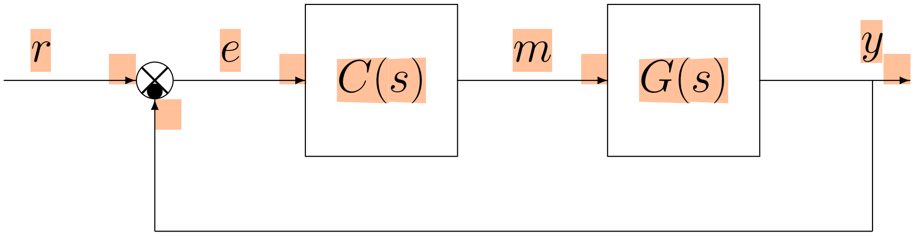
Lead/lag networks: inversion formulas

- Let us consider the following feedback system:



where $G(s)$ is the system to be controlled and $C(s)$ is an lead/lag network having the following structure:

$$C(s) = \frac{1+\tau_1 s}{1+\tau_2 s}$$

- You have a Lead network when $\tau_1 > \tau_2$:

$$C(s) = \frac{1+\tau s}{1+\alpha\tau s} \qquad \text{dove} \qquad \tau = \tau_1, \quad \alpha = \frac{\tau_2}{\tau_1} < 1$$

- You have a Lag network when $\tau_1 < \tau_2$:

$$C(s) = \frac{1+\alpha\tau s}{1+\tau s} \qquad \text{dove} \qquad \tau = \tau_2, \quad \alpha = \frac{\tau_1}{\tau_2} < 1$$

- The dynamic specifications for a feedback system are usually given in terms of phase margin M_φ and gain margin M_α.

- The parameters τ_1 and τ_2 of a lead/lag network that introduces an amplification M and a phase shift φ at frequency ω can be obtained using the following *inversion formulas*:

$$\tau_1 = \frac{M - \cos\varphi}{\omega \sin\varphi}, \qquad \tau_2 = \frac{\cos\varphi - \frac{1}{M}}{\omega\sin\varphi}$$

Inversion formulas: mathematical details

- Design problem: *Determine the parameters τ_1 and τ_2 of the lead/lag network $C(s)$ such that*

$$C(j\omega) = \frac{1 + j\,\tau_1\omega}{1 + j\,\tau_2\omega} = M\; e^{j\varphi}$$

that is, the network amplifies M and provides a phase shift φ at frequency ω.

- The design project is solved by using the following *inversion formulas*:

$$\tau_1 = \frac{M - \cos\varphi}{\omega \sin\varphi}, \qquad \tau_2 = \frac{\cos\varphi - \frac{1}{M}}{\omega \sin\varphi}$$

- The inversion formulas are obtained by rewriting the equation

$$C(j\omega) = \frac{1 + j\,\tau_1\omega}{1 + j\,\tau_2\omega} = M\; e^{j\varphi} = M\cos\varphi + jM\sin\varphi$$

in the form

$$(M\cos\varphi + jM\sin\varphi)(1 + j\;\tau_2\omega) = 1 + j\;\tau_1\omega$$

This equation can also be rewritten as follows:

$$\begin{bmatrix} 1 & -M\cos\varphi \\ 0 & M\sin\varphi \end{bmatrix} \begin{bmatrix} \tau_1\omega \\ \tau_2\omega \end{bmatrix} = \begin{bmatrix} M\sin\varphi \\ M\cos\varphi - 1 \end{bmatrix}$$

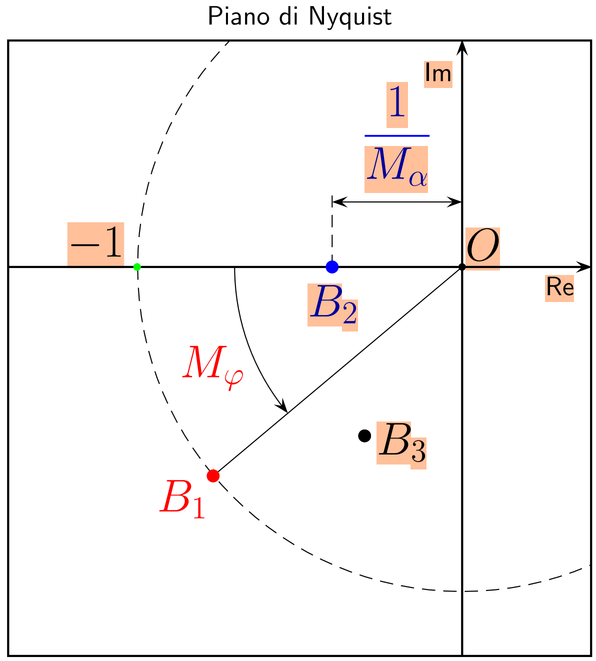
Solving with respect to the variables τ_1 and τ_2 one obtains

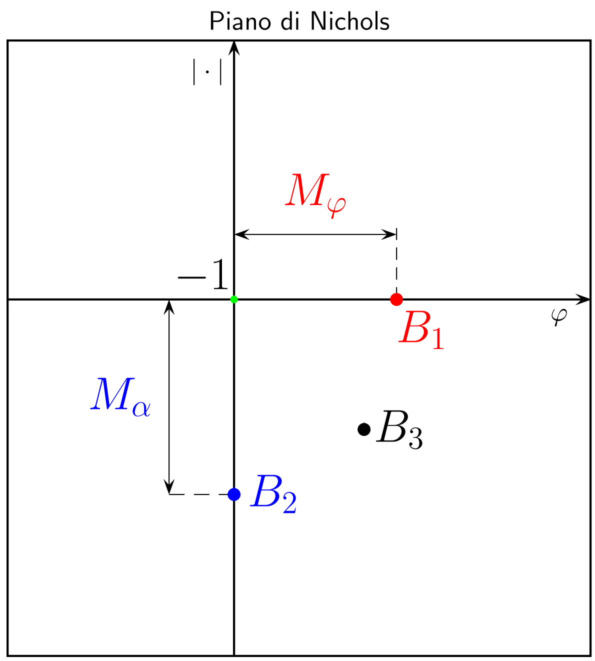
$$\tau_1 = \frac{\begin{vmatrix} M\sin\varphi & -M\cos\varphi \\ M\cos\varphi - 1 & M\sin\varphi \end{vmatrix}}{\omega M \sin\varphi} = \frac{M - \cos\varphi}{\omega\sin\varphi}$$

$$\tau_2 = \frac{\begin{vmatrix} 1 & M\sin\varphi \\ 0 & M\cos\varphi - 1 \end{vmatrix}}{\omega M \sin\varphi} = \frac{\cos\varphi - \dfrac{1}{M}}{\omega\sin\varphi}$$

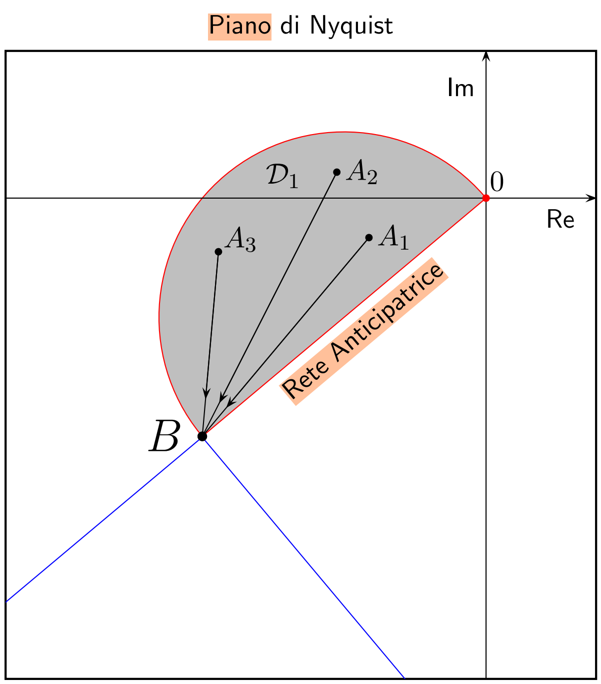
- These formulas are valid both for lead networks ($M > 1$ and $\varphi > 0$) and lag networks ($M < 1$ and $\varphi < 0$).

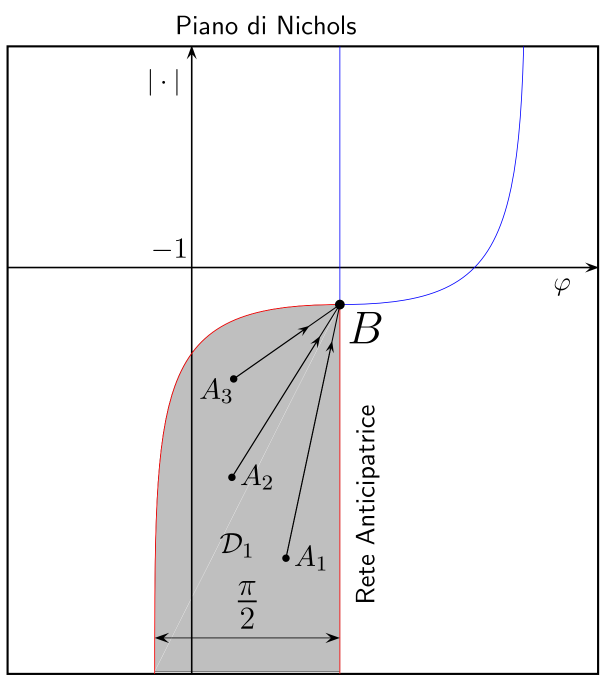
- From the design specifications to the determination of point B:



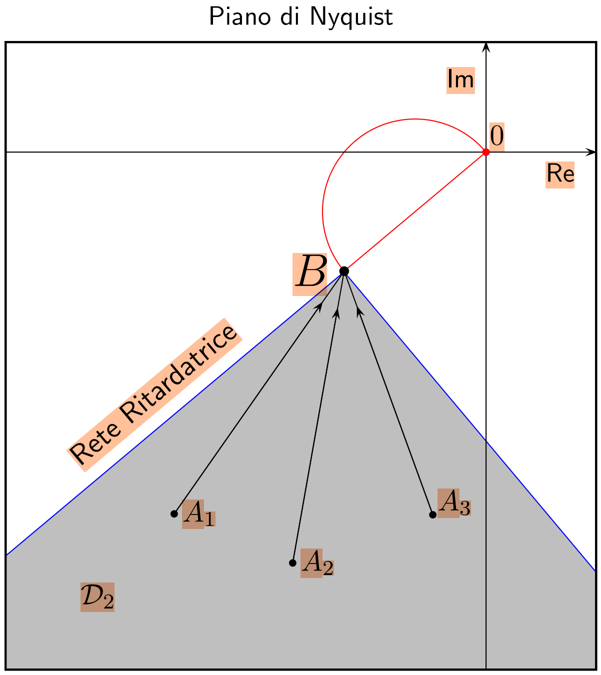


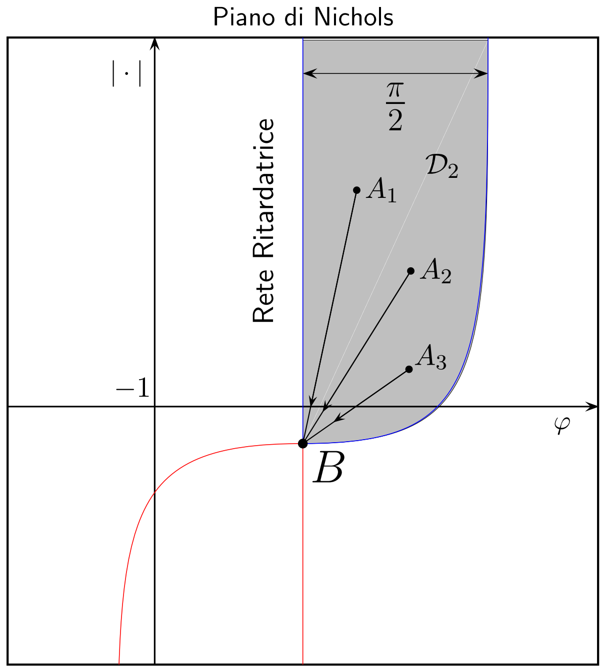
- Lead network: admissible regions for points A to be moved in B:



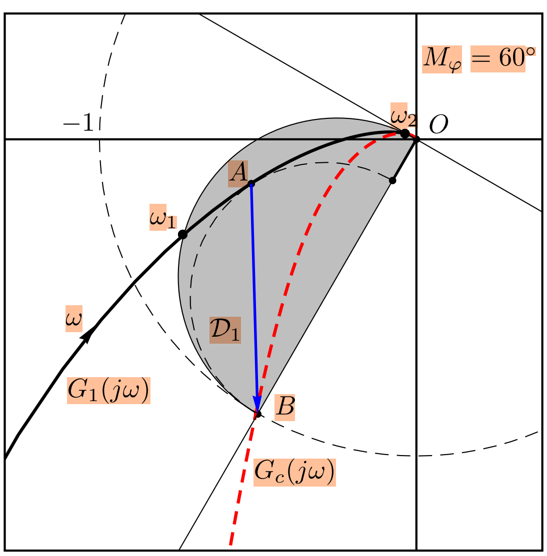


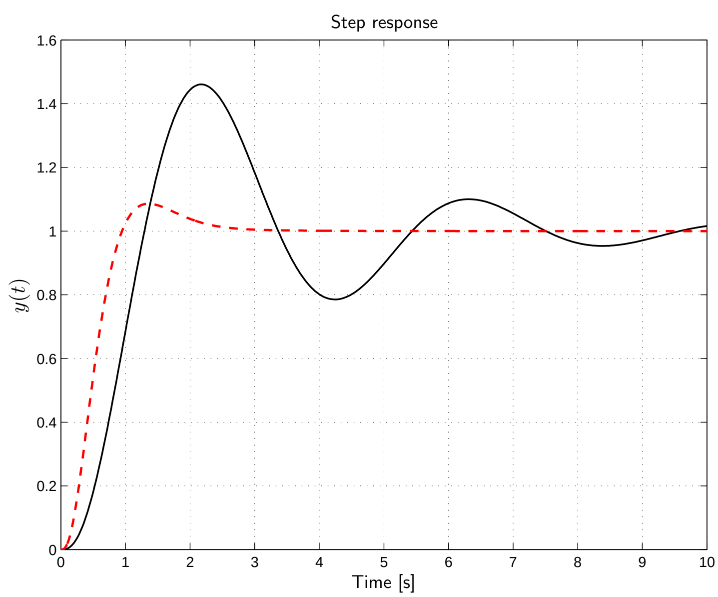
- Lag network: admissible regions for points A to be moved in B:



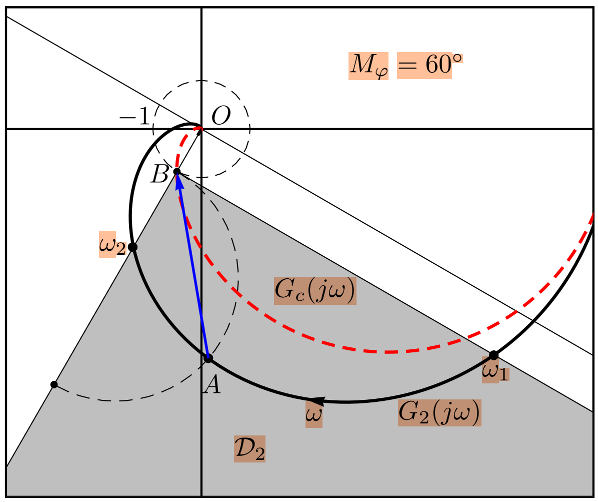


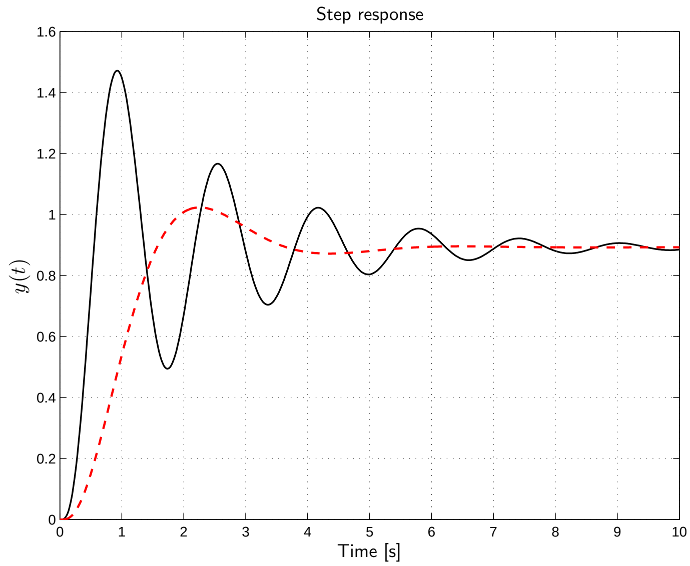
- Design of a lead network for system $G_1(s)$. Design specification on the phase margin: $M_\varphi = 60°$.





- Design of a lag network for system $G_2(s)$. Design specification on the phase margin: $M_\varphi = 60°$.





- Design of a lag network for system $G_2(s)$. Design specification on the gain margin: $M_\alpha = 5$.