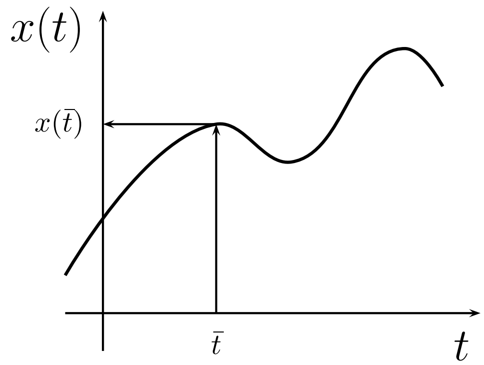
MATHEMATICAL RECALLS

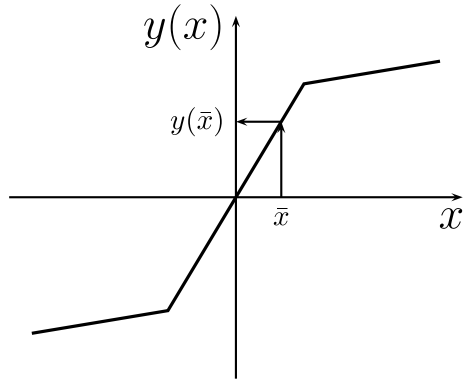
- **Real functions of time t:**

$$x(t) : t \to x(t)$$



- **Real functions of input x:**

$$y(x) : x \to y(x)$$



- **Complex numbers**. A complex number is a ordered couple of real numbers:

$$(x,\ y)$$

where x is the real part and y is the imaginary part of the complex number. There are many equivalent ways of representing complex numbers:

1) Using the imaginary number "j" a complex number can be expressed as follows:

$$(x,\ y) \equiv x + j\,y$$

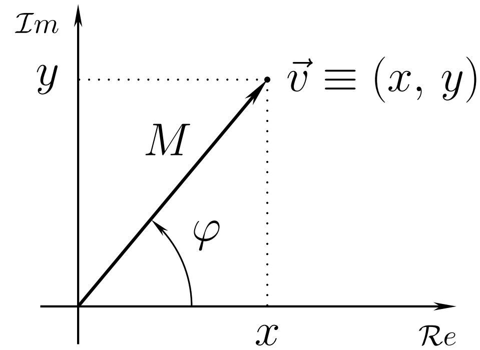
The number "j" denotes the imaginary part. The number "j" satisfies the following relations:

$$j = \sqrt{-1}, \qquad j^2 = -1, \qquad j^3 = -j, \qquad j^4 = 1, \qquad j^5 = j.$$

2) The complex numbers $(x,\, y)$ are in biunivocal correspondence with the points of a plane:

$$\vec{v} \equiv (x,\, y) \equiv x + j\, y$$

x denotes the real part
y denotes the imaginary part.



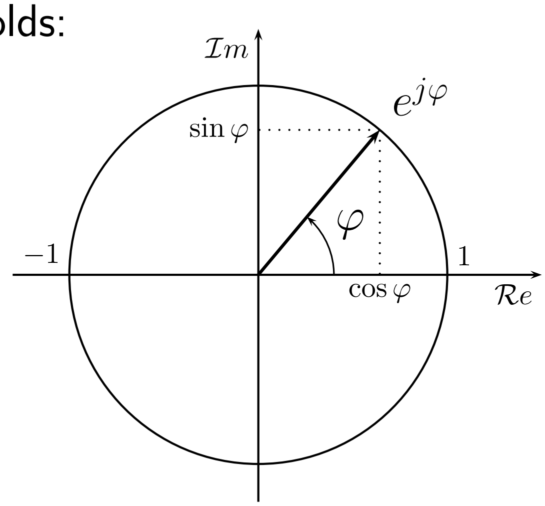
3) The points of the plane, in turn, are in biunivocal correspondence with the vectors $\vec{v}$ that link the point $(x,\ y)$ to the origin. The vector $\vec{v}$ can be expressed in "cartesian" or "polar" form:

$$\vec{v} = x + j\, y = M \angle \varphi = M\, e^{j\varphi}$$

Parameters M and φ denote the modulus and the phase of vector $\vec{v}$, respectively.

- The complex number $e^{j\varphi}$ represents a vector with unitary modulus and phase φ. The following relation holds:

$$e^{j\varphi} = \cos\varphi + j\, \sin\varphi$$

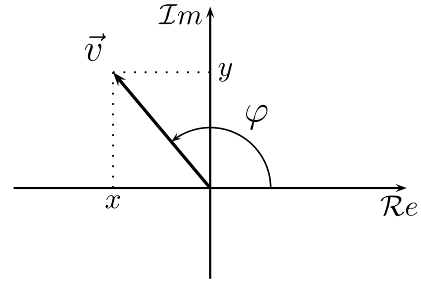


- At any time the complex number can be converted from "cartesian" to "polar" representation (and vice versa) using the following relations:

$$\begin{cases} M = \sqrt{x^2 + y^2} \\ \varphi = \arctan \dfrac{y}{x} \end{cases} \quad \leftrightarrow \quad \begin{cases} x = M\cos\varphi \\ y = M\sin\varphi \end{cases}$$

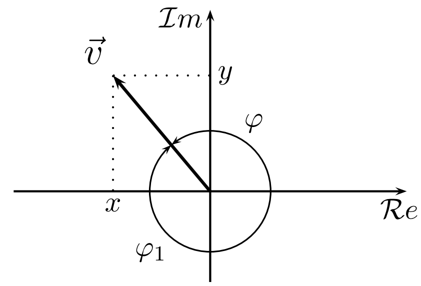
Warning: "arctan" provides φ values within the range $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$. When x is negative, a constant phase $\varphi_0 = \pi$ must be added to phase φ.

1) The phase φ of a plane vector $\vec{v}$ and the phase φ of the corresponding complex number $x+j\,y$ is measured in the **counterclockwise** direction with respect to the **real positive semiaxis**.



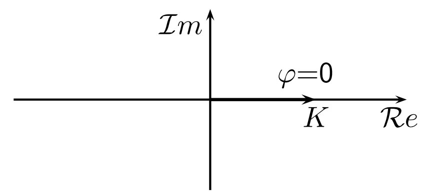
2) The phase φ of a plane vector $\vec{v}$ is defined up to a multiple of 2π:

$$\arg[\vec{v}] = \varphi + 2k\pi, \qquad \forall k \in \mathcal{Z}$$



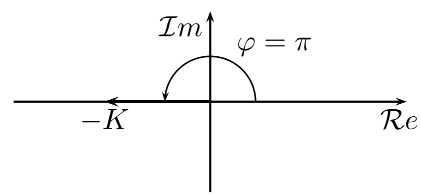
3) The phase of positive real numbers $K > 0$ is zero:

$$\varphi = \arg[K] = 0 + 2k\pi, \qquad \forall k \in \mathcal{Z}$$



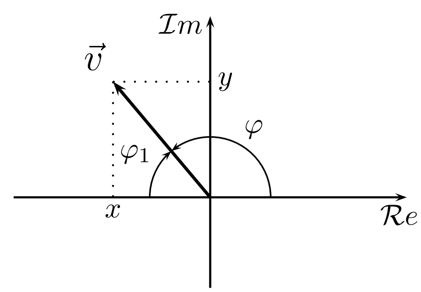
4) The phase of negative real numbers $-K < 0$ is $\varphi = \pi$:

$$\varphi = \arg[-K] = \pi + 2k\pi, \qquad \forall k \in \mathcal{Z}$$



5) The phase φ of a vector $\vec{v} = x + jy$ with negative real part, $x < 0$, can be computed using the following formula:

$$\varphi = \arg[\vec{v}] = \pi - \arctan\left[\frac{y}{|x|}\right]$$



6) The modulus of the product of complex numbers is equal to the **product of the moduli**. Example:

$$\left|\frac{(1+3j)}{(2-5j)(-4+j)}\right| = \frac{|1+3j|}{|2-5j|\,|-4+j|} = \frac{\sqrt{1^2+3^2}}{\sqrt{2^2+5^2}\sqrt{4^2+1^2}} = \frac{\sqrt{10}}{\sqrt{29}\sqrt{17}}$$

7) The phase of the product or the ratio of complex numbers is equal to the **sum or to the difference of the phases** of the considered complex numbers. Example:

$$\arg\left[\frac{(1+3j)}{(2-5j)(-4+j)}\right] = \arctan\frac{3}{1} - \left[\arctan\frac{-5}{2} + \pi - \arctan\frac{1}{4}\right]$$

Examples

Write the modulus $M(\omega) = |G(j\omega)|$ and the phase $\varphi(\omega) = \arg G(j\omega)$ of the following complex functions $G(j\omega)$ with $\omega > 0$:

1) $$G(j\omega) = \frac{(j\omega - 2)(j3\omega + 4)}{j\omega(j\omega + 2)}\, e^{-j5\omega}$$

$$\begin{cases} M(\omega) = \frac{\sqrt{16+9\omega^2}}{\omega} \\ \varphi(\omega) = \pi - \arctan\frac{\omega}{2} + \arctan\frac{3\omega}{4} - 5\,\omega - \frac{\pi}{2} - \arctan\frac{\omega}{2} \\ \qquad\;\; = \frac{\pi}{2} - 2\arctan\frac{\omega}{2} + \arctan\frac{3\omega}{4} - 5\,\omega \end{cases}$$

2) $$G(j\omega) = \frac{(2 + 3j\omega)(2j\omega - 1)}{(j\omega)^2(j\omega + 5)^2}\, e^{-4j\omega t_0}$$

$$\begin{cases} M(\omega) = \frac{\sqrt{4+9\omega^2}\,\sqrt{1+4\omega^2}}{\omega^2(25+\omega^2)} \\ \varphi(\omega) = \arctan\frac{3\omega}{2} + \pi - \arctan 2\omega - 4\omega t_0 - \pi - 2\arctan\frac{\omega}{5} \\ \qquad\;\; = \arctan\frac{3\omega}{2} - \arctan 2\omega - 4\omega t_0 - 2\arctan\frac{\omega}{5} \end{cases}$$

3) $$G(j\omega) = \frac{(j\omega + 3)(j\omega - 3)}{j\omega(2 - 5j\omega)}\, e^{-j3\omega t_0}$$

$$\begin{cases} M(\omega) = \frac{\omega^2+9}{\omega\sqrt{4+25\omega^2}} \\ \varphi(\omega) = \arctan\frac{\omega}{3} + \pi - \arctan\frac{\omega}{3} - 3\omega t_0 - \frac{\pi}{2} + \arctan\frac{5\omega}{2} \\ \qquad\;\; = \frac{\pi}{2} - 3\omega t_0 + \arctan\frac{5\omega}{2} \end{cases}$$

4) $$G(j\omega) = \frac{(1 - 5j\omega)^2}{(j\omega)^2(j\omega + 3)}\, e^{-2j\omega t_0}$$

$$\begin{cases} M(\omega) = \frac{1+25\omega^2}{\omega^2\sqrt{\omega^2+9}} \\ \varphi(\omega) = -2\arctan 5\,\omega - 2\omega t_0 - \pi - \arctan\frac{\omega}{3} \end{cases}$$

5) $$G(j\omega) = \frac{(3 - j\omega)}{j\omega(1 + 5j\omega)^2}\, e^{-2j\omega t_0}$$

$$\begin{cases} M(\omega) = \frac{\sqrt{\omega^2+9}}{\omega(1+25\omega^2)} \\ \varphi(\omega) = -\arctan\frac{\omega}{3} - 2\omega t_0 - \frac{\pi}{2} - 2\,\arctan 5\,\omega \end{cases}$$

- **Complex functions of a real variable**. Consider, for example, the following function $F(\omega)$:

$$F(\omega) = \frac{100}{(4 + j\,\omega)(13 - \omega^2 + j\,4\omega)} = M(\omega)\, e^{j\varphi(\omega)}$$

For each real value of ω, the function $F(\omega)$ provides a complex number $F(\omega)$ having modulus $M(\omega) = |F(\omega)|$ and phase $\varphi(\omega) = \arg[F(\omega)]$.

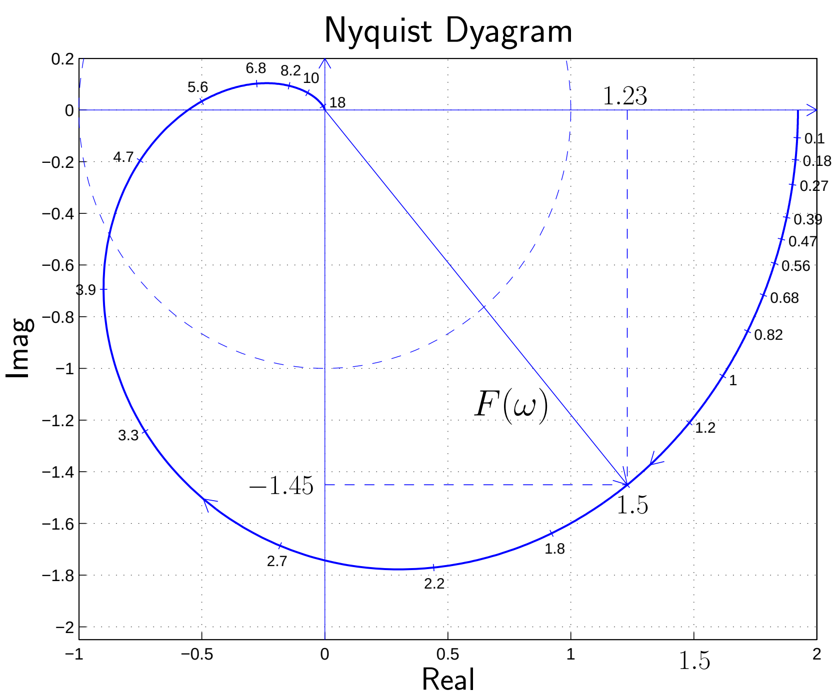
$$F(\omega)\;:\quad \omega \to F(\omega) = M(\omega)e^{j\varphi(\omega)}$$

Function $F(\omega)$ can be used for representing, for example, the output frequency response of a linear system:

$$x(t) = X\,\sin\omega t \longrightarrow \boxed{F(\omega)} \longrightarrow y(t) = \underbrace{M(\omega)\,X}_{Y(\omega)}\,\sin[\omega t + \varphi(\omega)]$$

In this case the module $M(\omega) = \frac{Y(\omega)}{X}$ has the meaning of "gain of the system" at frequency ω, while $\varphi(\omega)$ represents the offset of the sinusoidal output $y(t) = Y(\omega)\,\sin[\omega t + \varphi(\omega)]$ with respect to the sinusoidal input $x(t) = X\,\sin\omega t$.

- The functions $F(\omega)$ of this type can be graphically represented by a curve on the complex plane when ω varies from 0 to ∞:



In correspondence of frequency $\omega = 1.5$ it is:

$$F(\omega)\big|_{\omega=1.5} = 1.23 - 1.45\,j = \sqrt{1.23^2 + 1.45^2}\; e^{-j\arctan\frac{1.45}{1.23}} = M\,e^{-j\varphi}$$

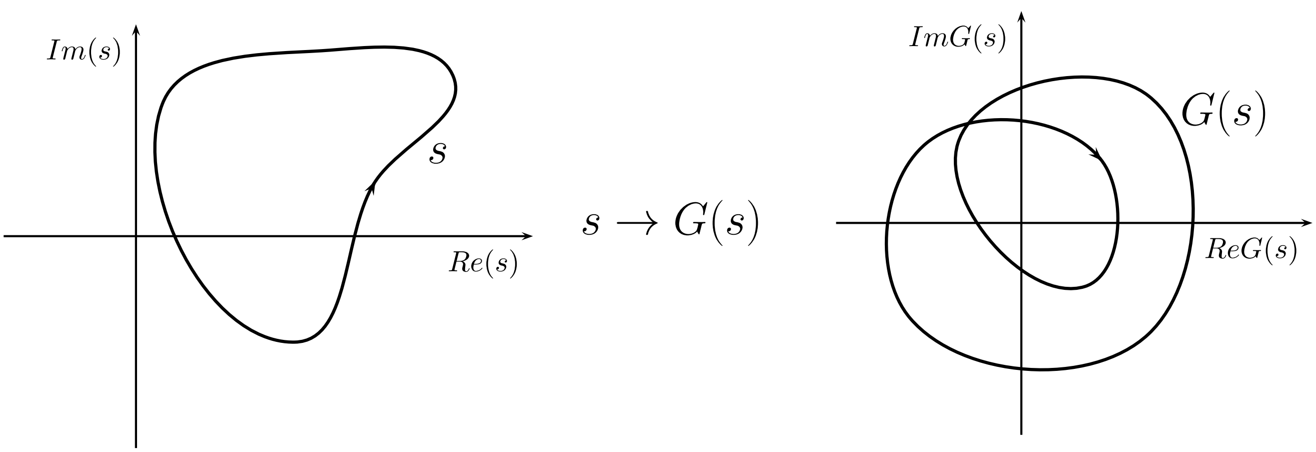
- **Complex functions of a complex variable**. An example is the Laplace transform $G(s)$ of a continuous-time signal $g(t)$:

$$g(t) \qquad \leftrightarrow \qquad G(s) = \int_0^\infty g(t)e^{-s\,t}\,dt$$

- For each value of the complex variable s, the function $G(s)$ provides a complex number $G(s)$:

$$G(s):\; s \to G(s) \qquad\qquad s \in \mathbb{C},\; G(s) \in \mathbb{C}$$

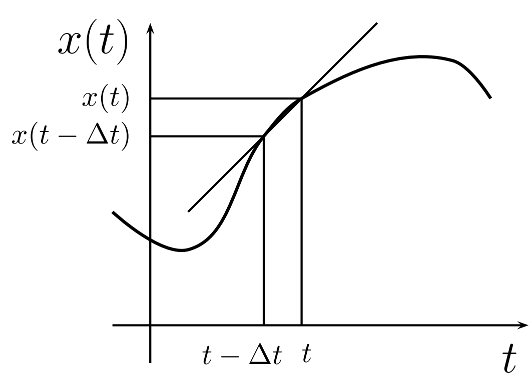
Each closed curve s on the complex plane is associated to a closed curve $G(s)$ on the transformed complex plane:



- A function of this type describes a “transformation” of the complex plane in itself.

- **Time-derivative of a function**. Given the function $x(t)$, with $\dot{x}(t)$, the symbol $\dot{x}(t)$ denotes the time-derivative of function $x(t)$ with respect to parameter t:

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{x(t) - x(t-\Delta t)}{\Delta t}$$



- The time-derivative $\dot{x}(t)$ has the physical meaning of “slope” of function $x(t)$ in the neighborhood of point t.

Differential equations

- **Differential equations**: are algebraic constraints between one or more functions, for example $x(t)$, $y(t)$, etc., and theirs time-derivatives $\dot{x}(t)$, $\dot{y}(t)$, $\ddot{x}(t)$, $\ddot{y}(t)$, etc.

- The differential equations can be:

  1) nonlinear:

$$f(x(t), y(t), \dot{x}(t), \dot{y}(t), \ddot{x}(t), \ddot{y}(t), \ldots) = 0$$

  2) linear time-varying (the coefficients $a_i(t)$ and $b_i(t)$ are time-varying):

$$b_2(t)\ddot{y}(t) + b_1(t)\dot{y}(t) + b_0(t)y(t) = a_1(t)\dot{x}(t) + a_0(t)x(t)$$

  3) linear time-invariant (the coefficients a_i and b_i are time-constant):

$$b_2\ddot{y}(t) + b_1\dot{y}(t) + b_0y(t) = a_1\dot{x}(t) + a_0x(t)$$

- In the following we will refer only to the linear time-invariant differential equations. Example:

$$\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) = \dot{x}(t) + 5\,x(t)$$

- The differential equations are used to describe the dynamic behavior of a physical system. If the differential equation of a dynamic system is known, then the dynamic system itself is completely known.

- Using the differential equation it is possible to forecast (that is compute) the future behavior of the output variable $y(t)$ of the system, when the time behavior of the input $x(t)$ is known.

$$\xrightarrow{x(t)} \boxed{\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) = \dot{x}(t) + 5\,x(t)} \xrightarrow{y(t)}$$

- If the input function $x(t)$ is known, the "unknown" of the differential equation if the output function $y(t)$. The differential equation is "solved" if function $y(t)$ is determined for each given input function $x(t)$.

- The differential equation provides the "static" description of the given system when all the time-derivatives are equal to zero ($\dot{y}(t) = 0$, $\ddot{y}(t) = 0, \ldots,$ $\dot{x}(t) = 0$, $\ddot{x}(t) = 0, \ldots$):

$$2\,y(t) = 5\,x(t) \qquad \to \qquad y(t) = \frac{5}{2}\,x(t)$$

- Eventually, it is possible to take into account also the initial conditions of the system (i.e. the energy stored within the system at time $t = 0$).

- In control problems, the initial conditions are often neglected because, for stable controlled systems, their influence tend to zero when $t \to \infty$.

- The differential equations can be solved in different ways. The more efficient way, from the control point of view, is the one that uses the **Laplace transform**.

- This method is based on the use of complex functions $X(s)$ and $Y(s)$ of the complex variable s which are in *biunivocal* correspondence with the continuous-time signals $x(t)$ e $y(t)$:

$$x(t) \qquad \leftrightarrow \qquad X(s) = \int_0^\infty x(t) e^{-s\,t}\, dt$$

$$y(t) \qquad \leftrightarrow \qquad Y(s) = \int_0^\infty y(t) e^{-s\,t}\, dt$$

- If the Laplace transform is used, the differential equation is transformed in a simple algebraic equation that can be easily solved.

- An important property of the Laplace transform is the following:

$$\mathcal{L}[x(t)] = X(s) \qquad \to \qquad \mathcal{L}[\dot{x}(t)] = sX(s) - x(0^-)$$

- When the initial conditions are zero, $x(0^-) = 0$, it is:

$$\mathcal{L}[\dot{x}(t)] = s\,X(s)$$

i.e., in the transformed space the time-derivative $\dot{x}(t)$ is obtained multiplying by "s" the Laplace transform $X(s)$ of signal $x(t)$.

- Example. Applying the Laplace transform to the following differential equation (if the initial conditions a zero)

$$\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) = \dot{x}(t) + 5\,x(t)$$

the following algebraic equation is obtained

$$s^2 Y(s) + 3\,s\,Y(s) + 2\,Y(s) = s\,X(s) + 5\,X(s)$$

$$(s^2 + 3\,s + 2)Y(s) = (s + 5)X(s)$$

from which one obtains the following relation between the input and the output functions $X(s)$ and $Y(s)$, respectively

$$Y(s) = \underbrace{\frac{(s+5)}{s^2 + 3\,s + 2}}_{G(s)} X(s) = G(s)\,X(s)$$

- Function $G(s)$ is called **transfer function** of the system. This function completely defines the dynamic of the given system and it can be directly obtained (in a biunivocal way) from the corresponding differential equation:

$$\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) = \dot{x}(t) + 5\,x(t) \qquad \leftrightarrow \qquad G(s) = \frac{(s+5)}{s^2 + 3\,s + 2}$$

- The relation $Y(s) = G(s)X(s)$ means that the solution of a differential equation in the Laplace transformed space is quite simple: the Laplace transform $Y(s)$ of the output signal can be easily obtained multiplying the transfer function $G(s)$ of the system by the Laplace transform $X(s)$ of the input signal $x(t)$:

$$Y(s) = G(s)X(s) \qquad \rightarrow \qquad y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)X(s)]$$

Finally, the output signal $y(t)$ is obtained computing the inverse Laplace transform of function $Y(s)$.