Linear differential equations

- From a dynamical point of view, the linear time-invariant systems are described by linear ordinary differential equations with constant coefficients:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0\, y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \ldots + b_0\, x$$

or in a compact form:

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{m} b_i \frac{d^i x(t)}{dt^i}$$

where $y(t)$ is the *output* function and $x(t)$ is the *input* function.

- *Conditions of physical realizability*: $n \geq m$.

$$\begin{cases} \text{if } n > m & \text{the system is } \textit{strictly proper} \\ \text{if } n = m & \text{the system is } \textit{proper} \\ \text{if } n < m & \text{the system is } \textit{improper} \end{cases}$$

- A differential equation can be solved if it is given:

  i) the *initial conditions*:

$$y(0^-)\ ,\quad \left.\frac{dy}{dt}\right|_{t=0^-}\ ,\ \ldots\ ,\ \left.\frac{d^{n-1}y}{dt^{n-1}}\right|_{t=0^-}.$$

  ii) the *input signal:*

$$x(t)\ ,\qquad 0 \leq t \leq T$$

- The function $x(t)$ is supposed to be *piecewise continuous*, *limited for each finite value of time* t, and the initial conditions of function $x(t)$ for $t = 0^-$ are supposed to be zero:

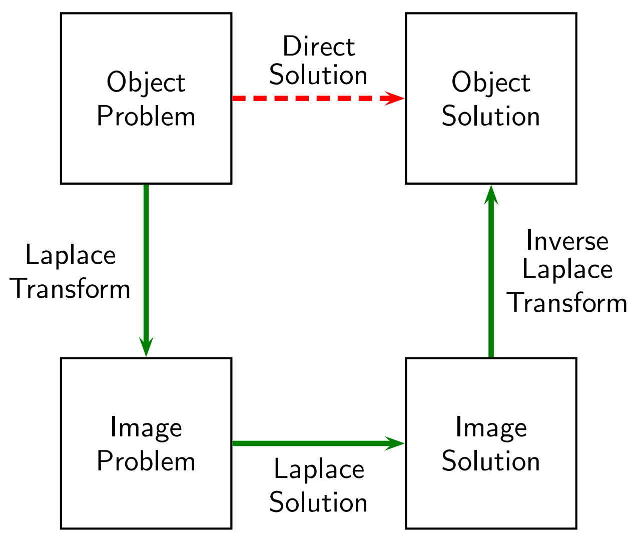
$$x(0^-) = 0\ ,\quad \left.\frac{dx}{dt}\right|_{t=0^-} = 0\ ,\ \ldots\ ,\ \left.\frac{d^{n-1}x}{dt^{n-1}}\right|_{t=0^-} = 0.$$

- The solution of a differential equation is the sum of two functions:

$$y(t) = y_0(t) + y_1(t)$$

1. *the free evolution* $y_0(t)$: the solution of the homogeneous differential equation obtained when the input signal is zero: $x(t) \equiv 0$, $0 \leq t \leq T$.
2. *the forced evolution* $y_1(t)$: the solution of the given differential equation when the initial conditions are zero.

- The linear time-invariant differential equations can be easily solved by using the *Laplace transform*.
- The Laplace transform defines a *biunivocal* correspondence between the continuous-time functions $x(t)$, $y(t)$ (Object Functions) and the corresponding complex functions $X(s)$, $Y(s)$ (Image Functions).

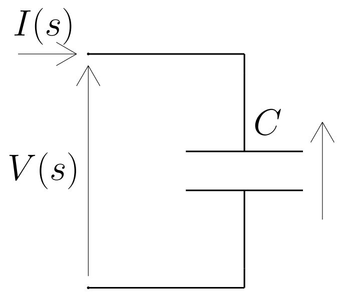


- The Laplace transform converts the *Object Problem* in a correspondent *Image Problem*. Typically, the *Image Problem* can be solved easily.
- The differential equations are transformed in algebraic equations (Image Problem) which can be solved very easily (Laplace Solution).
- From the "Image Solution" one obtains the "Object Solution" applying the *Inverse Laplace Transform*.

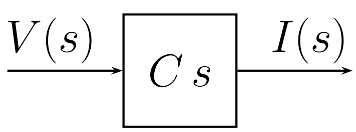
Orientation of a dynamic system

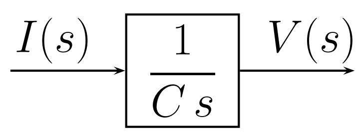
A differential equation is correctly oriented (with an *integral* causality) *if the output variable of the system is the variable which has the maximum degree of time-derivatives within the differential equation*. This rule is equivalent to the physical realization condition: $n \geq m$. This rule can be easily understood referring, for example, to the following linear system:

$$C\,\dot{V}(t) = I(t) \quad \leftrightarrow \quad C\,s\,V(s) = I(s) \quad \leftrightarrow$$

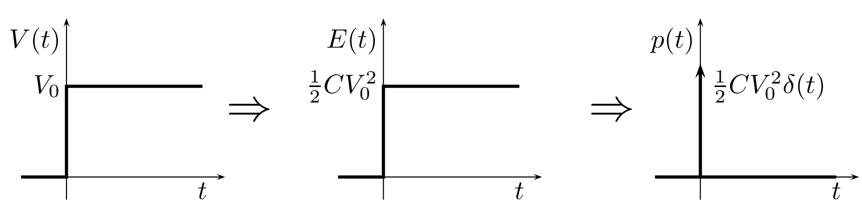


The energy stored within the system is: $E(t) = \frac{1}{2} C\,V^2(t)$. The system energy $E(t)$ varies in time only if a power flow $p(t) = \dot{E}(t) = V(t)\,I(t)$ enters or exits the system. From a *mathematical* point of view the system can be oriented in two different ways:

a) 

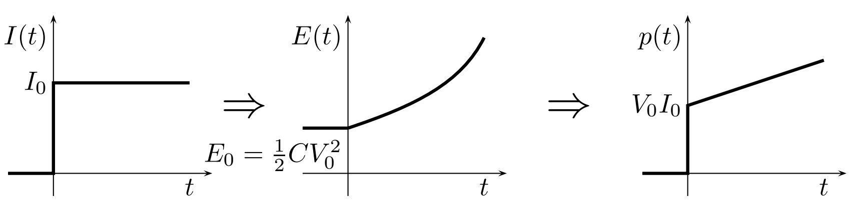
b) 

From an *energetic* point of view only the orientation b), that is $I(s)$ as input and $V(s)$ as output, is physically realizable because it is the only orientation which is compatible with an arbitrary choice of the input signal $I(s)$. The orientation a) is not compatible with an input voltage step $V(t)$ because this condition would imply, for $t = 0$, an instant variation of the energy $E(t)$ stored within the system, and therefore an infinite power flow $p(t)$ at $t = 0$.

a) 

$$\Rightarrow\; p(t)|_{t=0} = \infty.$$

Conversely, the orientation b) is compatible with an input current step $I(t)$:

b) 

$$\Rightarrow\; p(t)|_{t=0} = V_0 I_0.$$

Laplace transform

- The *Laplace transform* defines a *biunivocal correspondence* between a generic continuous-time variable $x(t)$ and a complex function $X(s)$ of the complex variable s:

$$X(s) = \mathcal{L}\big[x(t)\big]$$

The Laplace transform is defined as follows:

$$X(s) := \int_0^\infty x(t)\, e^{-st}\, dt$$

- The inverse transformation is called *Inverse Laplace transform*:

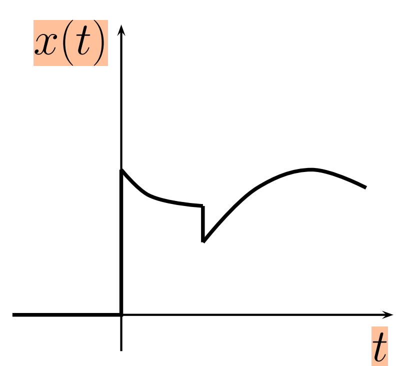
$$x(t) = \mathcal{L}^{-1}\big[X(s)\big]$$

The Inverse Laplace transform is defined as follows:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} X(s)\, e^{st}\, ds$$

- Function $X(s)$ is defined within a *convergence domain* which is a half plane of the complex plane s located on the right of a vertical straight line parallel to the imaginary axis.

- Function $x(t)$ can be Laplace transformed only if:
  1. $x(t) = 0$ for $t < 0$;
  2. $x(t)$ is a piecewise constant function which is limited for $t \geq 0$;
  3. the integral $\displaystyle\int_0^\infty |x(t)| e^{-\sigma t}\, dt$ exists for at least a real value of σ.



- The time behavior of variable $x(t)$ for $t < 0$ is completely taken into account by the *initial conditions* at instant $t = 0$.

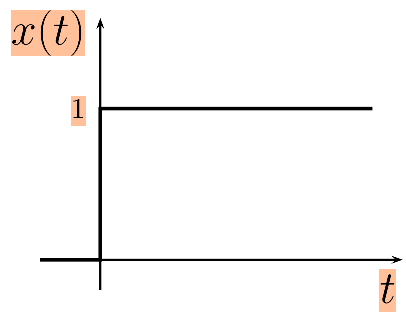
- **Laplace transform of the most common used signals:**

$$\mathcal{L}\left[t^n e^{-a\,t}\right] = \frac{n!}{(s+a)^{n+1}}$$

As particular cases of this relation, one obtains the Laplace transform of the following signals:

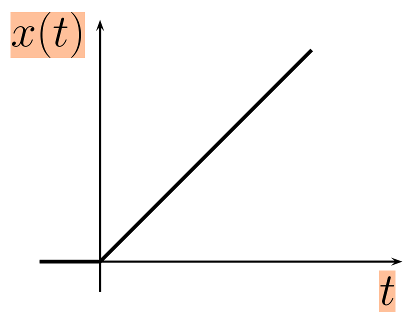
a) **Unitary Step** ($n = 0$, $a = 0$):

$$x(t) = u(t) \quad \leftrightarrow \quad X(s) = \frac{1}{s}$$



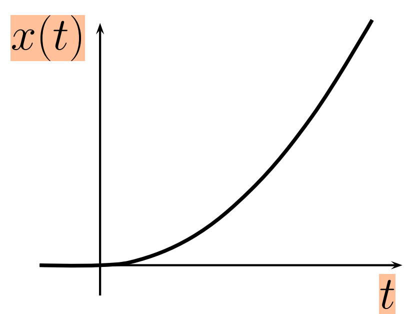
b) **Unitary Ramp** ($n = 1$, $a = 0$):

$$x(t) = t \quad \leftrightarrow \quad X(s) = \frac{1}{s^2}$$



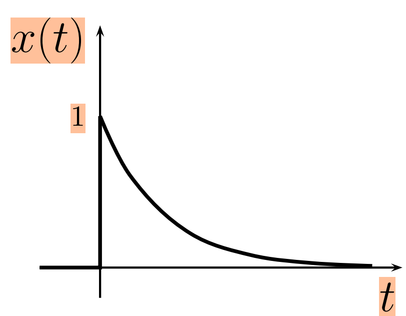
c) **Unitary Parabola** ($n = 2$, $a = 0$):

$$x(t) = \frac{t^2}{2} \quad \leftrightarrow \quad X(s) = \frac{1}{s^3}$$



d) **Exponential function** ($n = 0$, $a > 0$):

$$x(t) = e^{-a\,t} \quad \leftrightarrow \quad X(s) = \frac{1}{s+a}$$



e) **Sinusoidal function**: $x(t) = \sin \omega t$. This signal can be expressed as a linear combination of two exponentials:

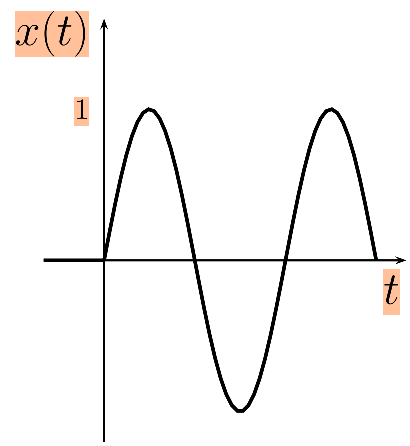
$$x(t) = \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Due to the linearity of the Laplace transform, it follows that:

$$\mathcal{L}[x(t)] = \mathcal{L}[\sin \omega t] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{1}{2j}\left[\frac{2\omega j}{s^2+\omega^2}\right]$$

from which one obtains:

$$x(t) = \sin \omega t \quad \leftrightarrow \quad X(s) = \frac{\omega}{s^2+\omega^2}$$

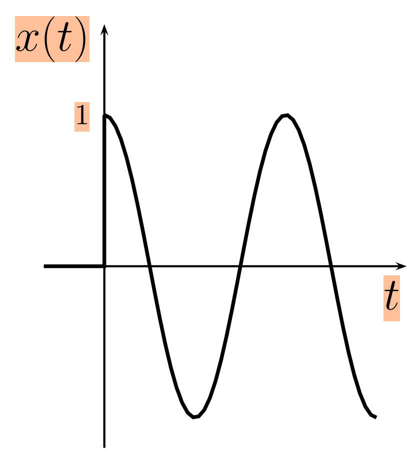


f) **Cosinusoidal function**: $x(t) = \cos \omega t$. For this function, the following relations hold:

$$\mathcal{L}[\cos \omega t] = \mathcal{L}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{1}{2}\left[\frac{2s}{s^2+\omega^2}\right]$$

from which one obtains:

$$x(t) = \cos \omega t \quad \leftrightarrow \quad X(s) = \frac{s}{s^2+\omega^2}$$



Poles and zeros of a function

- The Laplace transforms $X(s)$ of the most commonly used signals are rational functions:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

- The **zeros** of the function $X(s)$ are the solutions of the numerator polynomial:

$$N(s) = 0 \qquad \Leftrightarrow \qquad (s - z_1)(s - z_2)\ \ldots\ (s - z_m) = 0$$

- The **poles** of the function $X(s)$ are the solutions of the denominator polynomial:

$$D(s) = 0 \qquad \Leftrightarrow \qquad (s - p_1)(s - p_2)\ \ldots\ (s - p_n) = 0$$

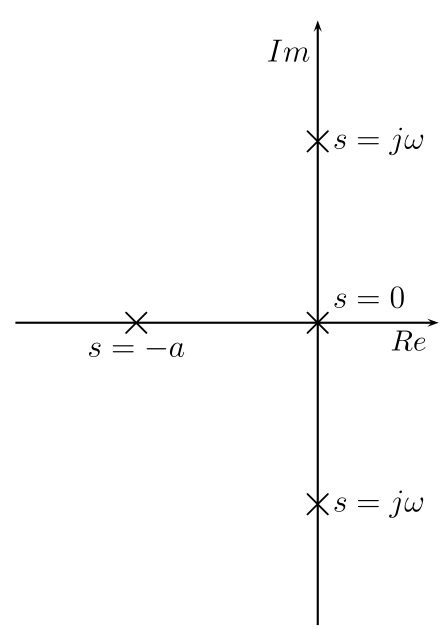
- The **relative degree** r of the function $X(s)$ is defined as follows:

$$r = n - m$$

where n and m are the degrees of the two polynomials $D(s)$ and $N(s)$.

- Position of the poles of the most commonly used Laplace functions:

$x(t)$	$X(s)$	poles	multiplicity
$u(t)$	$\frac{1}{s}$	$s = 0$	1
t	$\frac{1}{s^2}$	$s = 0$	2
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$s = 0$	3
$e^{-a\,t}$	$\frac{1}{s+a}$	$s = -a$	1
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$s_{1,2} = \pm j\omega$	1
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$s_{1,2} = \pm j\omega$	1



Properties of the Laplace transform

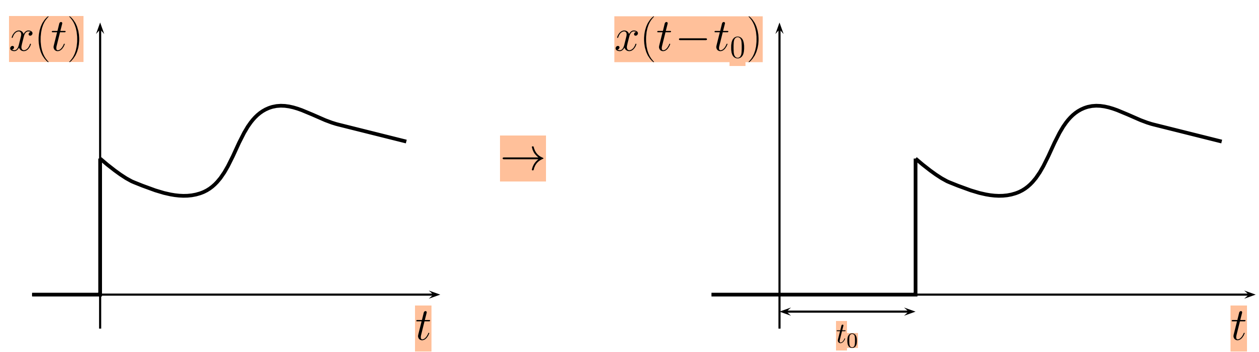
- **Linearity**. Given two arbitrary constants c_1 and c_2, two continuous-time functions $x_1(t)$ and $x_2(t)$, and the corresponding two Laplace transformed functions $X_1(s)$ and $X_2(s)$, the following relation holds:

$$\mathcal{L}\big[c_1\, x_1(t) + c_2\, x_2(t)\big] = c_1\, X_1(s) + c_2\, X_2(s)$$

- **Time shift**. Let $X(s)$ be the Laplace transform of function $x(t)$, which is zero for $t<0$. The following relation holds:

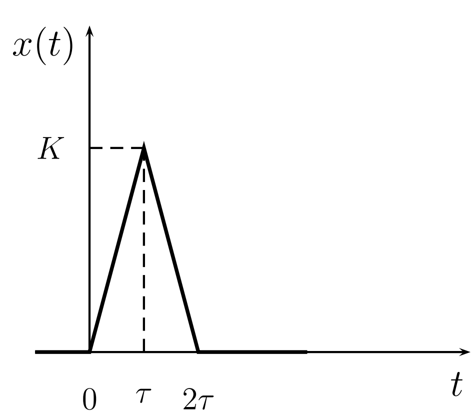
$$\mathcal{L}\big[x(t-t_0)\big] = e^{-t_0\, s}\, X(s)$$

that is, multiply $X(s)$ by function $e^{-t_0 s}$ in the Laplace transformed space corresponds, in the time space, to a time shift t_0 of function $x(t)$.

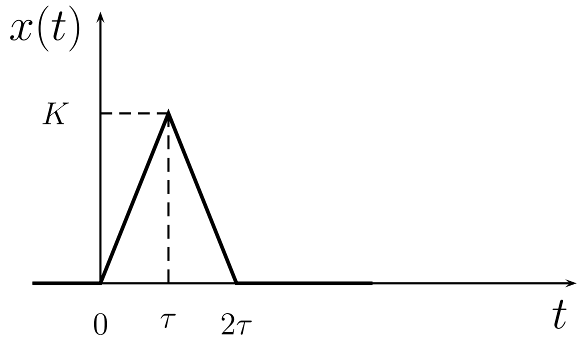


Example: The signal $x(t)$ can be expressed as the sum of three ramps with slope K/τ, $-2K/\tau$ and K/τ, respectively, applied at time $t=0$, $t=\tau$ and $t=2\,\tau$. Using the time shift theorem, one obtains

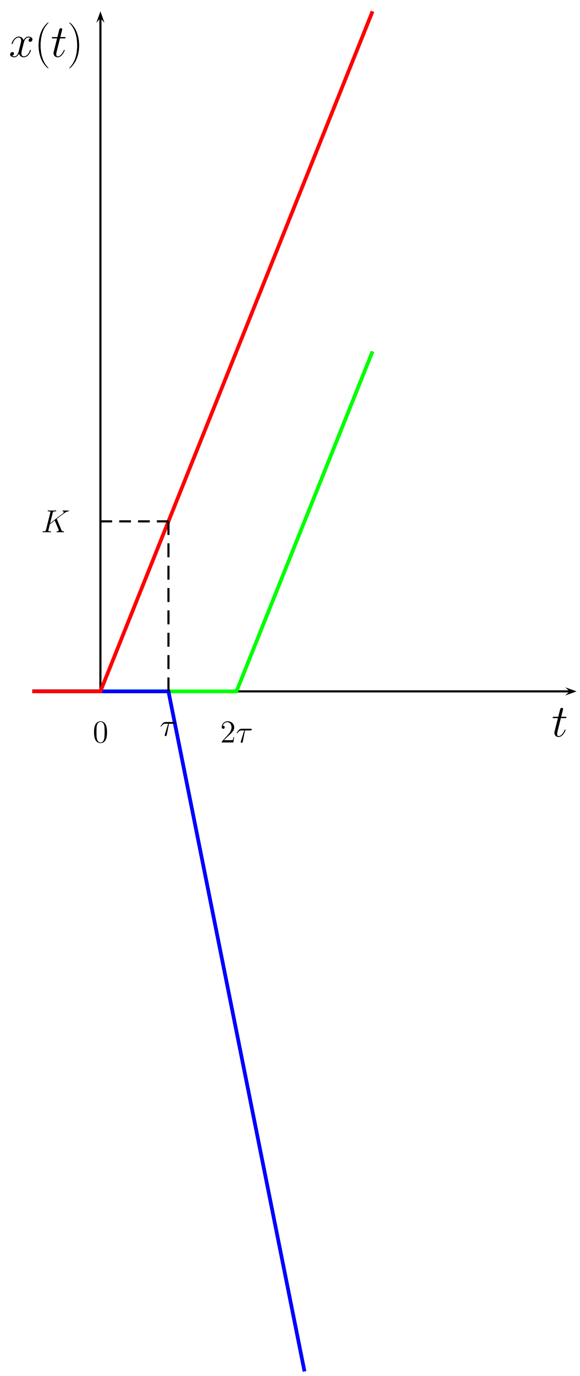
$$\begin{aligned} X(s) &= \frac{K}{\tau\, s^2}(1-2\,e^{-\tau s}+e^{-2\tau s}) \\ &= \frac{K}{\tau\, s^2}(1-e^{-\tau s})^2 \end{aligned}$$



The following signal:



is obtained as the sum of the following 3 ramp signals:



- **Transform of the integral function**. Let $X(s)$ be the Laplace transform of function $x(t)$. The following relation holds:

$$\mathcal{L}\left[\int_0^t x(\tau)\,d\tau\right] = \frac{1}{s}X(s)$$

Multiply the function $X(s)$ by $\frac{1}{s}$ is equivalent, in the time space, to compute the integral function of signal $x(t)$.

- **Transform of the time derivative function**. Let $X(s)$ be the Laplace transform of function $x(t)$. The following relation holds:

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = s\,X(s) - x(0^-)$$

where $x(0^-)$ is the value of function $x(t)$ at time $t=0^-$. When the initial conditions are zero, multiply function $X(s)$ by s is equivalent, in the time space, to compute the time-derivative of signal $x(t)$.

- **Initial value theorem**. Let $X(s) = \mathcal{L}[x(t)]$. The following relation holds:

$$\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} s\,X(s)$$

This theorem holds for any function $X(s)$.

- **Final value theorem**. Let $X(s) = \mathcal{L}[x(t)]$. The following relation holds:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s)$$

This theorem holds only for functions $X(s)$ having all the poles in the real negative half plane, except for a pole in the origin.

- The use of the <u>initial value theorem</u> is strictly related **to the relative degree** r of the function $X(s)$:

- 1) If the relative degree is zero, $r = n - m = 0$, the initial value $x(0^+)$ of the function $x(t)$ is infinite:

$$r = 0 \qquad \Rightarrow \qquad x(0^+) = \infty$$

- 2) If the relative degree is one, $r = n - m = 1$, the initial value $x(0^+)$ of the function $x(t)$ is constant:

$$r = 1 \qquad \Rightarrow \qquad x(0^+) = \lim_{s\to\infty} s\, X(s)$$

  3) If the relative degree is greater than one, $r = n - m > 1$, the initial value $x(0^+)$ of the function $x(t)$ is zero:

$$r > 1 \qquad \Rightarrow \qquad x(0^+) = 0$$

- The use of the <u>final value theorem</u> is strictly related to the **type** h of the function $X(s)$, i.e., the number h of zero poles of the function $X(s)$:

  1) If the function $X(s)$ is of type zero, $h = 0$, the final value $x(\infty)$ of the function $x(t)$ is zero:

$$h = 0 \qquad \Rightarrow \qquad x(\infty) = 0$$

  The final value theorem, in fact, assumes that all non-zero poles of the function $X(s)$ have a negative real part.

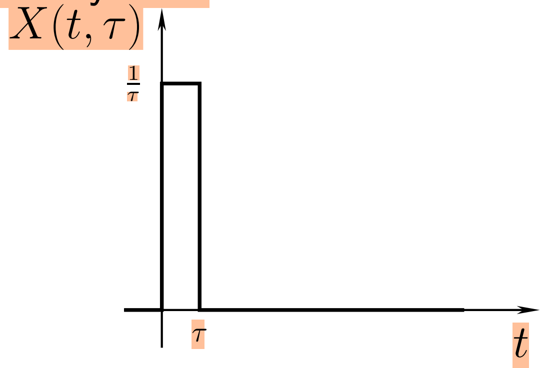
  2) If the function $X(s)$ is of type one, $h = 1$, the final value $x(\infty)$ of the function $x(t)$ is constant:

$$h = 1 \qquad \Rightarrow \qquad x(\infty) = \lim_{s\to 0} s\, X(s)$$

  3) If the function $X(s)$ is of type greater than one, $h > 1$, the final value $x(\infty)$ of the function $x(t)$ cannot be calculated using the final value theorem.

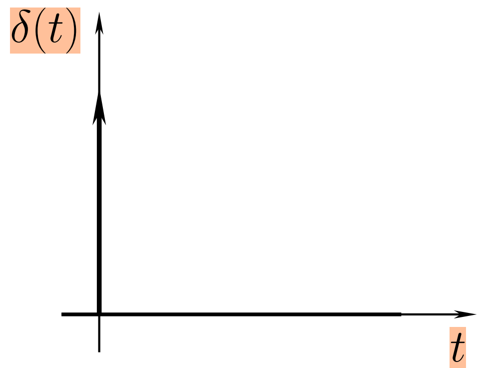
- **Dirac impulse** $\delta(t)$. It is an ideal signal which approximates the following impulse signal having unitary area:

$$\delta(t) = \lim_{\tau \to 0} X(t,\tau)$$



- The Dirac impulse is usually represented as follows:

Function $\delta(t)$



- The Laplace transform of the Dirac impulse is:

$$X(s) = \mathcal{L}[\delta(t)] = 1$$

The following relations hold:

$$X(s) = \mathcal{L}[\lim_{\tau \to 0} X(t,\tau)] = \lim_{\tau \to 0} \mathcal{L}[X(t,\tau)] = \lim_{\tau \to 0} X(s,\tau)$$

Since

$$X(s,\tau) = \frac{1}{\tau s} - \frac{1}{\tau s} e^{-\tau s} = \frac{1}{\tau s}\left(1 - e^{-\tau s}\right)$$

it follows that

$$X(s) = \lim_{\tau \to 0} X(s,\tau) = \lim_{\tau \to 0} \frac{\frac{d}{d\tau}(1 - e^{-\tau s})}{\frac{d}{d\tau}(\tau s)} = \lim_{\tau \to 0} \frac{s e^{-\tau s}}{s} = 1$$

- The time response of a system to the Dirac impulse is equal to the inverse Laplace transform of the transfer function $G(s)$:

$$Y(s) = G(s) \underbrace{X(s)}_{1} = G(s) \qquad \to \qquad y(t) = \mathcal{L}^{-1}[G(s)] = g(t)$$

- **Theorem of translation with respect to** s. Let $X(s)$ be the Laplace transform of function $x(t)$. The following relation holds:

$$\mathcal{L}\left[e^{-at}x(t)\right] = X(s+a)$$

- **Higher order time-derivatives**. Let $X(s)$ be the Laplace transform of function $x(t)$, and let $x(0^-)$, $\left.\frac{dx}{dt}\right|_{t=0^-}$, $\left.\frac{d^2x}{dt^2}\right|_{t=0^-}$, ... be the initial conditions of function $x(t)$ and its time-derivatives at time $t=0^-$. The following relations hold:

$$\begin{aligned}
\mathcal{L}\left[\frac{dx}{dt}\right] &= s\,X(s) - x(0^-)\\
\mathcal{L}\left[\frac{d^2x}{dt^2}\right] &= s^2\,X(s) - s\,x(0^-) - \left.\frac{dx}{dt}\right|_{t=0^-}\\
\mathcal{L}\left[\frac{d^3x}{dt^3}\right] &= s^3\,X(s) - s^2\,x(0^-) - s\left.\frac{dx}{dt}\right|_{t=0^-} - \left.\frac{d^2x}{dt^2}\right|_{t=0^-}\\
\dots &= \dots\\
\mathcal{L}\left[\frac{d^ix}{dt^i}\right] &= s^i\,X(s) - \sum_{j=0}^{i-1} s^{i-j-1}\left.\frac{d^jx}{dt^j}\right|_{t=0^-}
\end{aligned}$$

- **Transform of the integral product.** Let $X_1(s)$ and $X_2(s)$ be the Laplace transforms of functions $x_1(t)$ and $x_2(t)$, respectively. The following relation holds:

$$\mathcal{L}\left[\int_0^\infty x_1(\tau)\,x_2(t-\tau)\,d\tau\right] = X_1(s)\,X_2(s)$$

The convolution integral of functions $x_1(t)$ and $x_2(t)$ satisfies the commutative property:

$$\int_0^\infty x_1(\tau)\,x_2(t-\tau)\,d\tau = \int_0^\infty x_2(\tau)\,x_1(t-\tau)\,d\tau$$

Transfer function

- Let us refer to the following differential equation:

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{m} b_i \frac{d^i x(t)}{dt^i}$$

Applying the Laplace transform to this equation one obtains:

$$\sum_{i=0}^{n} a_i\, s^i\, Y(s) - \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} s^{i-j-1} \left.\frac{d^j y}{dt^j}\right|_{t=0^-} = \sum_{i=0}^{m} b_i\, s^i\, X(s)$$

where $X(s)$ and $Y(s)$ are the Laplace transforms of the input and output signals $x(t)$ and $y(t)$, respectively.

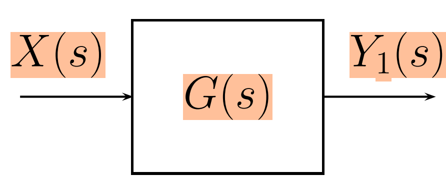
- The Laplace transform $Y(s)$ can be expressed as the sum of two functions:

$$Y(s) = \underbrace{\overbrace{\frac{\displaystyle\sum_{i=0}^{m} b_i\, s^i}{\displaystyle\sum_{i=0}^{n} a_i\, s^i}}^{G(s)} X(s)}_{Y_1(s)} + \underbrace{\frac{\displaystyle\sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} s^{i-j-1} \left.\frac{d^j y}{dt^j}\right|_{t=0^-}}{\displaystyle\sum_{i=0}^{n} a_i\, s^i}}_{Y_0(s)}$$

The functions $Y_0(s)$ and $Y_1(s)$ are the Lapace transforms of the *free evolution* $y_0(t)$ and the *forced evolution* $y_1(t)$, respectively.

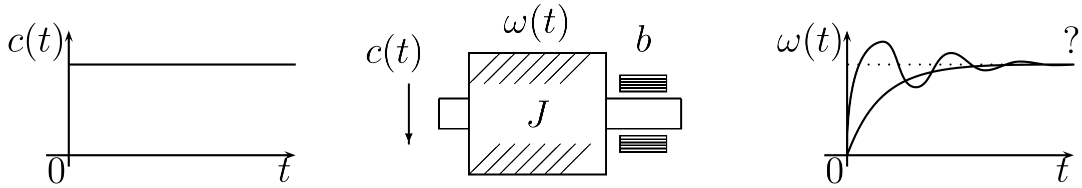
- The *Transfer Function* $G(s)$ of the system

$$G(s) = \frac{Y_1(s)}{X(s)} = \frac{\displaystyle\sum_{i=0}^{m} b_i\, s^i}{\displaystyle\sum_{i=0}^{n} a_i\, s^i}$$



is defined considering *all the initial conditions equal to zero*.

Example. Let us consider a mechanical element characterized by an inertia J, a linear friction coefficient b and an angular velocity ω. Let $c(t)$ be an external torque applied to the mechanical element.



Compute the unitary step response of the given system.

The problem can be solved only if the dynamic model of the system is known. The given system is characterized by the following differential equation:

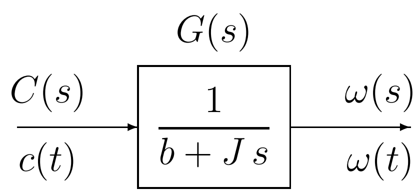
$$\frac{d[J\omega(t)]}{dt} = c(t) - b\,\omega(t) \qquad \leftrightarrow \qquad J\dot{\omega}(t) + b\,\omega(t) = c(t)$$

Applying the Laplace transform with zero initial conditions, one obtains:

$$J\,s\,\omega(s) + b\,\omega(s) = C(s) \qquad \leftrightarrow \qquad \omega(s) = \frac{1}{b + J\,s}\,C(s)$$

The transfer function $G(s)$ of the system is:

$$G(s) = \frac{1}{b + J\,s}$$



The coefficients of the transfer function $G(s)$ are in biunivocal correspondence with the coefficients of the differential equation. If $C(s) = \frac{1}{s}$, in the Laplace transformed space the step response of the system is:

$$\omega(s) = G(s)\,C(s) \qquad \rightarrow \qquad \omega(s) = \frac{1}{(b + J\,s)s}$$

The initial and the final values of function $\omega(t)$ can be obtained from $\omega(s)$ without using the inverse Laplace transform. Applying the initial value theorem, for example, one can compute the value of function $\omega(t)$ at time $t = 0^+$:

$$\omega(0^+) = \omega(t)|_{t\to 0} = \lim_{s\to\infty} s\,\omega(s) = \lim_{s\to\infty} s\,\frac{1}{(b + J\,s)s} = 0$$

Applying the final value theorem one can compute the value of function $\omega(t)$ for $t \to \infty$:

$$\omega(\infty) = \omega(t)|_{t\to\infty} = \lim_{s\to 0} s\,\omega(s) = \lim_{s\to 0} s\,\frac{1}{(b + J\,s)s} = \frac{1}{b}$$

Applying the initial value theorem to function $\dot\omega(s)$, one can also compute the value of the acceleration $\dot\omega(t)$ at time $t = 0^+$:

$$\dot\omega(0^+) = \dot\omega(t)|_{t\to 0} = \lim_{s\to\infty} s\,[\underbrace{s\,\omega(s)}_{\dot\omega(s)}] = \lim_{s\to\infty} \frac{s^2}{(b+J\,s)s} = \frac{1}{J}$$

In fact, in the Laplace transformed space the acceleration $\dot\omega(s)$ can be easily obtained multiplying the angular velocity $\omega(s)$ for the s variable (which represents the "time-derivative" in the Laplace transformed space).
To obtain exactly the time behavior $\omega(t)$ it is necessary to compute the inverse Laplace transform of function $\omega(s)$. The easiest way to do this is to use the simple fractions decomposition. There are always two coefficients α and β such that function $\omega(s)$ can be rewritten as follows:

$$\omega(s) = \frac{1}{(b+J\,s)s} \qquad \leftrightarrow \qquad \omega(s) = \frac{\alpha}{b+J\,s} + \frac{\beta}{s}$$

The coefficients α e β can be determined (for example) imposing equal the two previous expressions:

$$\omega(s) = \frac{\alpha}{b+J\,s} + \frac{\beta}{s} = \frac{\alpha s + \beta(b+J\,s)}{(b+J\,s)s} = \frac{(\alpha+\beta J)\,s + \beta b}{(b+J\,s)s} = \frac{1}{(b+J\,s)s}$$

Solving the system one obtains:

$$\begin{cases} \alpha + \beta J &= 0 \\ \beta b &= 1 \end{cases} \qquad \to \qquad \begin{cases} \alpha &= -\frac{J}{b} \\ \beta &= \frac{1}{b} \end{cases}$$

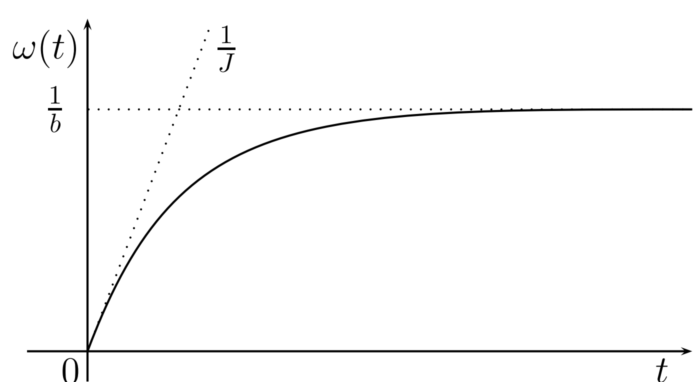
from which it is

$$\omega(s) = \frac{1}{(b+J\,s)s} = \frac{1}{b}\left[\frac{1}{s} - \frac{J}{b+J\,s}\right] = \frac{1}{b}\left[\frac{1}{s} - \frac{1}{s+\frac{b}{J}}\right]$$

Applying the inverse Laplace transform to the two elements, one obtains function $\omega(t)$:

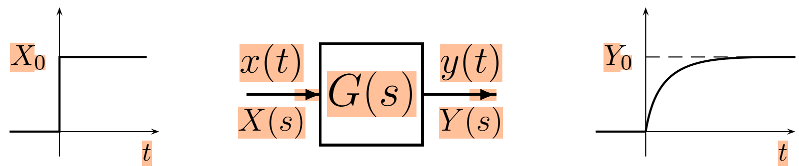
$$\omega(t) = \frac{1}{b}\left(1 - e^{-\frac{b}{J}\,t}\right)$$

Function function $\omega(t)$ has an exponential time behavior:



Risposta al gradino unitario di un sistema $G(s)$

- The **static gain** K_0 of a system $G(s)$ is defined as the ratio Y_0/X_0 between the steady-state value Y_0 of the output signal $y(t)$ of system $G(s)$ when the input is a step signal with amplitude X_0.



$$\Rightarrow \qquad K_0 = \frac{Y_0}{X_0} = G(0).$$

Using the final value theorem it is easy to prove that the static gain K_0 of a system $G(s)$ is equal to $G(0) = G(s)|_{s=0}$, that is the value of function $G(s)$ when $s = 0$:

$$Y_0 = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\,G(s)\,\frac{X_0}{s} = G(0)\,X_0 \quad \Rightarrow \quad K_0 = \frac{Y_0}{X_0} = G(0).$$

From a theoretical point of view, the static gain $G(0)$ can be defined only for asymptotically stable systems $G(s)$, but from a practical point of view the static gain $G(0)$ is used also for unstable or simply stable systems.

- **The time-response** $y(t)$ **of a dynamic system** $G(s)$ **to a unitary input step** $x(t) = X_0 = 1$, $X(s) = \frac{1}{s}$, **satisfies the following properties**:

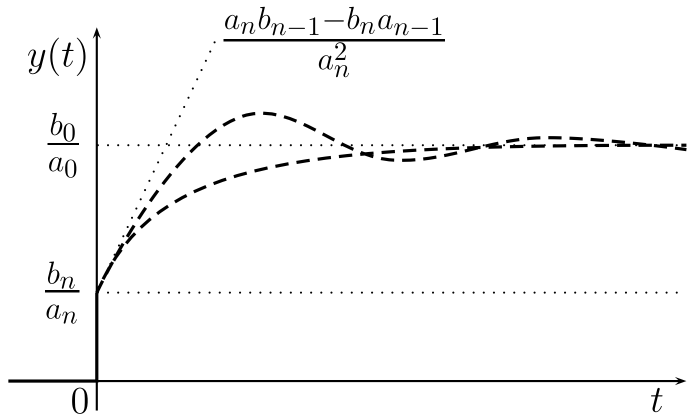
$$\begin{aligned}
1)\ \ y(\infty) &= \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\,G(s)\,\frac{1}{s} = \lim_{s\to 0} G(s) = G(0)\\
2)\ \ y(0^+) &= \lim_{t\to 0^+} y(t) = \lim_{s\to\infty} s\,Y(s) = \lim_{s\to\infty} s\,G(s)\,\frac{1}{s} = \lim_{s\to\infty} G(s) = G(\infty)\\
3)\ \ \dot{y}(0^+) &= \lim_{t\to 0^+} \dot{y}(t) = \lim_{s\to\infty} s^2\,Y(s) = \lim_{s\to\infty} s^2\,G(s)\,\frac{1}{s} = \lim_{s\to\infty} s\,G(s)\\
\vdots\ &= \ \vdots
\end{aligned}$$

These properties can be directly obtained applying the initial and final value theorems. The property 1) holds only for asymptotically stable systems $G(s)$, while all the other properties hold for all systems $G(s)$.

• Let us consider a dynamic system characterized by the following transfer function $G(s)$:

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

• where $m = n$. If the system is asymptotically stable (all the pole of function G(s) have negative real part), the qualitative behavior of the time response $y(t)$ of system $G(s)$ to a unitary input step $x(t) = 1$, is the following:



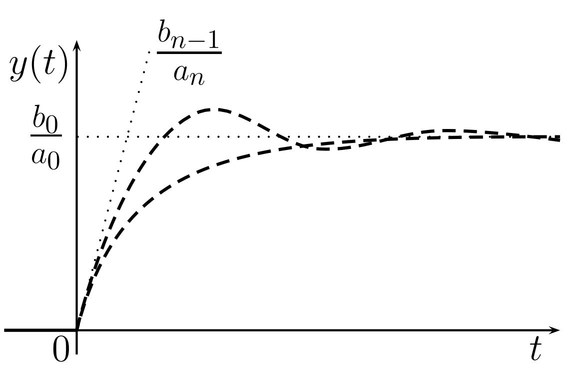
In fact function $G(s)$ can be rewritten in the following way:

$$G(s) = \frac{b_n}{a_n} + \frac{(b_{n-1} - \frac{b_n}{a_n}a_{n-1})s^{n-1} + (b_{n-2} - \frac{b_n}{a_n}a_{n-2})s^{n-2} + \ldots}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

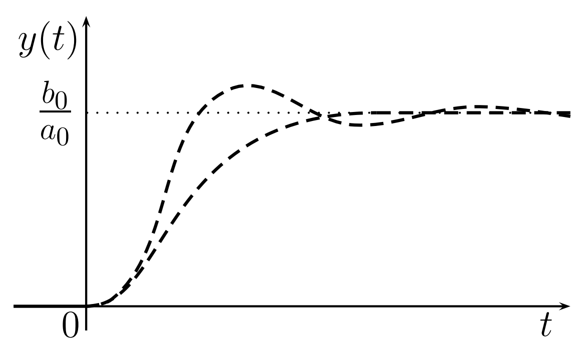
where $\frac{b_n}{a_n}$ is the gain which multiplies the unitary input step.

• Particular cases:

1) $(b_n = 0) \leftrightarrow (n = m + 1)$:



2) $(b_n = b_{n-1} = 0) \leftrightarrow (n = m + 2)$:



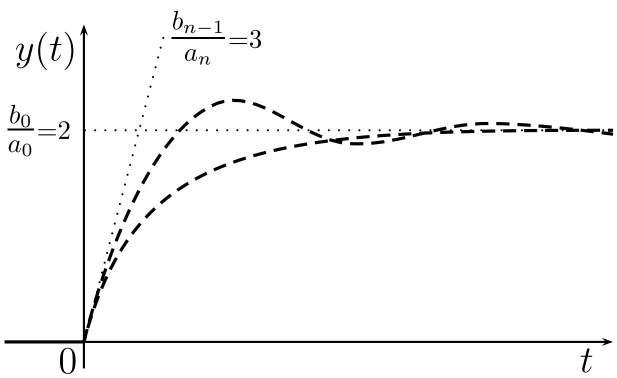
Example. Compute the qualitative behavior of the time-response of the following differential equation to the unitary input step:

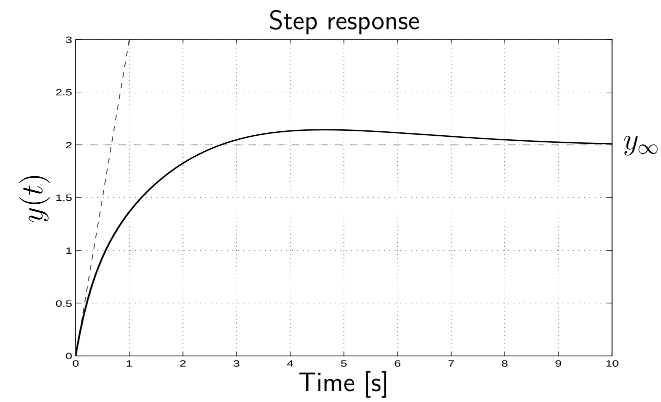
$$\dddot{y}(t) + 4\,\ddot{y}(t) + 3\,\dot{y}(t) + y(t) = 3\,\ddot{x}(t) + 5\,\dot{x}(t) + 2\,x(t)$$

Using the Laplace transform one directly obtains the transfer function $G(s)$:

$$G(s) = \frac{3\,s^2 + 5\,s + 2}{s^3 + 4\,s^2 + 3\,s + 1}$$

In this case the relative degree of function $G(s)$ is $r = n - m = 1$, therefore the qualitative behavior of the time-response $y(t)$ to the unitary input step is:

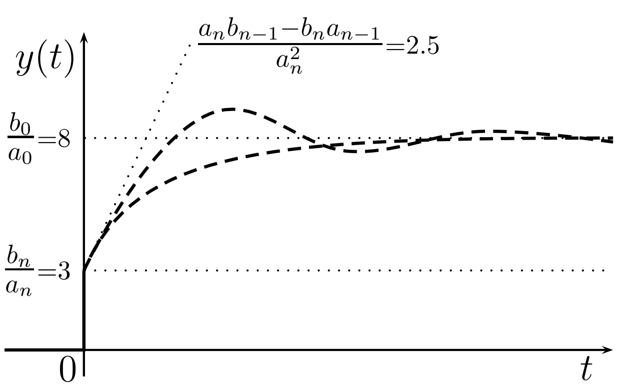


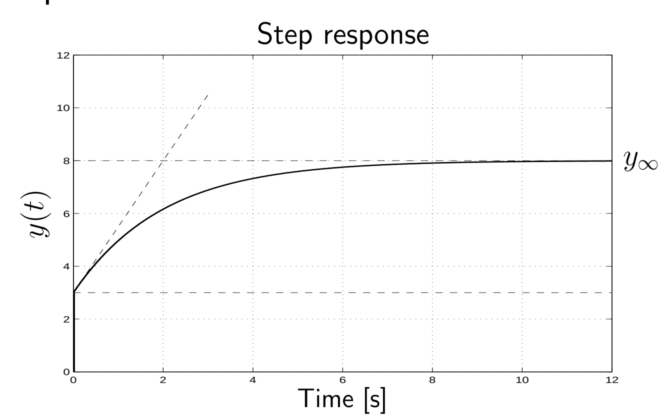


Example. Compute the qualitative behavior of the time-response of the following transfer function $G(s)$ to the unitary input step:

$$G(s) = \frac{6\,s + 8}{2\,s + 1}$$

In this case the relative degree of function $G(s)$ is zero, therefore the qualitative behavior of the time-response $y(t)$ to the unitary input step is:





Function $G(s)$ can be rewritten as follows:

$$G(s) = 3 + \frac{5}{2\,s + 1}$$

The first term corresponds to the output step of amplitude 3, while the second term represents a first order system having static gain 5 and initial slope $\frac{5}{2} = 2.5$.

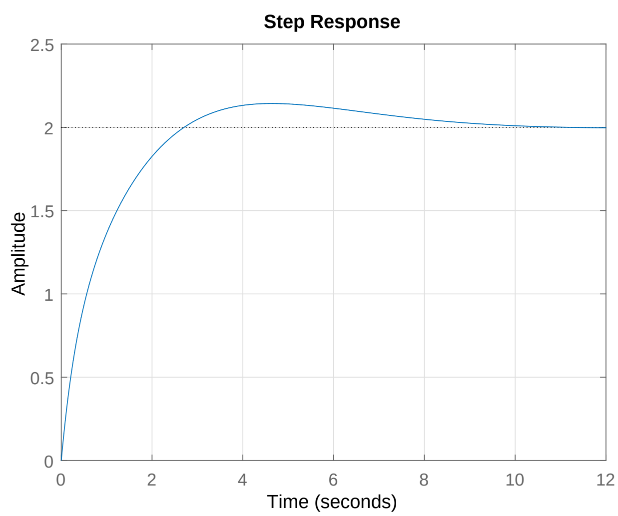
Laplace Transforms in Matlab:

```
-- Matlab commands -------------------------------------------------------
echo on
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%  Laplace transforms of time signals x(t)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms t n a b c w J s             %%% "syms" defines symbolic variables
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%  "laplace" computes the Laplace transform of the given function x(t)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
G1 = laplace(sym(1))          %%%  Unitary step function:
G2 = laplace(t)               %%%  Unitary ramp function:
G3 = laplace(t^2/2)           %%%  Unitary parabola function:
G4 = laplace(exp(-a*t))       %%%  Exponential function:
G5 = laplace(sin(w*t))        %%%  Sin function:
G6 = laplace(cos(w*t))        %%%  Cos function:
 -- Matlab output ---------------------------------------------------------
G1 =
1/s
G2 =
1/s^2
G3 =
1/s^3
G4 =
1/(a + s)
G5 =
w/(s^2 + w^2)
G6 =
s/(s^2 + w^2)

-- Matlab commands -------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%  Laplace transforms of other funcitons
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
G1 = simplify(laplace(exp(-a*t)*cos(w*t))) %%%  Exponential-cosinus function:
G2 = simplify(laplace((1-exp(-b*t/J))/b))  %%%  Time function:
G3 = laplace(2*(1+t^2)*exp(5*t))           %%%  Time function:
G4 = laplace(4+3*exp(-3*t)*sin(7*t))       %%%  Time function:
G5 = laplace(2*exp(5*t)*sin(8*t))          %%%  Time function:
G6 = laplace(2*t^2*exp(-4*t))              %%%  Time function:
 -- Matlab output ---------------------------------------------------------
G1 =
(a + s)/((a + s)^2 + w^2)
G2 =
1/(s*(b + J*s))
G3 =
2/(s - 5) + 4/(s - 5)^3
G4 =
4/s + 21/((s + 3)^2 + 49)
G5 =
16/((s - 5)^2 + 64)
G6 =
4/(s + 4)^3
```

Step response of a function $G(s)$ in Matlab:

```
 -- Matlab commands ---------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%  "step" computes the step response of the given s function
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
s=tf('s');                          %%% Defines "s" as the Laplace variable
Gs=(3*s^2+5*s+2)/(s^3+4*s^2+3*s+1)  %%% Defines the Gs function:
figure(1)                           %%% Open a new figure
step(Gs)                            %%% Compute the step response of function Gs
grid on                             %%% Add the grid to the figure
print -depsc -painters Risposta_al_gradino_Gs.eps
 -- Matlab output -----------------------------------------------------
```




```
 -- Matlab commands ---------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%  "step" computes the step response of the given s function
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Gs=(6*s+8)/(2*s+1)                  %%% Defines the Gs function:
figure(1)                           %%% Open a new figure
step(Gs)                            %%% Compute the step response of function Gs
grid on                             %%% Add the grid to the figure
ylim([0 10])                        %%% Defines the limits of the y axis
print -depsc -painters Risposta_al_gradino_Gs2.eps
 -- Matlab output -----------------------------------------------------
```