0^+)$ to $G(j\infty)$ the Nyquist diagram rotates π clockwise.

6. *The "complete" Nyquist diagram*. Being $h = 1$, the Nyquist diagram must be closed with a clockwise infinitely large semi-circumference.

Diagramma di Nyquist