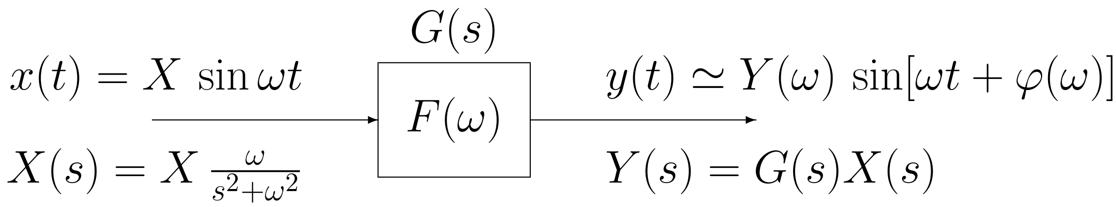
The frequency response function

- If a sinusoidal signal $x(t) = X\ \sin\omega t$ is applied to a *asymptotically stable* linear system $G(s)$:



after the transient, the output signal $y(t)$ is sinusoidal with the same frequency ω of the input signal:

$$y(t) = Y(\omega)\ \sin\left[\omega t + \varphi(\omega)\right]$$

- The amplitude $Y(\omega)$ of the output signal and the offset $\varphi(\omega)$ with respect to the input signal are functions of the frequency ω.

- The *frequency response function* is a complex function $F(\omega)$ of real variable ω defined as follows:

$$\boxed{F(\omega) := \frac{Y(\omega)}{X}\, e^{j\varphi(\omega)}} \qquad \Rightarrow \qquad \begin{cases} |F(\omega)| = \frac{Y(\omega)}{X} \\ \arg F(\omega) = \varphi(\omega) \end{cases}$$

- It describes the behavior of the system in periodic conditions at frequency ω. Function $F(\omega)$ is defined for $0 \leq \omega < \infty$. Due to the system's linearity, the function $F(\omega)$ is independent of X.

- **Theorem.** *A time-invariant linear system $G(s)$ with all the poles within the negative real half-plane, in presence of a sinusoidal input signal shows, in steady-state conditions, a sinusoidal output having the same frequency ω of the input signal. The frequency response function $F(\omega)$ can be obtained from the transfer function $G(s)$ as follows:*

$$\boxed{F(\omega) = G(s)|_{s=j\omega} = G(j\omega)}$$

- **Proof.** The Laplace transform of the output signal, starting from a zero initial conditions, is given by the following relation:

$$Y(s) = G(s)\,X(s) = G(s)\,\frac{X\,\omega}{s^2+\omega^2} = G(s)\,\frac{X\,\omega}{(s-j\omega)\,(s+j\omega)}$$

The poles of function $Y(s)$ are the poles of the transfer function $G(s)$ plus the poles $p_{1,2} = \pm j\omega$ of the input signal $X(s)$.

- The inverse Laplace transform of function $Y(s)$ shows a transient term $y_0(t)$, due to the poles of $G(s)$, and a sinusoidal permanent term $y_p(t)$:

$$y(t) = y_0(t) + y_p(t) = y_0(t) + M_1 \cos[\omega t + \varphi_1]$$

The parameters M_1 and φ_1 are functions of the residues $K_{1,2}$ of the poles $p_{1,2}$:

$$K_1 = G(s)\,\frac{X\,\omega}{s+j\omega}\bigg|_{s=j\omega} = \frac{X}{2\,j}\,G(j\omega), \qquad K_2 = K_1^*.$$

- Recalling that system $G(s)$ is asymptotically stable, and that

$$M_1 = 2\,|K_1| = X|G(j\omega)|, \qquad \varphi_1 = \arg(K_1) = \arg G(j\omega) - \frac{\pi}{2}$$

for t sufficiently high one obtains that:

$$\begin{aligned} y(t) \simeq y_p(t) &= X|G(j\omega)|\cos[\omega t + \arg G(j\omega) - \tfrac{\pi}{2}] \\ &= \underbrace{X|G(j\omega)|}_{Y(\omega)}\ \sin\left[\omega t + \underbrace{\arg G(j\omega)}_{\varphi(\omega)}\right] \end{aligned}$$

- The frequency response function can also be defined for unstable systems, but in this case it cannot be measured experimentally.

- *The impulse response function $g(t)$ of an asymptotically stable linear system completely defines the frequency response $F(\omega)$:*

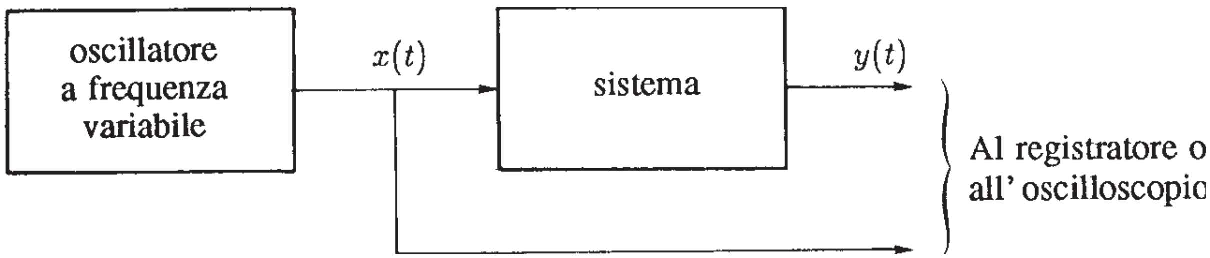
$$g(t) \longleftrightarrow G(s) \stackrel{s=j\omega}{\longrightarrow} G(j\omega) = F(\omega)$$

- *The frequency response $F(\omega)$ of an asymptotically stable linear system completely defines the impulse response function $g(t)$:*

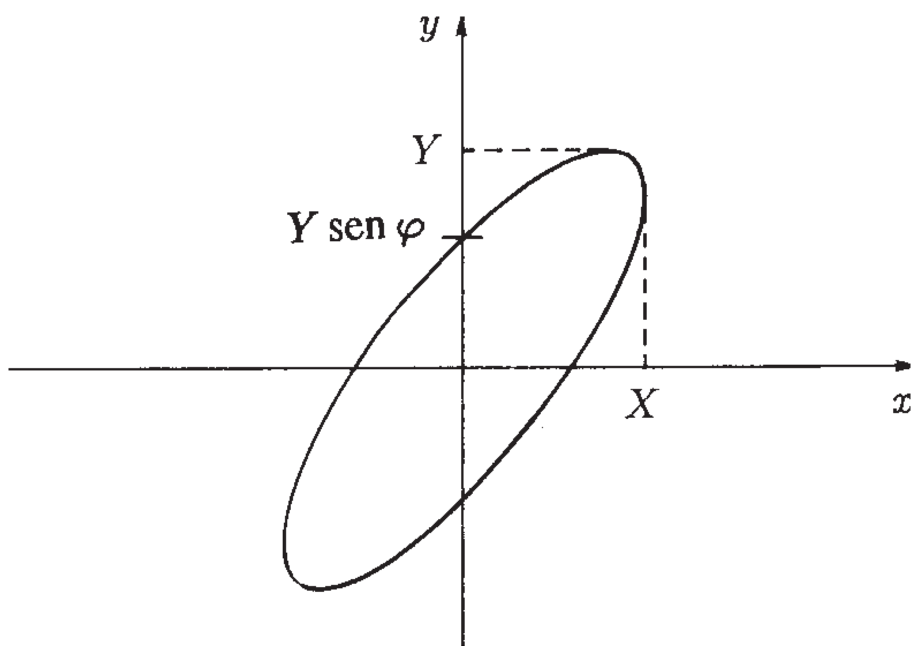
$$g(t) = \frac{1}{2\pi j}\int_{\sigma_0-j\infty}^{\sigma_0+j\infty} G(s)\,e^{st}\,ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{G(j\omega)}_{F(\omega)}\,e^{j\omega t}\,d\omega$$

Detection of the frequency response function

- The frequency response function can be measured experimentally by using an input variable frequency sinusoidal oscillator and an oscilloscope:



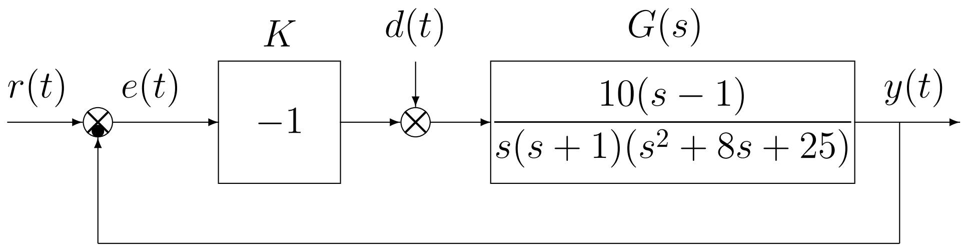
- The amplitude Y and the φ phase of the output signal $y(t)$ with respect to the input signal $x(t)$ can be determined analyzing the Lissajous figure obtained by connecting the input $x(t)$ and the output $y(t)$ signals to the two axes of the oscilloscope:



- The frequency response function is:

$$F(\omega) = \frac{Y}{X} e^{j\varphi}$$

• ***Example.*** Referring to the following system:



determine the steady-state error $e_\infty(t)$ of the system in presence of the constant disturbance $d(t) = 2$ and the sinusoidal input signal $r(t) = 3 + \cos t$.

• In this case the linearity property of the system is applied: the steady-state error $e_\infty(t)$ is the sum of the two terms given by the input $R(s)$ and the disturbance $D(s)$:

$$E(s) = G_d(s)\,D(s) + G_r(s)\,R(s) = \frac{-G(s)}{1 + K\,G(s)}D(s) + \frac{1}{1 + K\,G(s)}R(s)$$

Substituting $K = -1$ and the $G(s)$ function, one obtains:

$$E(s) = \underbrace{\frac{-10(s-1)}{s(s+1)(s^2+8s+25)-10(s-1)}}_{G_d(s)}D(s) + \underbrace{\frac{s(s+1)(s^2+8s+25)}{s(s+1)(s^2+8s+25)-10(s-1)}}_{G_r(s)}R(s)$$

The contributions on the steady-state error $e_\infty(t)$ due to the "constant" components ($d_0 = 2$ and $r_0 = 3$) of the disturbance $d(t)$ and the input signal signal $r(t)$ can be determined, in steady-state conditions, by computing the static gains $G_d(0)$ and $G_r(0)$ of the transfer functions $G_d(s)$ and $G_r(s)$:

$$e_0 = \left.\frac{-10(s-1)}{s(s+1)(s^2+8s+25)-10(s-1)}\right|_{s=0} d_0 + \left.\frac{s(s+1)(s^2+8s+25)}{s(s+1)(s^2+8s+25)-10(s-1)}\right|_{s=0} r_0 = 2$$

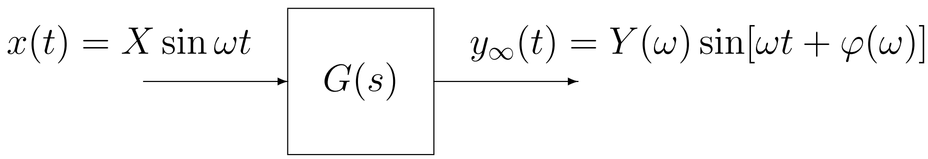
The sinusoidal term $\cos t$ of the reference signal $r(t)$ generates, in steady-state condition, a sinusoidal term $e_\omega(t)$ on the steady-state error $e_\infty(t)$. The $e_\omega(t)$ signal can be easily determined by computing the frequency response function $G_r(j\omega)$ at the frequency $\omega = 1$:

$$G_r(j) = \left.\frac{s(s+1)(s^2+8s+25)}{s(s+1)(s^2+8s+25)-10(s-1)}\right|_{s=j} = 1.569\,e^{-j\,0.1974} = 1.569\angle - 11.31^o$$

Therefore, the steady-state error $e_\infty(t)$ is:

$$e_\infty(t) = e_0 + |G_r(j)|\cos(t + Arg[G_r(j)]) = 2 + 1.569\cos(t - 0.1974)$$

- **Note**: the frequency response function is useful to calculate the steady-state response of an asymptotically stable linear system when an input sinusoidal signal is given:



- The frequency response function is defined as follows:

$$F(\omega) = \frac{Y(\omega)}{X} \, e^{j\varphi(\omega)} = G(j\omega)$$

- For problems of this type it is not appropriate to use the method of the Laplace transform because in this case the calculations to be done would be much more heavy, and the final result would be the same.

- A common error is to calculate the steady-state output signal $y_\infty(t)$ by applying the final value theorem to the Laplace transform $Y(s)$ of the output signal $y(t)$

$$x(t) = X \, \sin(\omega t) \;\; \rightarrow \;\; R(s) = \frac{X\,\omega}{s^2 + \omega^2} \qquad \Longrightarrow \qquad Y(s) = G(s)\frac{X\,\omega}{s^2 + \omega^2}$$

- It is good to remember that this computation is not correct because the final value theorem can only be applied to functions $Y(s)$ that have all the poles within the real negative half plane, with, eventually, one pole in the origin. In the present case the function $Y(s)$ has a couple of complex conjugate poles on the imaginary axis and therefore it is not possible to apply the final value theorem. Moreover, it is evident that the output signal, being sinusoidal, does not have a limit for $t \rightarrow \infty$.

- On the contrary, using the frequency response function, the solution of the problem is straightforward. In fact, using the theoretical relation $F(\omega) = G(j\omega)$ the steady-state output signal $y_\infty(t)$ can be directly computed as follows:

$$x(t) = X \sin(\omega t) \qquad \Rightarrow \qquad y_\infty(t) = y(t)|_{t \to \infty} = X\,|G(j\omega)| \, \sin(\omega t + \arg G(j\omega))$$

- The steady-state response $y_\infty(t)$ of a linear system $G(s)$ to a generic sinusoidal input $x(t) = X \cos(\omega t + \alpha)$ can always be computed as follows:

$$x(t) = X \cos(\omega t + \alpha) \qquad \Rightarrow \qquad y_\infty(t) = X\,|G(j\omega)| \, \cos(\omega t + \alpha + \arg G(j\omega))$$

- The offset $\varphi = \arg G(j\omega)$ given by the frequency response function is always the offset of the output signal $y(t)$ with respect to the input signal $x(t)$.