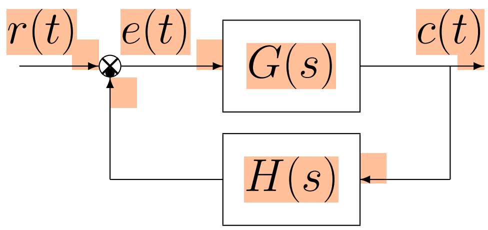
The Nyquist criterion

- Given the frequency response of a linear system $F(s)$, the Nyquist criterion provides a necessary and sufficient condition to determine the stability of the corresponding closed loop system $G_0(s)$:



$$F(s) = G(s)H(s)$$

$$G_0(s) = \frac{G(s)}{1 + F(s)}$$

- Due to its graphical nature, the Nyquist criterion is useful for the design of the controller because it provides information: 1) on the stability and robustness of the controlled system; 2) on the type of controller which is useful to use to improve the dynamic behavior of the controlled system.

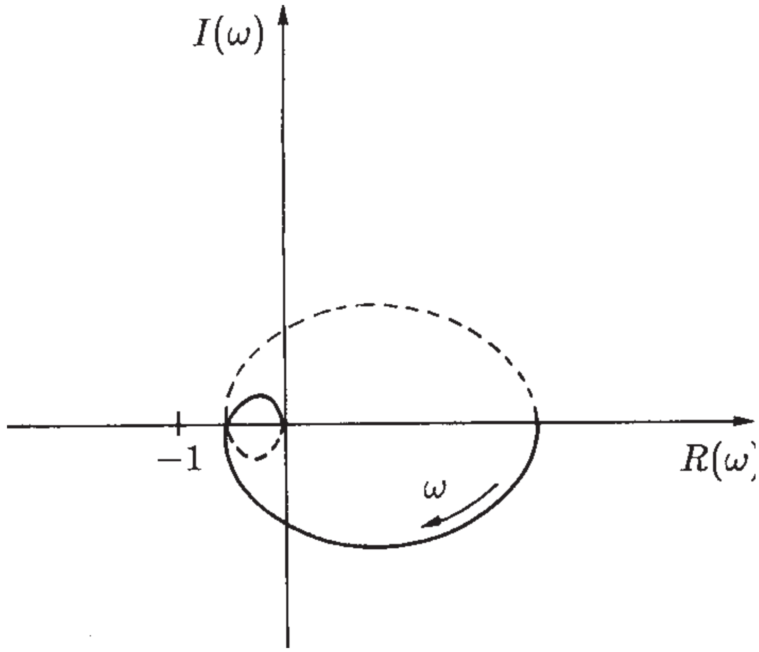
- **Nyquist criterion (systems stable in open loop).** *If all the poles of the closed loop gain $F(s)$ are asymptotically stable, except for one or two poles in the origin, a necessary and sufficient condition for the feedback system $G_0(s)$ to be asymptotically stabile is that the complete Nyquist diagram of function $F(s)$ does not touch nor encircle the critical point $-1+j0$.*

- The Nyquist criterion refers to the "complete Nyquist diagram", i.e. the Nyquist diagram must be plotted for $\omega \in [-\infty, +\infty]$. Since:

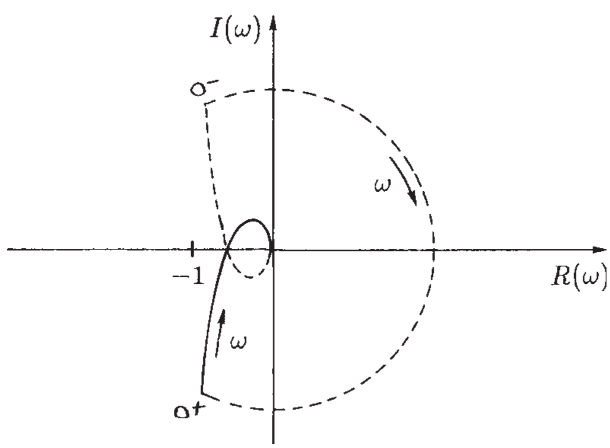
$$F(-j\omega) = F^*(j\omega)$$

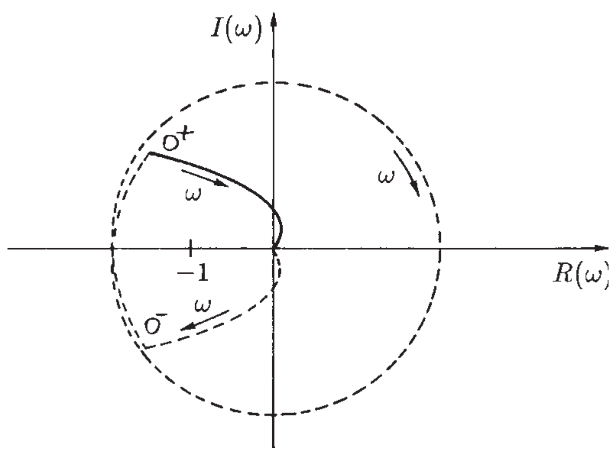
the Nyquist diagram for negative frequencies can easily be obtained by flipping upsidedown, around the real axis, the Nyquist diagram plotted for positive frequencies.

- For systems of "type 0" the "complete Nyquist diagram" is a closed curve:



- For systems of "type 1" or "type 2" (which have branches which pass through the infinity) the Nyquist diagram must be closed at the infinity with a semi o full circumference, respectively, plotted in the clockwise direction starting from $\omega = 0^-$ and arriving at $\omega = 0^+$

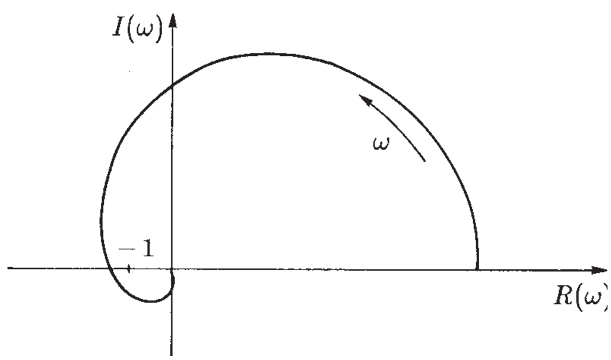




- The Nyquist criterion for the open loop stable systems covers most of the cases of interest. However, the Nyquist criterion can be given in a more general version which applies also to the systems which are unstable in open loop.

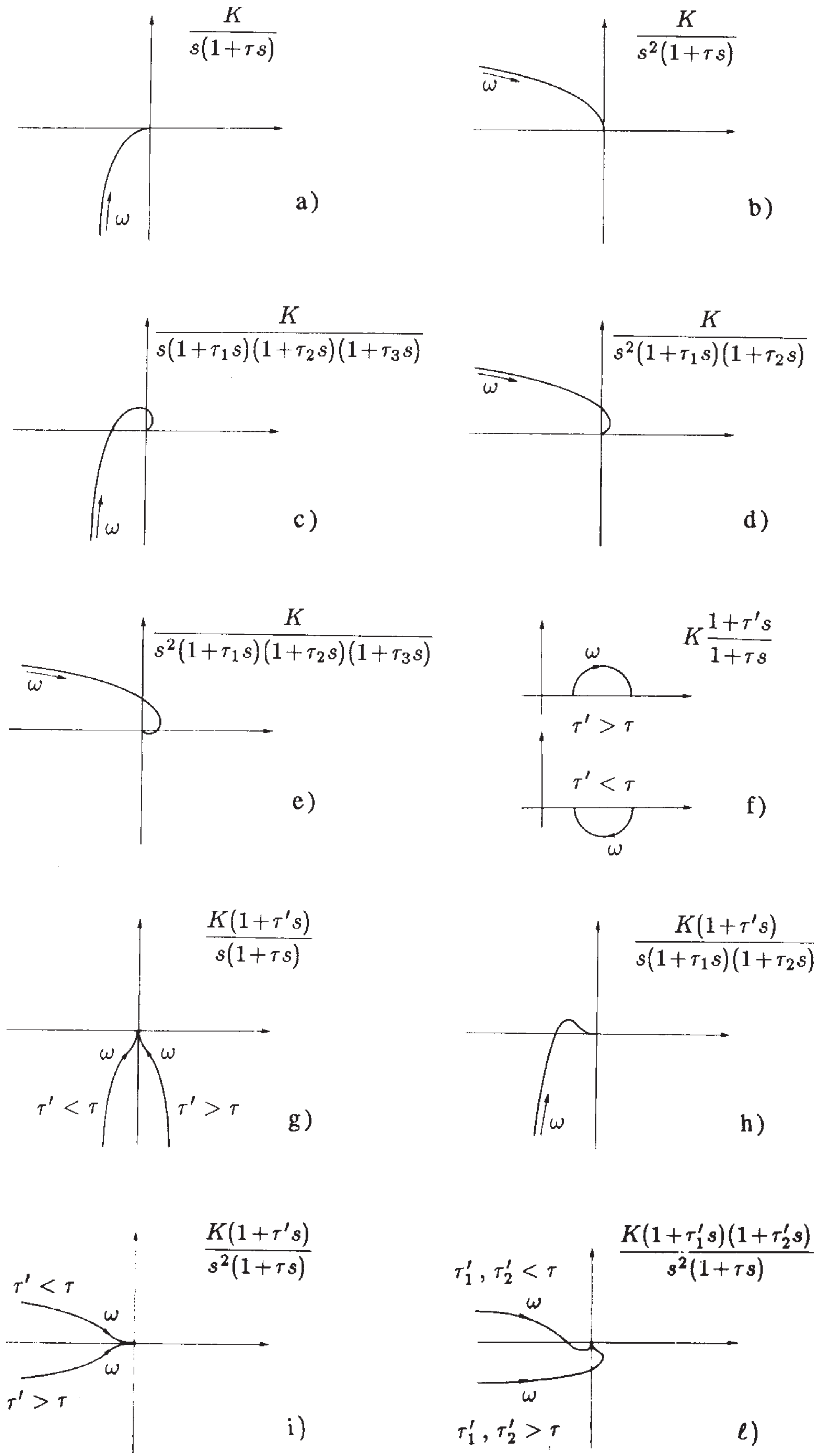
- **Nyquist criterion (for systems unstable in open loop).** If the closed loop gain $F(s)$ does not have poles on the imaginary axis, except for one or two poles in the origin, *necessary and sufficient condition* for the feedback system $G_0(s)$ to be asymptotically stabile is that the *complete Nyquist diagram* of function $F(s)$ encircles the critical point $-1+j0$ as many times in counterclockwise direction as is the number of unstable poles of function $F(s)$.

- **Example** Consider the following system:



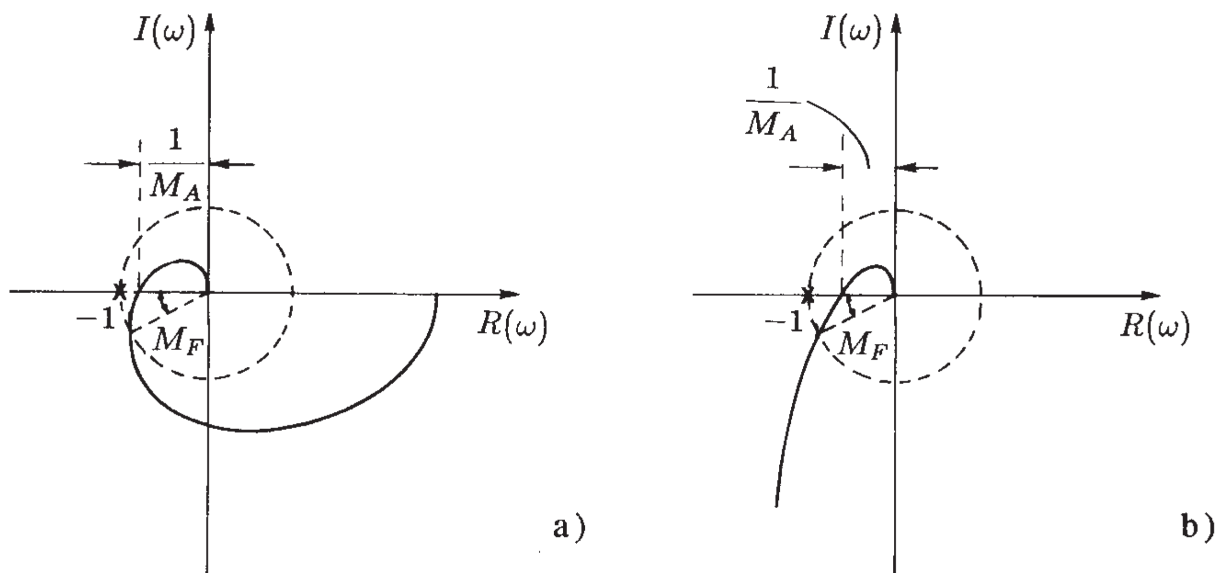
$$F(s) = \frac{K\,(1+\tau\,s)}{(1-\tau_1\,s)\,(1-\tau_2\,s)}$$

The feedback system is stable for high values of constant K because, in this case, the Nyquist diagram of function $F(s)$ encircles two times counterclockwise the critical point $-1+j0$ and function $F(s)$ has two unstable poles. On the contrary, for small values of constant K, the Nyquist diagram of function $F(s)$ does not encircle the critical point $-1+j0$ and therefore the feedback system is unstable.


$\frac{K}{s(1+\tau s)}$
ω
a)
$\frac{K}{s^2(1+\tau s)}$
ω
b)
$\frac{K}{s(1+\tau_1 s)(1+\tau_2 s)(1+\tau_3 s)}$
ω
c)
$\frac{K}{s^2(1+\tau_1 s)(1+\tau_2 s)}$
ω
d)
$\frac{K}{s^2(1+\tau_1 s)(1+\tau_2 s)(1+\tau_3 s)}$
ω
e)
$K\frac{1+\tau' s}{1+\tau s}$
ω
$\tau' > \tau$
$\tau' < \tau$
ω
f)
$\frac{K(1+\tau' s)}{s(1+\tau s)}$
ω ω
$\tau' < \tau$
$\tau' > \tau$
g)
$\frac{K(1+\tau' s)}{s(1+\tau_1 s)(1+\tau_2 s)}$
ω
h)
$\frac{K(1+\tau' s)}{s^2(1+\tau s)}$
$\tau' < \tau$
ω
ω
$\tau' > \tau$
i)
$\frac{K(1+\tau_1' s)(1+\tau_2' s)}{s^2(1+\tau s)}$
$\tau_1', \tau_2' < \tau$
ω
ω
$\tau_1', \tau_2' > \tau$
ℓ)

Gain and Phase Margins

- The Nyquist diagram of a given function $G(s)$ is useful also because it provides a measure of the "robustness" of the feedback systems, that is the distance of the Nyquist diagram $G(j\omega)$ from the critical point $-1+j\,0$.
- If the Nyquist diagram $G(j\omega)$ is far from the critical point (and the Nyquist criterion is satisfied) then the feedback system is stable and far from the instability.



- The "measure of the distance" of the Nyquist diagram $G(j\omega)$ from the critical point is usually given by means of the *Gain and Phase Margins*:

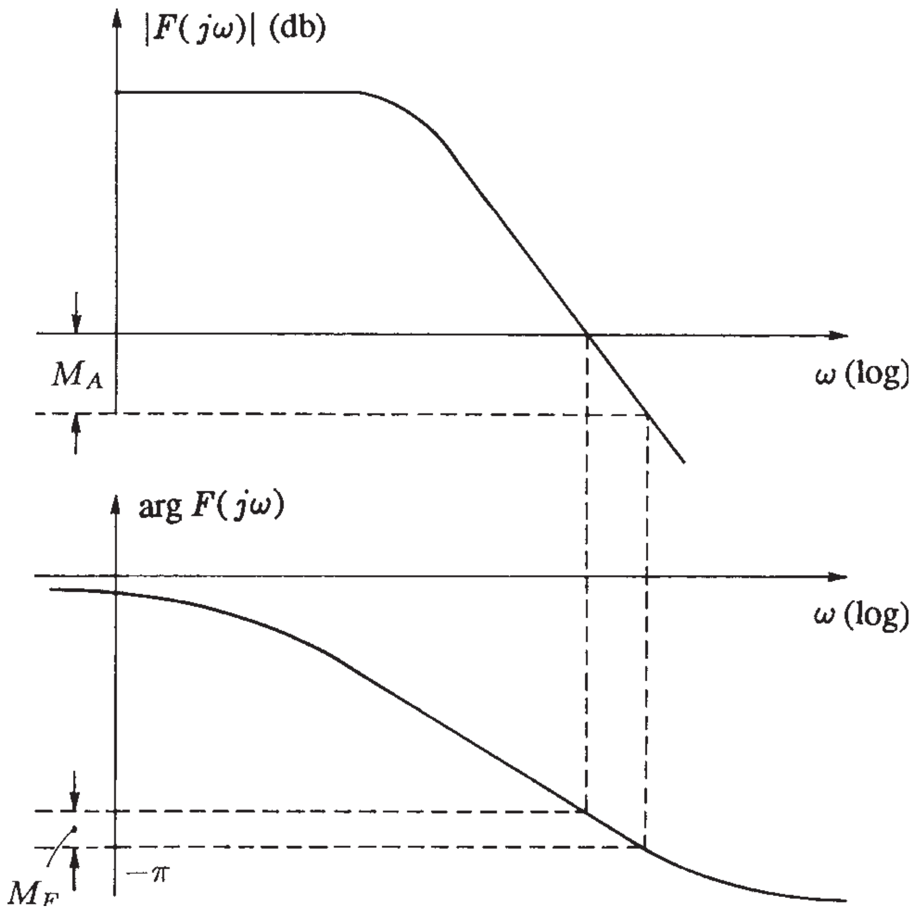
Gain Margin M_A: is the inverse of the gain of function $G(j\omega)$ at the frequency $\bar{\omega}$ for which $\arg G(j\bar{\omega}) = -\pi$:

$$M_A = \frac{1}{|G(j\bar{\omega})|} \qquad \text{where} \qquad \arg G(j\bar{\omega}) = -\pi.$$

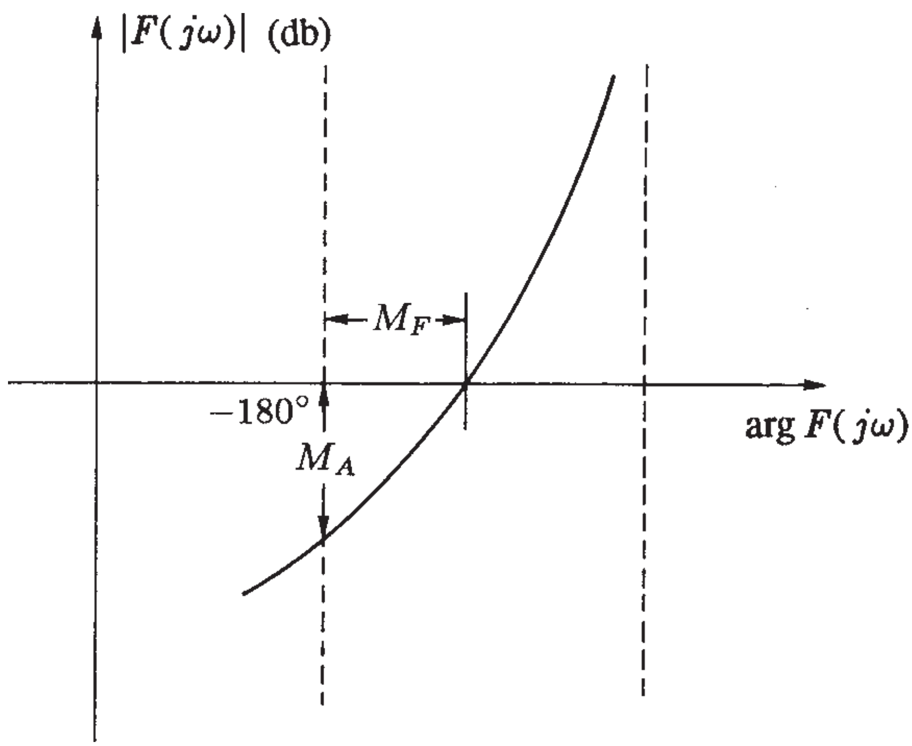
Phase Margin M_F: is the phase of function $G(j\omega)$ at the frequency ω^* for which $|G(j\omega^*)| = 1$ (i.e. when $G(j\omega)$ intersects the unitary circle) minus π (i.e. the phase of the critical point $-1+j\,0$).

$$M_F = \arg G(j\omega^*) - \pi \qquad \text{where} \qquad |G(j\omega^*)| = 1.$$

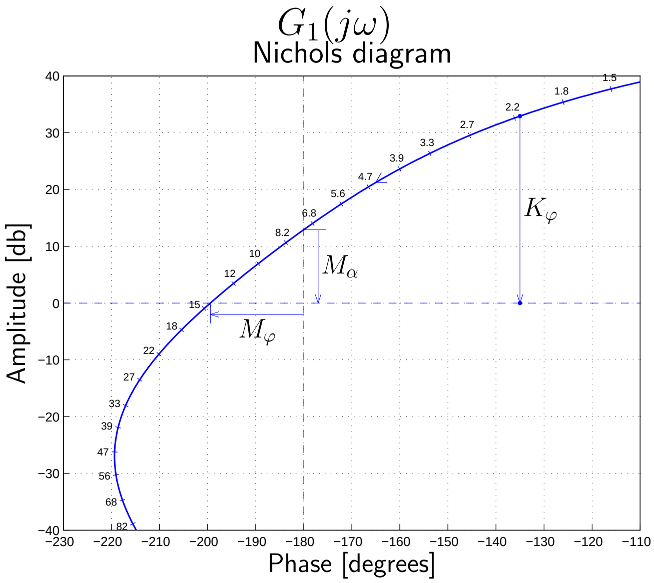
- Gain and Phase margins on Bode diagrams:

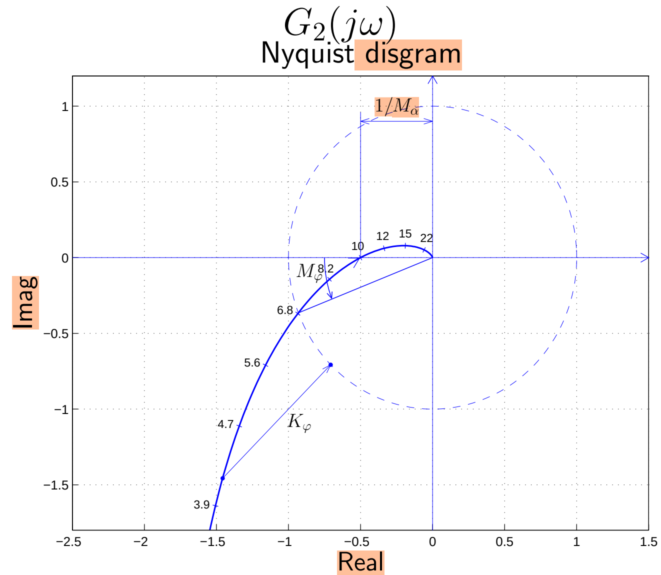


- Gain and Phase margins on Nichols diagrams:



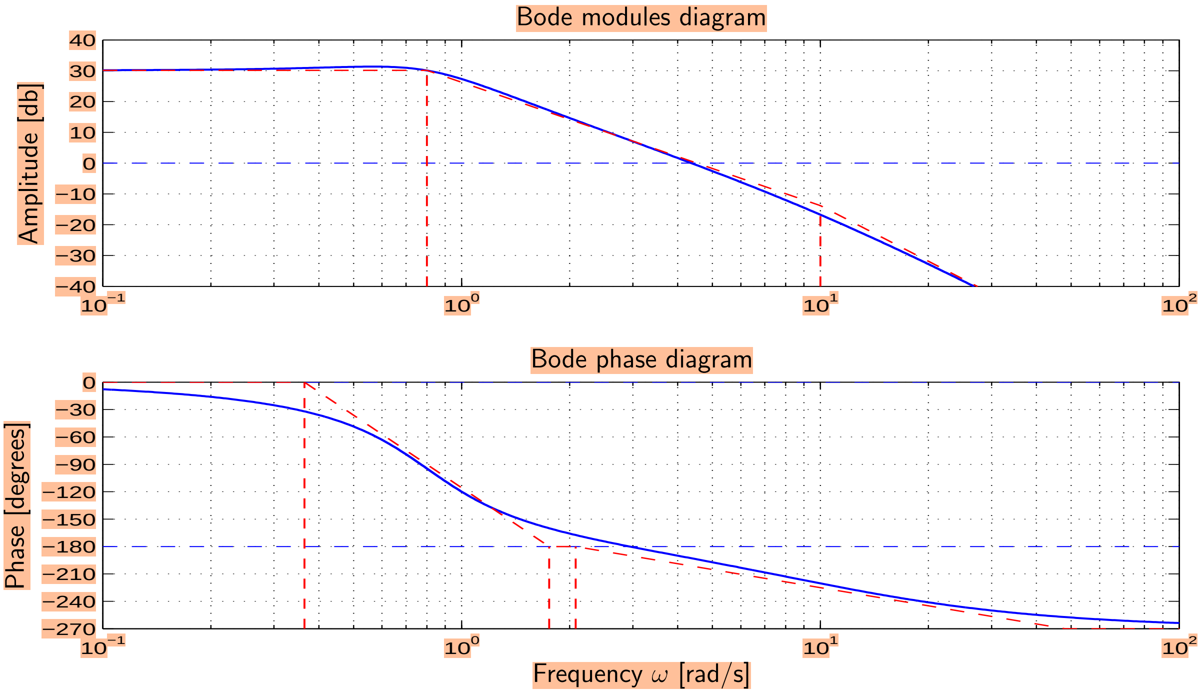
- The two figures show the frequency response functions of two minimum phase linear systems $G_1(s)$ and $G_2(s)$.
  For each system and with the precision allowed by the graphs, determine:
  c.1) the gain margin M_a of the system;
  c.2) the phase margin M_φ of the system;
  c.3) the gain K_φ the system $K_\varphi G(s)$ has a phase margin $M_\varphi = 45°$;
  c.4) the steady-state output response $y_r(t)$ of system $G(s)$ when the input signal is $x(t) = 3\sin(5.6t)$;





c.1) $M_a = -12.91$ db $= 0.226$	c.1) $M_a = 2$
c.2) $M_\varphi = -19.42°$	c.2) $M_\varphi = 21.4°$
c.3) $K_\varphi = -32.92$ db $= 0.022$	c.3) $K_\varphi = 0.49$
c.4) $y_r(t) = 3 \cdot 7.4511 \sin(5.6t - 172.2°)$	c.4) $y_r(t) = 3 \cdot 1.36 \sin(5.6t - 148.5°)$

- Given the following system:



Compute:

a) The position of the two dominant pole $p_{1,2}$ of system $G(s)$:

$$p_{1,2} \simeq -0.4 \pm j0.693.$$

b) The settling time T_a of the step response of system $G(s)$:

$$T_a \simeq \frac{3}{0.4} \text{ s} = 7.5 \text{ s}.$$

c) The stability margins of system $G(s)$:

$$M_\varphi \simeq -10^\circ, \qquad M_a \simeq 0.5,$$