Exercises on the use of the Laplace transform

1. Compute the Laplace transform $X(s)$ of the following signals $x_1(t)$ and $x_2(t)$:

$$x_1(t) = 2(1+t^2)\,e^{5\,t}, \qquad x_2(t) = 4 + 3\,e^{-3t}\sin(7\,t)$$

Solution:

$$X_1(s) = \frac{2}{(s-5)} + \frac{4}{(s-5)^3}, \qquad X_1(s) = \frac{4}{s} + \frac{21}{(s+3)^2+7^2}$$

2. Compute the Laplace transform $X(s)$ of the following signals $x_1(t)$ and $x_2(t)$:

$$x_1(t) = 2\,e^{5t}\sin(8\,t), \qquad x_2(t) = 2\,t^2\,e^{-4\,t}$$

Solution:

$$X_1(s) = \frac{16}{(s-5)^2+64}, \qquad X_2(s) = \frac{4}{(s+4)^3}$$

3. Write, as a function of signals $x(t)$ and $y(t)$, the differential equation associated to the following transfer function $G(s)$:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{3\,s^2+b}{s(s^2+2\,s+a)}$$

Solution:

$$\dddot{y}(t) + 2\,\ddot{y}(t) + a\,\dot{y}(t) = 3\,\ddot{x}(t) + b\,x(t)$$

4. Compute the Laplace transform $Y(s)$ of the output signal $y(t)$ of the differential equation $3\,\dot{y}(t) + 4\,y(t) = 0$ having initial condition $y(0)$. Solution. Applying the Laplace transform to the differential equation one obtains:

$$\mathcal{L}\,[3\,\dot{y}(t) + 4\,y(t) = 0] \qquad \to \qquad 3\,[s\,Y(s) - y(0)] + 4\,Y(s) = 0$$

from which it follows that:

$$Y(s) = \frac{3\,y(0)}{3\,s+4} = \frac{y(0)}{s+\frac{4}{3}} \qquad \to \qquad y(t) = y(0)\,e^{-\frac{4}{3}t}$$

5. Compute the initial value $y_0 = \lim_{t\to 0^+}$ of signal $y(t)$ corresponding to the following complex Laplace function $Y(s)$:

$$Y(s) = \frac{b\,s+3}{(s+b)(s-a)+2} \qquad \to \qquad y_0 = b$$

6. Compute the final value $y_\infty = \lim_{t\to\infty}$ of signal $y(t)$ corresponding to the following complex Laplace function $Y(s)$:

$$Y(s) = \frac{s-a}{(b\,s+3)\,s} \qquad \to \qquad y_\infty = -\frac{a}{3}$$