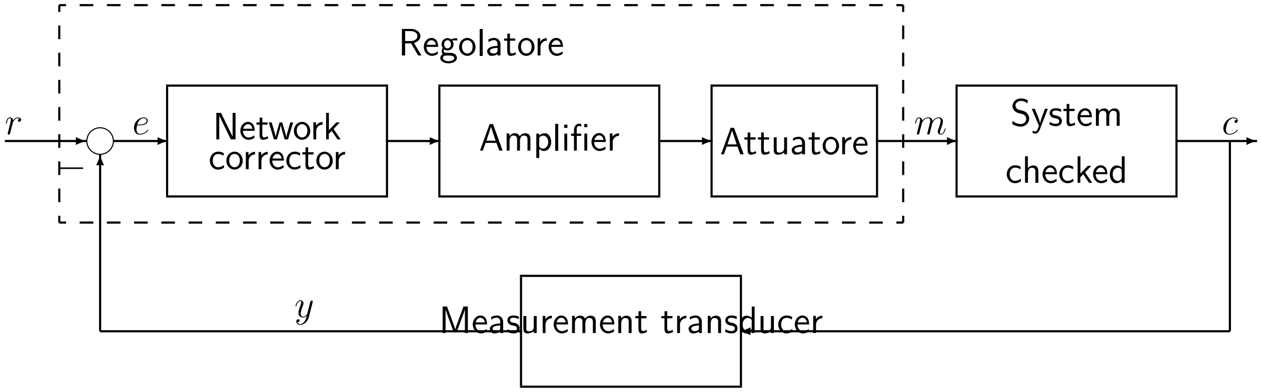
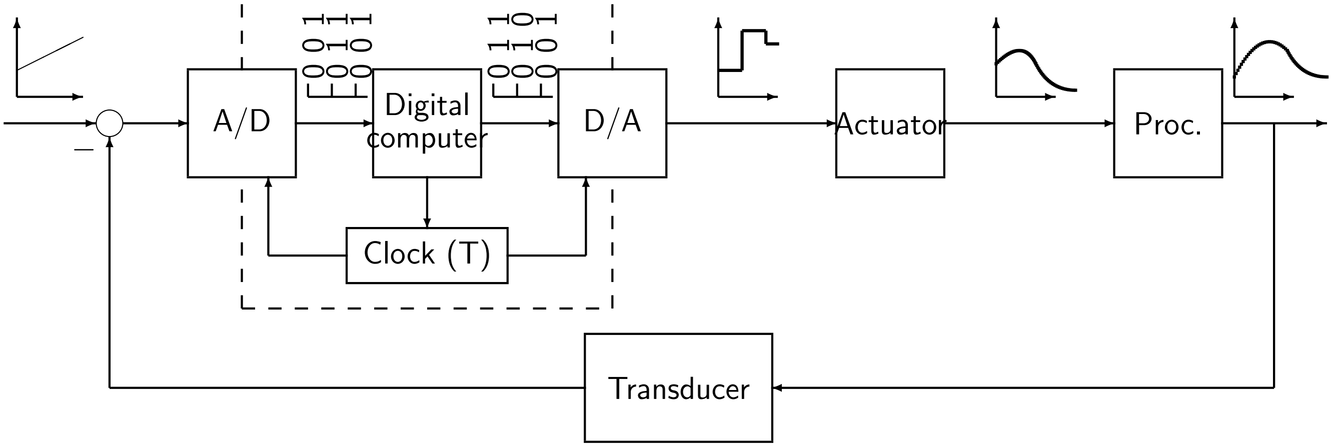
DIGITAL CONTROL

- PROCESS: a set of operations or transformations that must take place in an appropriate sequence in a plant or in a physical system
- CONTROL OF PROCESSES: set of methodologies, techniques and technologies oriented to the automated management of industrial plants
- DIGITAL CONTROL SYSTEMS: feedback control systems in which there is a digital uncalculator and therefore a discrete time processing of the control law
- ANALOG CONTROL SYSTEM:

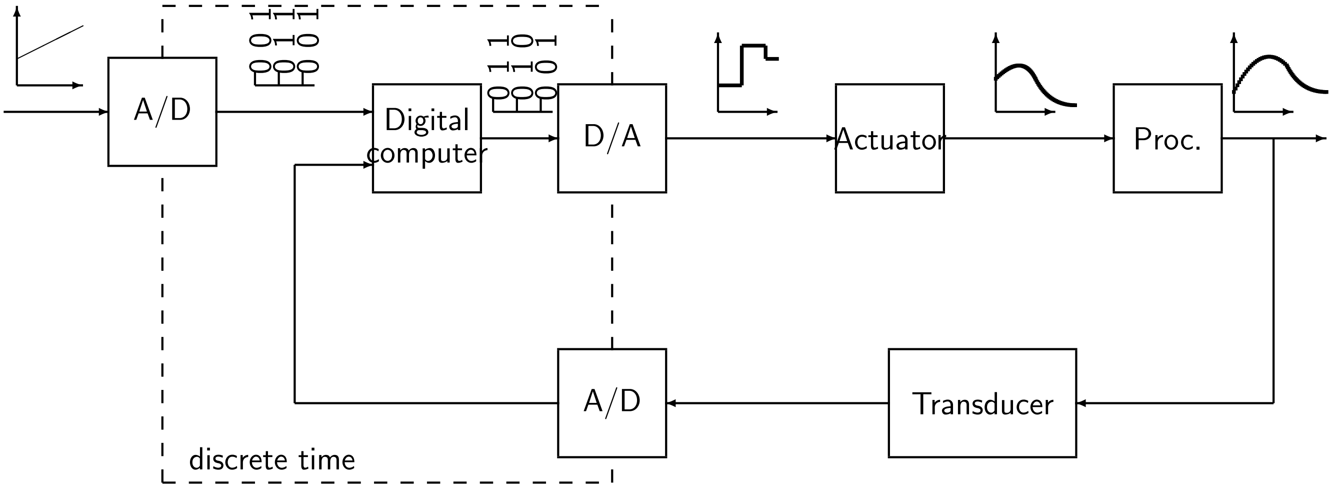


TYPICAL DIAGRAMS OF A DIGITAL CONTROL SYSTEM

- Sampling of the error signal:



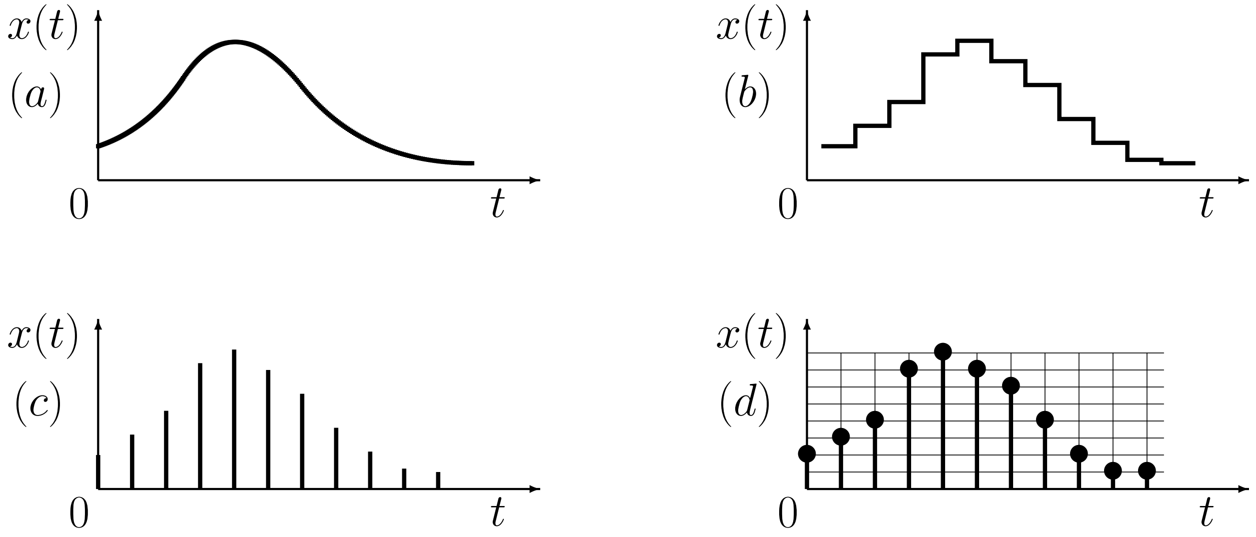
- Sampling of the feedback signal:



- DIGITAL CONTROL/ANALOG CONTROL:

  + Greater capacity and processing accuracy
  + Increased flexibility
  + Greater reliability and repeatability
  + More transmissibility of signals
  - More difficult and articulated design
  - Stabilitability is more precarious
  - Possibility of unanticipated arrests
  - You need to use electricity

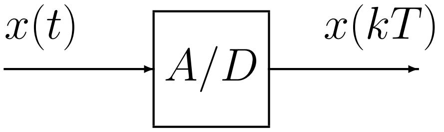
INTEREST SIGNALS: a) Analogue of a continuous type; b) Quantized time-continuous; c) A sampled data; d) Digital ;

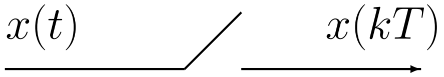


INTERFACE DEVICES

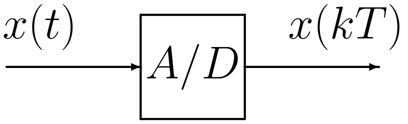
- A/D: Analog/Digital converter. Two possible descriptions:

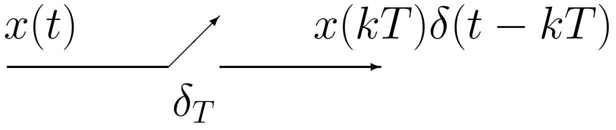
  1) Generating a sequence of numeric values:



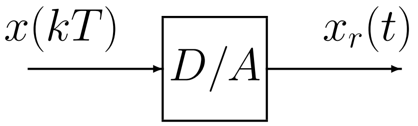


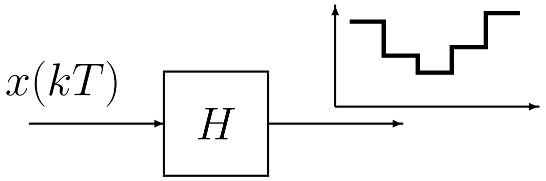
  2) Dirac Pulse Sampling:



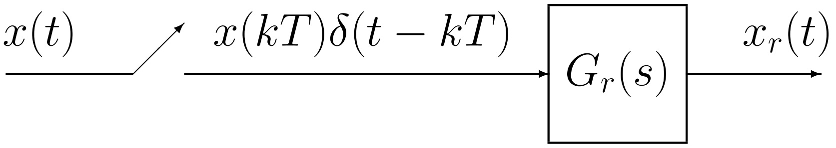


- D/A, Digital/Analog converter





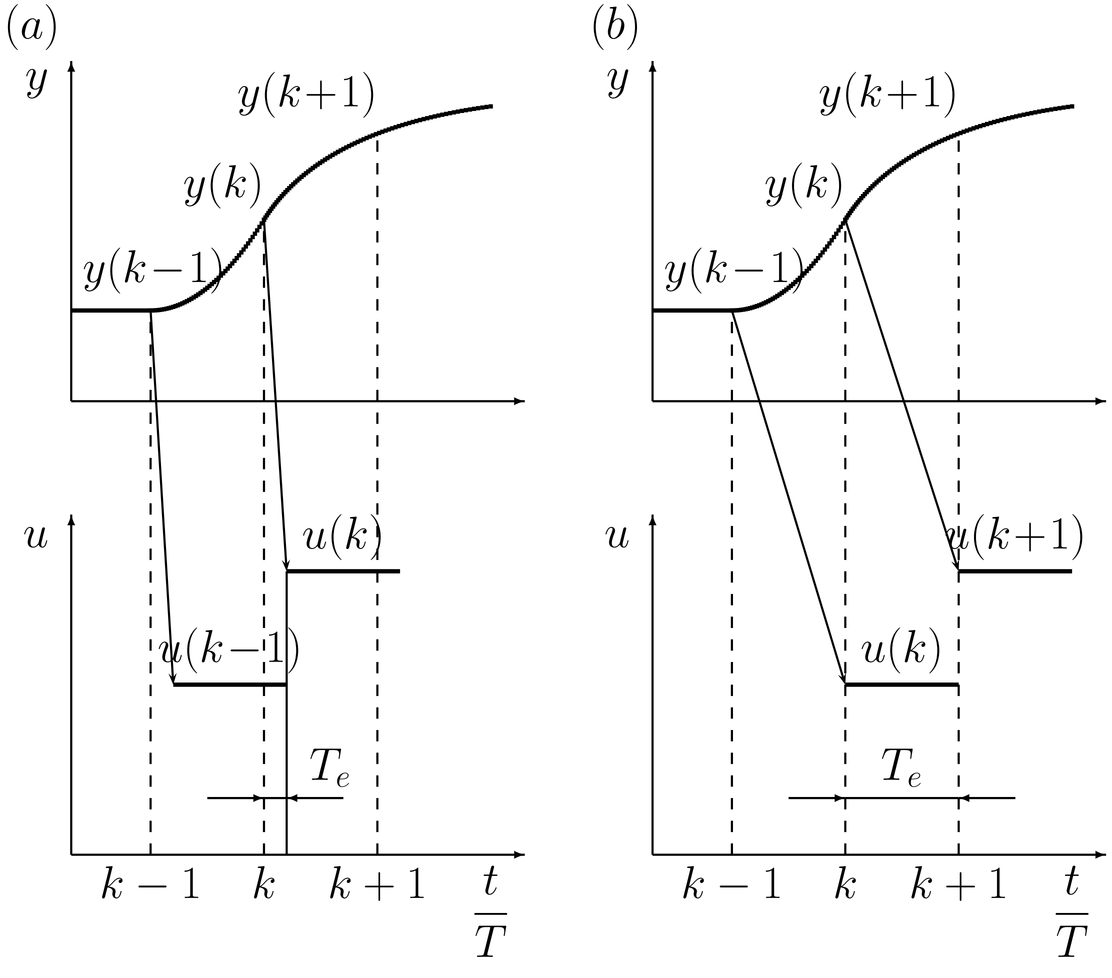
If you use the Dirac pulse sampler, the rebuilder (ie the D/A converter) can be represented by a simple transfer function $G_r(s)$:



huge Zero Order Reconstructor:

$$\boxed{G_r(s) = \frac{1-e^{-sT}}{s}}$$

PROCESSING AND SYNCHRONIZATION TIME



- Differences equations. They are static bonds that bind current values (instantly k) and pass (at $k-1$, $k-2$, etc.) of the e_k and exit u_k:

$$u_k = f(e_0, e_1, \ldots, e_k; u_0, u_1, \ldots, u_{k-1})$$

The difference equation is linear if $f(\cdot)$ is linear:

$$u_k = -a_1 u_{k-1} - \ldots - a_n u_{k-n} + b_0 e_k + \ldots + b_m e_{k-m}$$

The solution of a difference equation is given by the sum of the "free response" (null input and initial null conditions) and of the "forced response" (initial null and non-zero input) of the system:

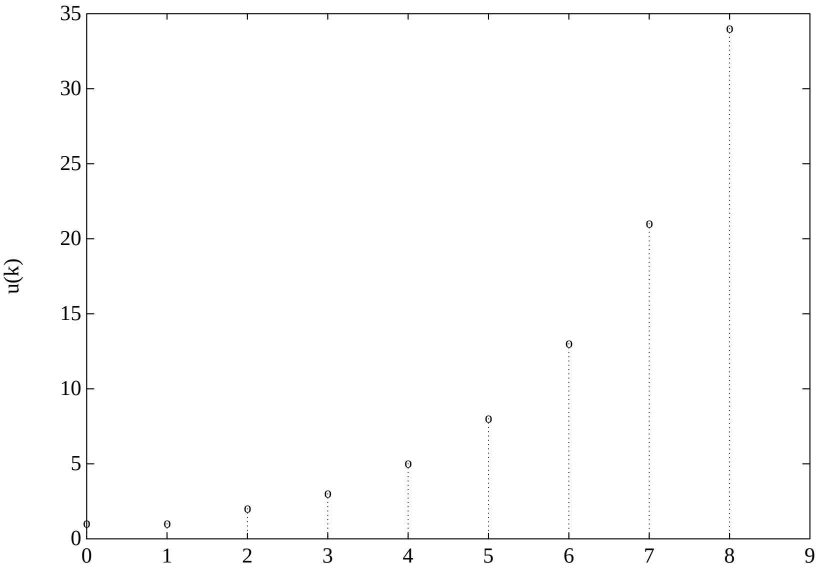
$$u_k = u_k^l + u_k^f$$

To determine the free response, many initial conditions exist as well as the order of the difference equation.

- Solution of equations to differences to coef -fi -cien -ti constants

$$u_k = u_{k-1} + u_{k-2} \qquad\qquad k \geq 2$$

$u_0 = u_1 = 1.$



To solve the difference equations you can use the $\mathcal{Z}$-transformed method.

CZ-trasformata

- Let us give a sequence of values $x_k \in \mathbb{R}$, defined for $k = 0, 1, 2, \ldots$ and nothing for $k < 0$. The $\mathcal{Z}$-transformed (unilatera) of the sequence x_k is the complex variable function z defined as

$$X(z) = \mathcal{Z}[x_k] = \sum_{k=0}^{\infty} x_k z^{-k} = x_0 + x_1\, z^{-1} + \cdots + x_k\, z^{-k} + \cdots$$

- In the case where the sequence of values x_k is obtained by sampling uniformly with period T a continuous signal described by the function $x(t)$, $t \geq 0$, we will have that $x_k = x(kT)$:

$$\boxed{X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}}$$

- To indicate that a sequence has been obtained by sampling a continuous time signal, the following notation is often used:

$$X(z) = \mathcal{Z}[X(s)]$$

meaning:

$$X(z) = \mathcal{Z}\Big[\Big\{ \mathcal{L}^{-1}[X(s)]\Big|_{t=kT} \Big\}\Big]$$

- In engineering applications, the function $X(z)$ generally assumes an expression rational fract of the type

$$X(z) = \frac{b_0\, z^m + b_1\, z^{m-1} + \cdots + b_m}{z^n + a_1\, z^{n-1} + \cdots + a_n}$$

which can also be expressed in powers of z^{-1}:

$$X(z) = \frac{b_0\, z^{-(n-m)} + b_1\, z^{-(n-m+1)} + \cdots + b_m\, z^{-n}}{1 + a_1\, z^{-1} + \cdots + a_n z^{-n}}$$

- Example:

$$X(z) = \frac{z(z+0.5)}{(z+1)(z+2)} = \frac{1 + 0.5\, z^{-1}}{(1+z^{-1})(1+2\, z^{-1})}$$

- Unit discrete pulse, also called Kronecker function $\delta_0(t)$:

$$\boxed{x(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad \leftrightarrow \qquad X(z) = 1}$$

In fact:

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k) z^{-k} = 1 + 0z^{-1} + 0z^{-2} + 0z^{-3} + \cdots = 1$$

- Unitary step. Let the unit step function be given

$$x(k) = h(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

The function $h(k)$, called unitary sequence, is the following:

$$\boxed{h(k) = \begin{cases} 1 & k = 0, 1, 2, \ldots \\ 0 & k < 0 \end{cases} \qquad \leftrightarrow \qquad X(z) = \frac{z}{z-1}}$$

In fact:

$$\begin{aligned} H(z) &= \mathcal{Z}[h(k)] = \sum_{k=0}^{\infty} h(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \end{aligned}$$

The series is convergent for $|z| > 1$.

- Unit ramp. Consider the unit ramp function:

$$\boxed{x(k) = \begin{cases} k & k \geq 0 \\ 0 & k < 0 \end{cases} \qquad \leftrightarrow \qquad X(z) = \frac{z}{(z-1)^2}}$$

In fact, the $\mathcal{Z}$-transform of $x(k) = k$ is

$$\begin{aligned} X(z) &= \mathcal{Z}[k] = \sum_{k=0}^{\infty} x(k) z^{-k} = \sum_{k=0}^{\infty} k z^{-k} = (z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) \\ &= z^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots) = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z-1)^2} \end{aligned}$$

convergent for $|z| > 1$.

- Power function a^k. Be given the function

$$\boxed{x(k) = \begin{cases} a^k & k = 0, 1, 2, \ldots \\ 0 & k < 0 \end{cases} \qquad \leftrightarrow \qquad X(z) = \frac{z}{z-a}}$$

with a real or complex constant. From the definition of $\mathcal{Z}$-transformed we have that

$$\begin{aligned} X(z) &= \mathcal{Z}\big[a^k\big] = \sum_{k=0}^{\infty} x(k) z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k} \\ &= 1 + a\, z^{-1} + a^2\, z^{-2} + a^3\, z^{-3} + \cdots = \frac{1}{1 - a\, z^{-1}} = \frac{z}{z-a} \end{aligned}$$

This geometric series converges for $|z| > |a|$.

Discretization of continuous time signals

- Unitary step:

$$\mathcal{Z}\left[\frac{1}{s}\right] = \mathcal{Z}[h(t)|_{t=kT}] = \frac{z}{z-1}$$

- Unitary ramp:

$$\mathcal{Z}\left[\frac{1}{s^2}\right] = \mathcal{Z}[t|_{t=kT}] = \mathcal{Z}[k\,T] = T\frac{z}{(z-1)^2}$$

- Exponential function:

$$\mathcal{Z}\left[\frac{1}{s+b}\right] = \mathcal{Z}\big[e^{-b\,t}\big|_{t=kT}\big] = \mathcal{Z}\big[e^{-b\,k\,T}\big] = \mathcal{Z}\big[(e^{-b\,T})^k\big] = \frac{z}{z - e^{-bT}}$$

- Sinusoidal function. Both the sinusoid is given

$$\boxed{x(t) = \begin{cases} \sin\omega t & t \ge 0 \\ 0 & t < 0 \end{cases} \qquad \leftrightarrow \qquad X(z) = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}}$$

From Euler's formulas it is known that

$$\sin\omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$\begin{aligned} X(z) &= \mathcal{Z}[\sin\omega t] = \frac{1}{2j}\left(\frac{1}{1-e^{j\omega T}\,z^{-1}} - \frac{1}{1-e^{-j\omega T}\,z^{-1}}\right) \\ &= \frac{1}{2j}\frac{(e^{j\omega T} - e^{-j\omega T})z^{-1}}{1-(e^{j\omega T}+e^{-j\omega T})z^{-1}+z^{-2}} \\ &= \frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T + z^{-2}} = \frac{z\sin\omega T}{z^2-2z\cos\omega T+1} \end{aligned}$$

convergent for $|z| > 1$.

- cosinusoidal function. Be given the function

$$\boxed{x(t) = \begin{cases} \cos\omega t & t \ge 0 \\ 0 & t < 0 \end{cases} \qquad \leftrightarrow \qquad X(z) = \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}}$$

$$\begin{aligned} X(z) &= \mathcal{Z}[\cos\omega t] = \frac{1}{2}\left(\frac{1}{1-e^{j\omega T}\,z^{-1}} + \frac{1}{1-e^{-j\omega T}\,z^{-1}}\right) \\ &= \frac{1}{2}\frac{2-(e^{-j\omega T} + e^{j\omega T})z^{-1}}{1-(e^{j\omega T}+e^{-j\omega T})z^{-1}+z^{-2}} \\ &= \frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T + z^{-2}} \\ &= \frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1} \qquad\qquad |z| > 1 \end{aligned}$$

- Example: $X(s) = \frac{1}{s(s+1)}$
- First technique: $x(t) = 1 - e^{-t}$

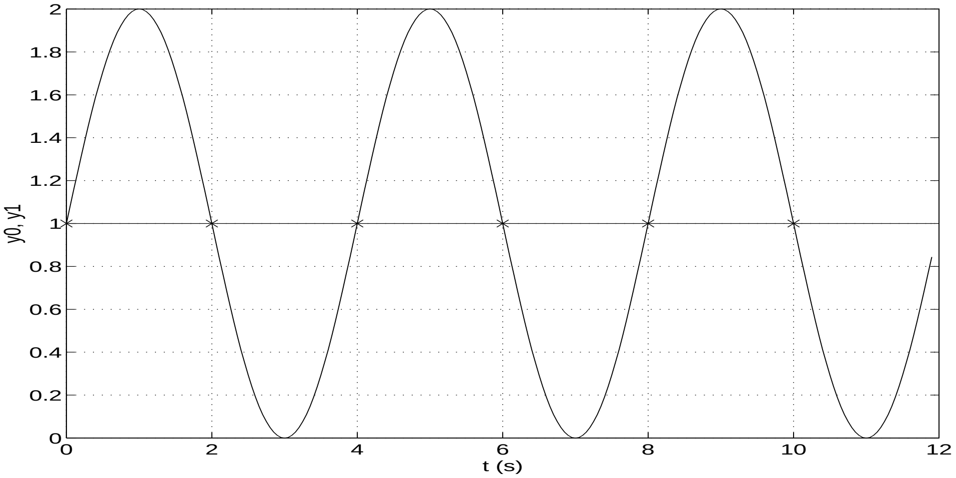
$$\begin{aligned} X(z) &= \mathcal{Z}\left[1 - e^{-t}\right] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T} z^{-1}} \\ &= \frac{(1 - e^{-T}) z^{-1}}{(1 - z^{-1})(1 - e^{-T} z^{-1})} = \frac{(1 - e^{-T}) z}{(z - 1)(z - e^{-T})} \end{aligned}$$

- Second technique:

$$X(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{1+s}$$

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T} z^{-1}}$$

- The $\mathcal{Z}$-transformed $X(z)$ and its corresponding sequence $x(k)$ are bound by a two-way match
- This does not typically occur between the $\mathcal{Z}$-transformed $X(z)$ and its "reverse" $x(t)$
- Given a $X(z)$ you can usually have HHeBb many $x(t)$
- This ambiguity does not subsist if the restrictive conditions on T dictated by the Shannon's theorem are verified
- Diverse continuous time functions can have the same values $x(k)$



- PROPRIET 'A AND THEOREMIS OF $\mathcal{Z}$-TRANSFORMED

- Linearity 'a:

$$x(k) = af(k) + bg(k)$$

$$\boxed{X(z) = aF(z) + bG(z)}$$

- Multiplication for a^k. Let $X(z)$ la $\mathcal{Z}$-transform $x(t)$, a constant.

$$\boxed{\mathcal{Z}\left[a^k x(k)\right] = X(a^{-1}z)}$$

$$\mathcal{Z}\left[a^k x(k)\right] = \sum_{k=0}^{\infty} a^k x(k) z^{-k} = \sum_{k=0}^{\infty} x(k)(a^{-1}z)^{-k} = X(a^{-1}z)$$

- Theorem of translation over time. If $x(t) = 0, t < 0$, $X(z) = \mathcal{Z}[x(t)]$, and $n = 1, 2, \ldots$, then

$$\boxed{\mathcal{Z}[x(t - nT)] = z^{-n} X(z)} \qquad \text{(delay)}$$

$$\boxed{\mathcal{Z}[x(t + nT)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}\right]} \qquad \text{(advance)}$$

- Theorem of the initial value.
If $X(z)$ is the $\mathcal{Z}$-transform of $x(t)$ and if the $\lim_{z\to\infty} X(z)$ exists , then the initial value $x(0)$ of $x(t)$ is given by:

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

- Final value theorem. Let all the poles of $X(z)$ be inside the unit circle, with at most one simple pole for $z = 1$.

$$\boxed{\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[(1 - z^{-1})X(z)\right]}$$

- Example: Consider the signal described by

$$X(z) = \frac{Tz(z+1)}{2(z-0.5)(z-1)}$$

The final value of the sequence $x(kT)$ is therefore given by

$$\begin{aligned} \lim_{k\to\infty} x(kT) &= \lim_{z\to 1}(1-z^{-1})\frac{Tz(z+1)}{2(z-0.5)(z-1)} \\ &= \lim_{z\to 1}\frac{T(z+1)}{2(z-0.5)} \\ &= 2T \end{aligned}$$

- Transformation of periodic functions.

Let be given a periodic $x_p(k)$ periodic pT and $x(k)$ the sequence of the samples of the first period and null for $k > p$

$$\boxed{x(k) = \begin{cases} x_p(k) & k = 0, \ldots, p \\ 0 & k > p \end{cases}}$$

If $X(z)$ is the $\mathcal{Z}$-transform of $x(k)$ then it is worth

$$\mathcal{Z}[x_p(k)] = \frac{z^p}{z^p - 1}X(z) = \frac{1}{1-z^{-p}}X(z)$$

- THE ANTI-AFRIPPY ZT
- It allows to pass from a $\mathcal{Z}$-transformed $X(z)$ to the corresponding sequence x_k and possibly to the continuous function $x(t)$ which corresponds to the sequence x_k for sampling.

$$\begin{tikzcd}
X(z) \arrow[rr, leftrightarrow, "\text{bijective}"'] & & x(k) & & x(t) \arrow[ll, "\text{not bijective}"']
\end{tikzcd}$$

- If the Shannon Theorem is satisfied, the continuous function $x(t)$ can be uniquely determined from the sequence x_k.
- There are several methods to turn out to be a $X(z)$ function. Among these the simplest is the ‘ ‘Method of breaking down into simple fractions”:

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

- Case 1. If all the poles are simple, $X(z)$ can be rewritten as follows:

$$X(z) = \frac{\bar{c}_1}{z-p_1} + \frac{\bar{c}_2}{z-p_2} + \cdots + \frac{\bar{c}_n}{z-p_n} = \sum_{i=1}^{n} \frac{\bar{c}_i}{z-p_i}$$

where the coefficients $\bar{c}_i$ are calculated using the “residuals” rule:

$$\bar{c}_i = \Big[(z-p_i)X(z)\Big]_{z=p_i}$$

- Antitrasforming is obtained:

$$X(z) = z^{-1} \sum_{i=1}^{n} \frac{\bar{c}_i\, z}{z-p_i} \qquad \Rightarrow \qquad x(k) = \begin{cases} 0 & k = 0 \\ \displaystyle\sum_{i=1}^{n} \bar{c}_i\, p_i^{k-1} & k \geq 1 \end{cases}$$

- If in the expression of $X(z)$ there is at least a zero in the origin, it is appropriate to decompose the function $X(z)/z$ in simple fracts:

$$\frac{X(z)}{z} = \frac{c_1}{z-p_1} + \cdots + \frac{c_n}{z-p_n} \qquad c_i = \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}$$

- In this case, anti-transforming is obtained:

$$X(z)=\sum_{i=1}^{n}\frac{c_i\,z}{z-p_i} \qquad \Rightarrow \qquad x(k)=\sum_{i=1}^{n}c_i\,p_i^k$$

When complex poles co -niu -ga -ti are present, the coefficients c_i are also com -ples -si. In this case, Euler formulas are used to obtain trigonometric functions.

- Case 2. If the function $X(z)$, or the function $X(z)/z$, has multiples:

$$X(z)=\frac{B(z)}{A(z)}=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{(z-p_1)^{r_1}(z-p_2)^{r_2}\cdots(z-p_h)^{r_h}}$$

then the decomposition is carried out as follows:

$$X(z)=\sum_{i=1}^{h}\sum_{k=1}^{r_i}\frac{c_{ik}}{(z-p_i)^{r_i-k+1}}$$

where the residues are calculated as follows:

$$c_{ik}=\left[\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z-p_i)^{r_i}X(z)\right]_{z=p_i}$$

where $i=\{1,\ldots,h\}$ and $k=\{1,\ldots,r_i\}$.

- Example. To reverse the function:

$$X(z)=\frac{1}{z^4+6z^3+13z^2+12z+4}=\frac{1}{(z+2)^2(z+1)^2}$$

In this case we have:

$$X(z)=\frac{c_{11}}{(z+2)^2}+\frac{c_{12}}{(z+2)}+\frac{c_{21}}{(z+1)^2}+\frac{c_{22}}{(z+1)}$$

where

$$c_{11}=[(z+2)^2X(z)]|_{z=-2}=1 \qquad c_{12}=\left[\tfrac{d}{dz}(z+2)^2X(z)\right]_{z=-2}=2$$

$$c_{21}=\left[(z+1)^2X(z)\right]_{z=-1}=1 \qquad c_{22}=\left[\tfrac{d}{dz}(z+1)^2X(z)\right]_{z=-1}=-2$$

Antitrasforming is obtained:

$$x(k)=\begin{cases} 0 & k=0 \\ \frac{c_{11}(k-1)(-2)^{k-1}}{2}+c_{12}(-2)^{k-1}+c_{21}(k-1)(-1)^{k-1}+c_{22}(-1)^{k-1} & k\geq 1\end{cases}$$

Example. Calculate the $c(n)$ solution of the following difference equation:

$$c(n+1) = c(n) + i\,c(n)$$

starting from the initial condition $c(0) = c_0$. This equation can be interpreted as the law of capitalization of an initial capital c_0 at the interest rate i.

[Solution.] Applying the Z-transformed to the previous equation is obtained

$$z\,C(z) - z\,c_0 = (i+1)C(z) \qquad \rightarrow \qquad [z-(1+i)]C(z) = z\,c_0$$

from which

$$C(z) = \frac{z\,c_0}{z-(1+i)} = \frac{c_0}{1-(1+i)z^{-1}} \qquad \rightarrow \qquad c(n) = c_0(1+i)^n$$

Example. Determine the analytical expression of the Fibonacci sequence described by the following equation to differences

$$y(n+2) = y(n+1) + y(n)$$

starting from the initial conditions $y(0) = y(1) = 1$.

[Solution.] Applying the Z-transformed method

$$z^2Y(z) - z^2y(0) - z\,y(1) = z\,Y(z) - z\,y(0) + Y(z)$$

you get it

$$Y(z) = \frac{z[zy(0) + y(1) - y(0)]}{z^2 - z - 1}$$

By imposing the initial conditions $y(0) = y(1) = 1$ is obtained

$$Y(z) = \frac{z^2}{z^2-z-1} = \frac{z^2}{(z-a)(z-b)} = \frac{1}{a-b}\left[\frac{a\,z}{(z-a)} - \frac{b\,z}{(z-b)}\right]$$

where a and b are the roots of the $Y(z)$ function

$$a = \frac{1+\sqrt{5}}{2}, \qquad\qquad b = \frac{1-\sqrt{5}}{2}$$

Antitrasforming is obtained

$$y(n) = \frac{1}{a-b}\left[a\,a^n - b\,b^n\right] = \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Example. Calculate the impulse response $g(n)$ of the following discrete dynamic system

$$G(z) = \frac{z(z+1)}{(z-0.8)(z+0.5)}$$

[Solution.] To calculate the impulse response $g(n)$ of the system $G(z)$ proceed using the method of decomposition into simple fractions

$$\frac{G(z)}{z} = \frac{(z+1)}{(z-0.8)(z+0.5)} = \frac{1.8}{1.3}\frac{1}{(z-0.8)} - \frac{0.5}{1.3}\frac{1}{(z+0.5)}$$

from which

$$G(z) = 1.3846\frac{z}{(z-0.8)} - 0.3846\frac{z}{(z+0.5)}$$

Antitrasforming is obtained:

$$g(n) = 1.3846(0.8)^n - 0.3846(-0.5)^n$$

The discrete system $G(z)$ is stable.

Example. Determine the response $y(n)$ to the unit step $u(n) = 1$ of the following discrete dynamic system

$$y(n) = 0.5\, y(n-1) + u(n)$$

starting from initial condition null $y(0) = 0$.

[Solution.] The discrete transfer function $G(z)$ associated with the assigned difference equation is as follows:

$$G(z) = \frac{1}{1-0.5\, z^{-1}}$$

The Z-transform of the input signal $u(n) = 1$ is:

$$U(z) = \frac{1}{1-z^{-1}}$$

Thus, the Z-transform of the output signal $y(n)$ is

$$Y(z) = G(z)U(z) = \frac{1}{(1-0.5\, z^{-1})}\,\frac{1}{(1-z^{-1})} = \frac{z^2}{(z-0.5)(z-1)}$$

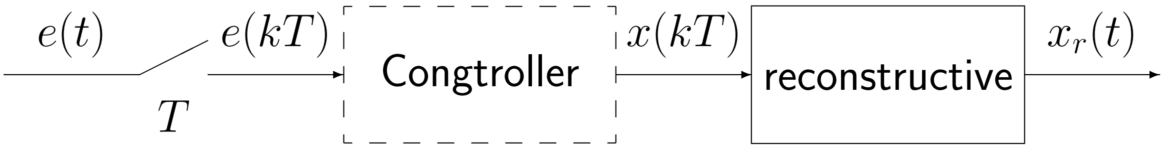
By performing the simple fractional decomposition of the function $Y(z)/z$ we obtain

$$\frac{Y(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

from which we get $y(n)$

$$Y(z) = \frac{2\,z}{z-1} - \frac{z}{z-0.5} \qquad \rightarrow \qquad y(n) = 2 - 0.5^n$$

- SAMPLING AND RECONSTRUCTION
- Feedback systems with digital control are characterized by a part continues(the process to be controlled) and a discrete part(the digital controller)
- There are therefore both time-based variables of -scre -to and time-based variables continuous
- The interface devices are the sampler and the rebuilder



- Zero Order Reconstructor:

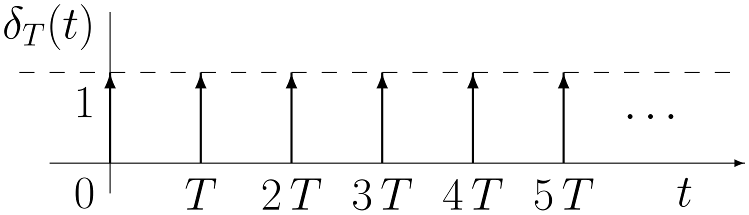
$$x_r(t) = \sum_{k=0}^{\infty} x(kT)[h(t-kT) - h(t-(k+1)T)]$$

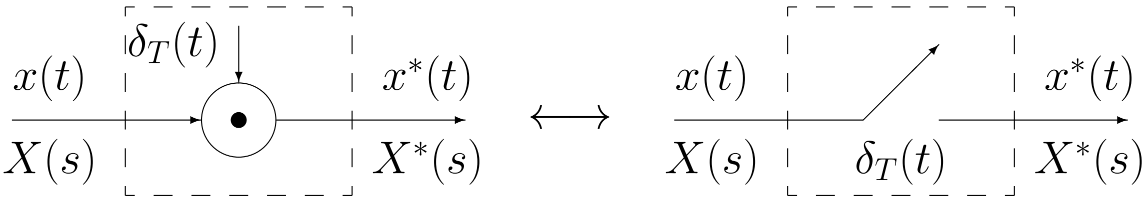
$$X_r(s) \;=\; \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs} - e^{-(k+1)Ts}}{s}\right] = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$H_0(s) = \frac{1-e^{-Ts}}{s} \qquad\qquad X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$x^*(t) = \mathcal{L}^{-1}[X^*(s)] = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

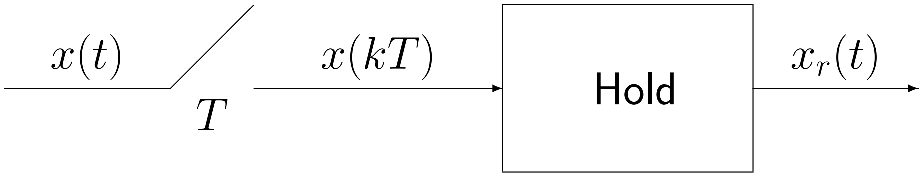
$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t-kT)$$

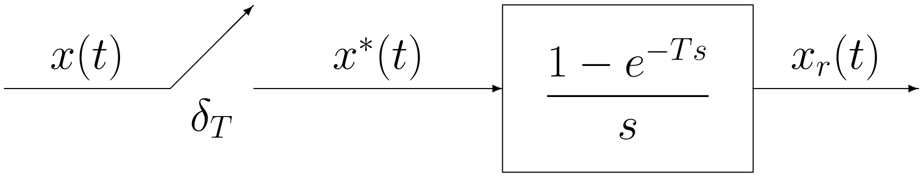




- The impulsive sampler is an ideal model of the real sampler (A/D converter) considered appropriate for the needs of analysis and design of digital controls
- The zero order rebuilder's output is:

$$X_r(s) = H_0(s)\, X^*(s) = \frac{1 - e^{-Ts}}{s}\, X^*(s)$$





$$X^*(s) = \sum_{k=0}^{\infty} x(kT) e^{-kTs}$$

$$\boxed{z = e^{sT}} \qquad \longleftrightarrow \qquad \boxed{s = \frac{1}{T}\ln z}$$

$$X^*(s)\Big|_{s = \frac{1}{T}\ln z} = \sum_{k=0}^{\infty} x(kT)\, z^{-k} = X(z)$$

- The zeta transform of the sequence $x(kT)$ instead of the Laplace transform of the signal $x^*(t)$ allows to operate with fractional rational functions.

$$\begin{aligned} x^*(t) &= x(t)\,\delta_T(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \\ \delta_T(t) &= \sum_{n=-\infty}^{\infty} c_n\, e^{j\,n\omega_s t} \\ c_n &= \frac{1}{T}\int_0^T \delta_T(t)\, e^{-j\,n\omega_s t} dt = \frac{1}{T} \end{aligned}$$

it follows

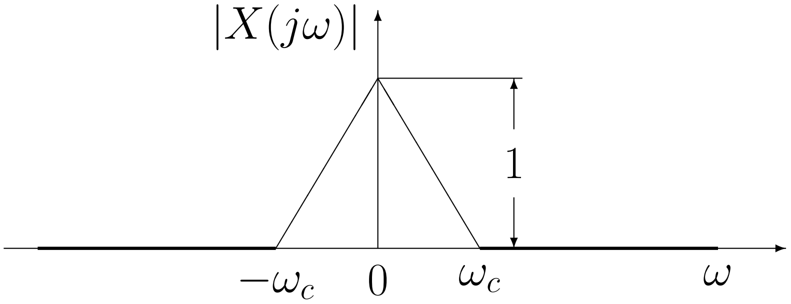
$$\begin{aligned} x^*(t) &= x(t)\,\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j\,n\omega_s t} \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{j\,n\omega_s t} \end{aligned}$$

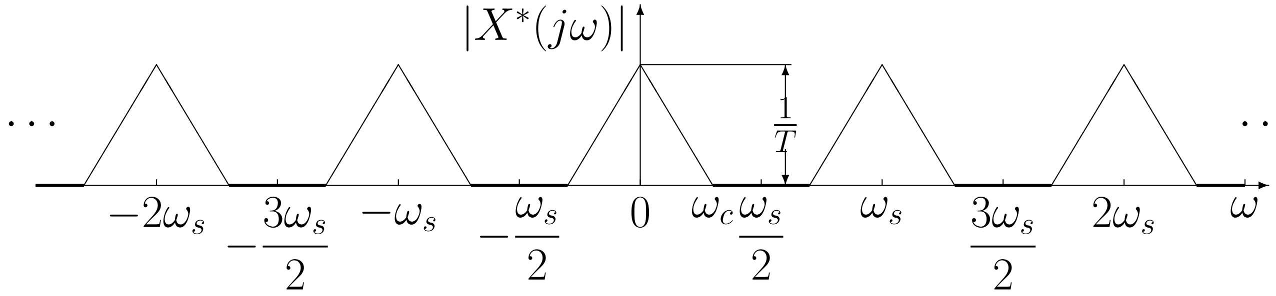
$$X^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\mathcal{L}\big[x(t)\, e^{j\,n\omega_s t}\big] = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(s-j\,n\omega_s)$$

- Unless the $1/T$ multiplicative constant, the Laplace transform $X^*(s)$ of the sampled signal is obtained from the sum of the infinite terms, $X(sj\,n\omega_s)$, each of which is obtained from the $X(s)$ by translation of $j\,n\omega_s$ in the complex field.

- The spectral trend of the sampled signal is valid:

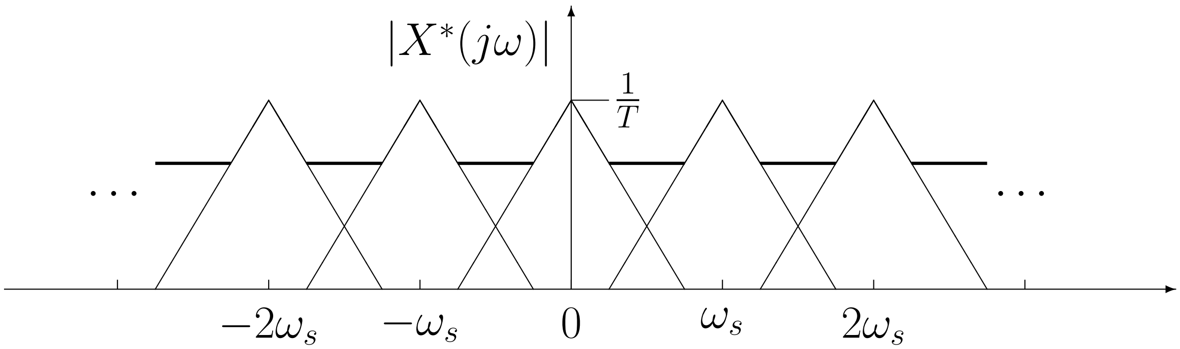
$$X^*(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega - j\,n\omega_s)$$





- The condition $\omega_s > 2\omega_c$ keeps the original spectrum separate from the complementary components so that, by filtering, it is possible to completely reconstruct the signal $x(t)$

- In the event that the condition $\omega_s > 2\omega_c$ is not respected:



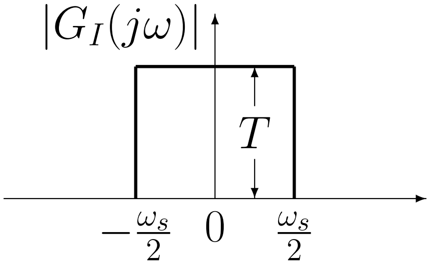
- The original spectrum is partially superimposed on the adjoining complementary components so that by filtering it is no longer possible to derive the original signal starting from the se-gna

- SHANNON THEOREM
- Let $\omega_s = \frac{2\pi}{T}$ the sampling pulsation (T is the sampling period), and let ω_c la pi u high spectral component of the time-continuous signal $x(t)$. The signal $x(t)$ is completely rebuildable starting from the sampled signal $x^*(t)$ if and only if the pulsation ω_s is greater than twice the pulsation ω_c:

$$\omega_s > 2\omega_c$$

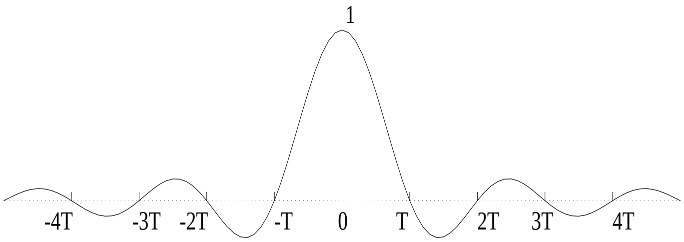
- Reconstruction by an ideal filter

$$G_I(j\omega) = \begin{cases} T & -\dfrac{\omega_s}{2} \le \omega \le \dfrac{\omega_s}{2} \\ 0 & \text{altrove} \end{cases}$$



- The ideal filter $G_i(j\omega)$ is not physically real. His impulse response is:

$$g_I(t) = \frac{\sin(\omega_s t/2)}{\omega_s t/2}$$



- Shannon's reconstruction formula:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} x^*(\tau)\, g_I(t-\tau)\, d\tau \\ &= \sum_{k=-\infty}^{\infty} x(kT) \int_{-\infty}^{\infty} \delta(\tau - kT) \frac{\sin(\omega_s(t-\tau)/2)}{\omega_s(t-\tau)/2}\, d\tau \\ &= \sum_{k=-\infty}^{\infty} x(kT) \frac{\sin(\omega_s(t-kT)/2)}{\omega_s(t-kT)/2} \end{aligned}$$

- To rebuild $x(t)$ you need all past and future $x(kT)$ samples.
- Caustic and easily achievable retreads are used in the controls.

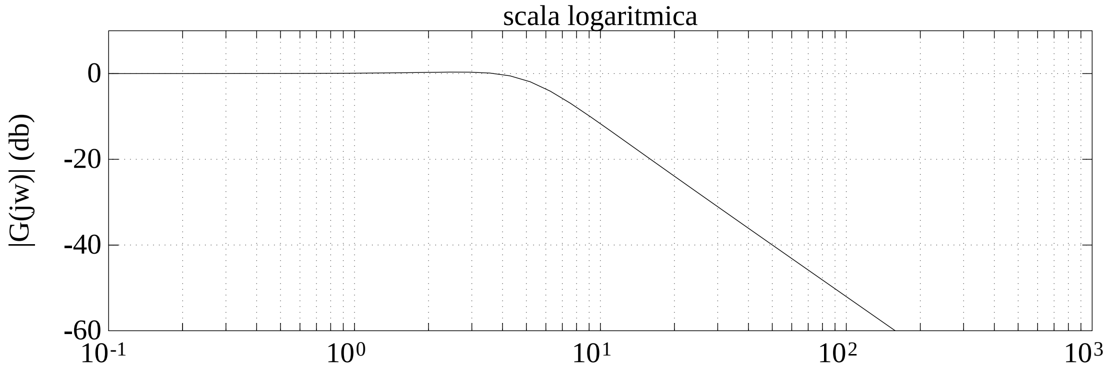
- Sampling of the impulse response of a second-order system:

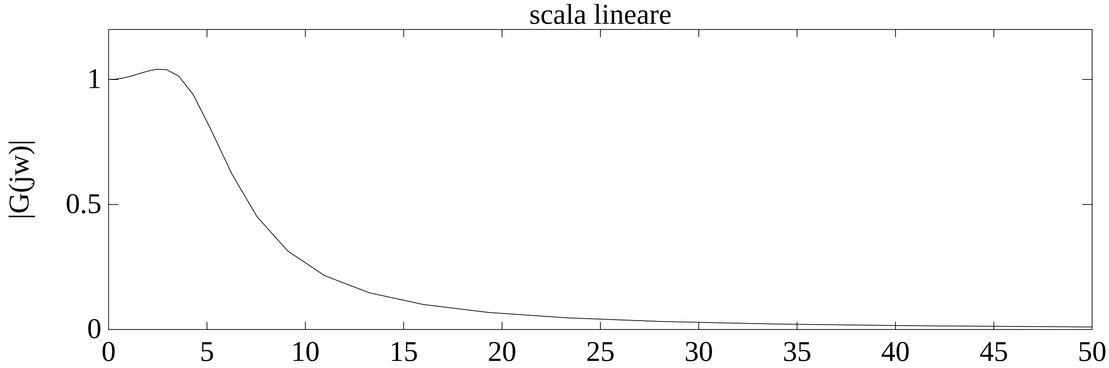
$$G(s) = \frac{25}{s^2 + 6\,s + 25}$$

- The $G(s)$ system has a unitary static gain, has two complex conjugate poles $p_{1,2} = -3 \pm j4$, natural pulsation $\omega_n = 5\ rad/s$ and damping coefficient $\delta = 3/5$:

$$G(s) = \frac{25}{(s+3)^2 + 4^2}$$

- Diagram of amplitudes of $G(j\omega)$:





- For $\omega > 10\omega_n = 50\ rad/s = \bar{\omega}$, the width of $G(j\omega)$ is less than one cent (-40 , db) of the static gain.

- Although the spectrum is theoretically unlimited, it appears to be practically neglected by pulsations greater than $\bar{\omega} = 50\ rad/s$.

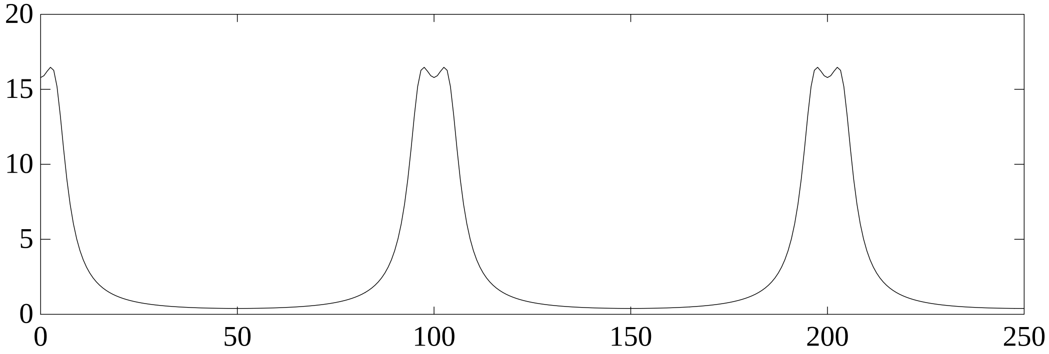
- Applying the $\mathcal{Z}$-transformed one has:

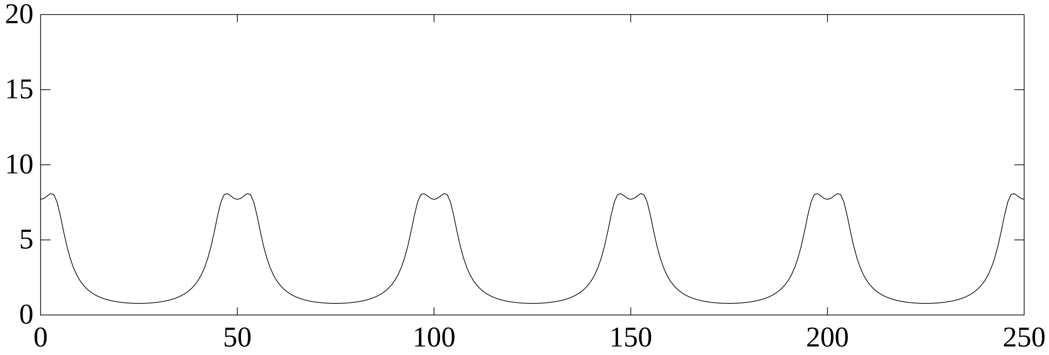
$$G(z) = \frac{25}{4} \frac{e^{-3T}\sin(4T)\, z}{z^2 - 2e^{-3T}\cos(4T)\, z + e^{-6T}}$$

- Spectral response is given by:

$$G^*(j\omega) = G(z)|_{z=e^{j\omega T}} \qquad\qquad 0 \le \omega \le \frac{\pi}{T}$$

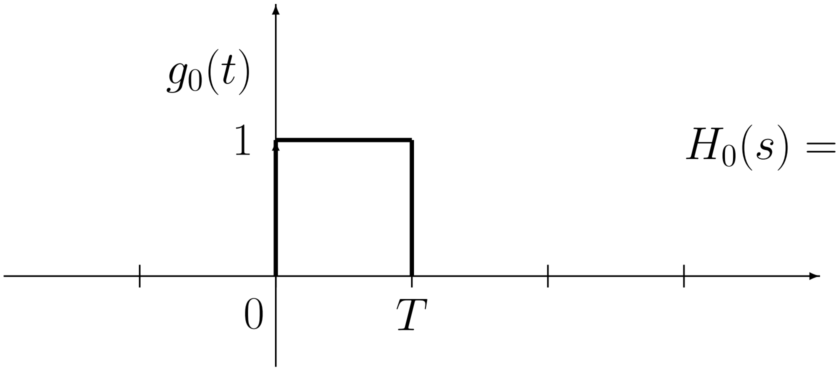
- Spectral trend of $G^*(j\omega)$ when $T = \frac{\pi}{50}$ and $T = \frac{\pi}{25}$





- Zero Order Reconstructor

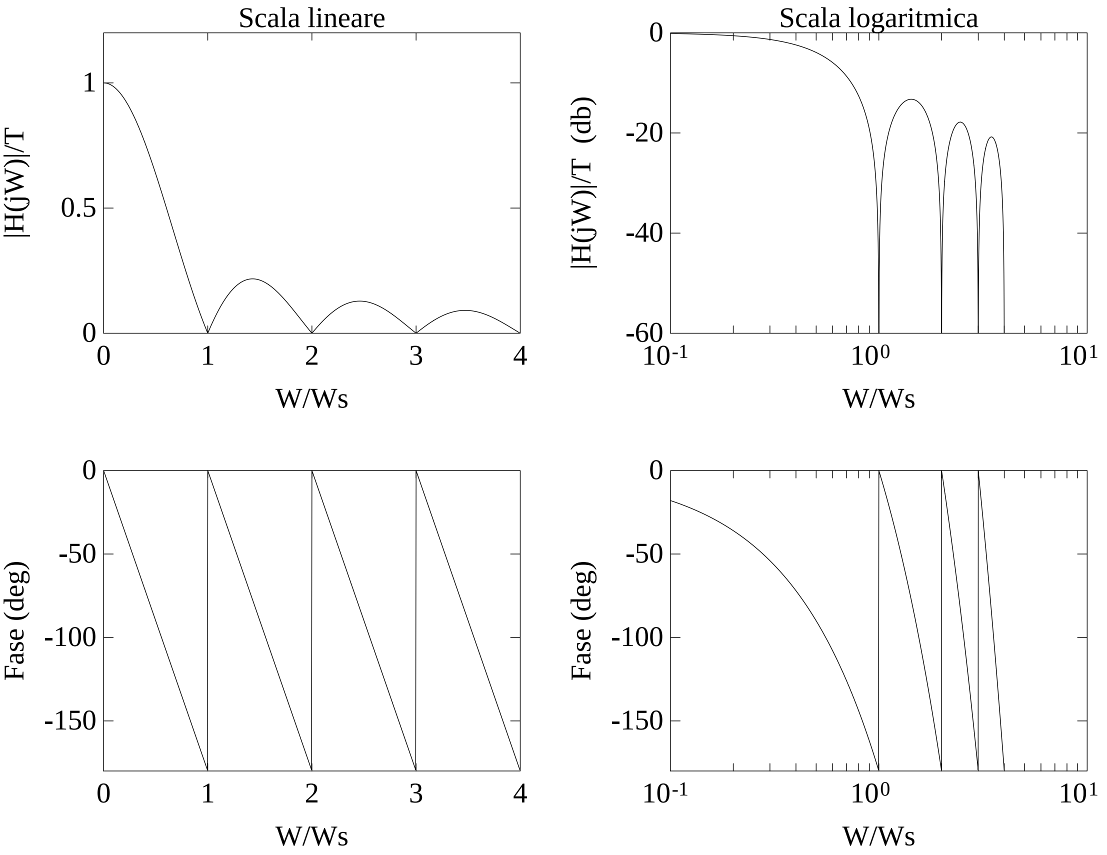
$$x_0(t) = x(k\,T) \qquad\qquad k\,T \leq t < (k+1)T$$



$$H_0(s) = \frac{1 - e^{-s\,T}}{s}$$

- The frequency response of the zero order reconstructor:

$$H_0(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = T\frac{\sin(\omega T/2)}{\omega T/2}\, e^{-j\omega T/2} \simeq T\, e^{-j\omega T/2}$$



- Correspondence between plan s and plan z:

$$X^*(s) = X(z)|_{z=e^{sT}}$$

- The complex variables s and z are bound by the relation:

$$\boxed{z = e^{sT}}$$

- Place $s = \sigma + j\omega$ we have:

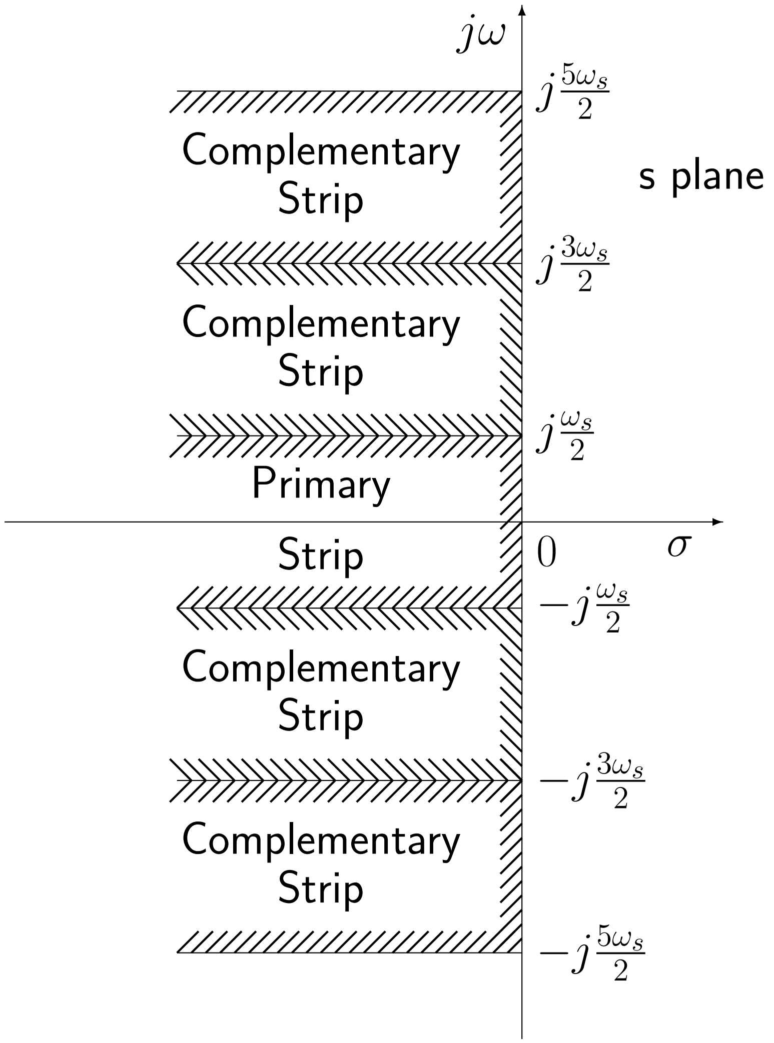
$$z = e^{T(\sigma+j\omega)} = e^{T\sigma}e^{jT\omega} = e^{T\sigma}e^{jT(\omega+\frac{2k\pi}{T})}$$

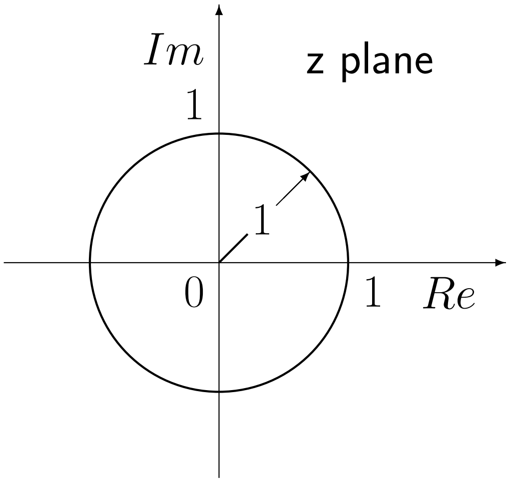
- Each point in the z plane is in correspondence with infinite points in the s plane.
- The points of the negative real s plane $(\sigma < 0)$ correspond to the points of the z plane within the unit circle:

$$|z| = e^{T\sigma} < 1$$

- The points on the imaginary axis $(\sigma = 0)$ are mapped on the unit circle $(|z| = 1)$, while those on positive real side $(\sigma > 0)$ are mapped outside the circle unitary $(|z| > 1)$.
- The plan strip s bounded by the horizontal lines $s = j\omega_s/2$ and $s = -j\omega_s/2$ takes the name of **primary strip**.

- Primary strip and complementary stripes:





- The complex variables s and z are bound by the relation:

$$z = e^{sT}$$

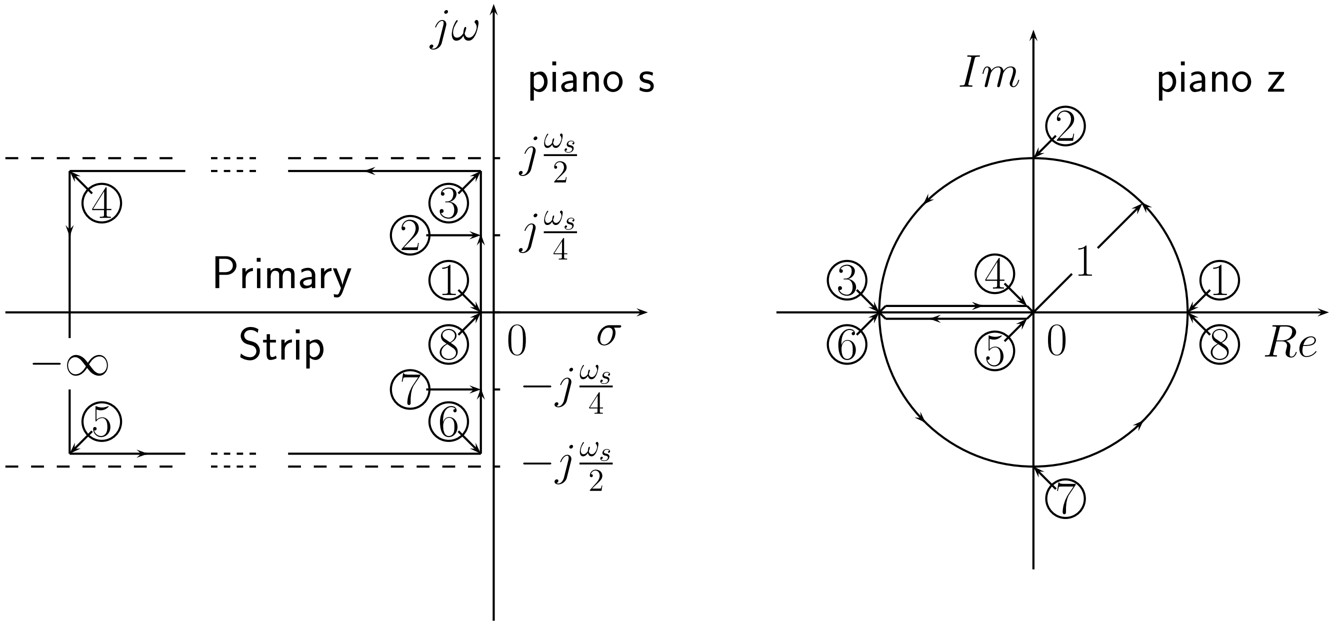
- Place $s = \sigma + j\omega$ we have:

$$z = e^{T(\sigma+j\omega)} = e^{T\sigma}e^{jT\omega}$$

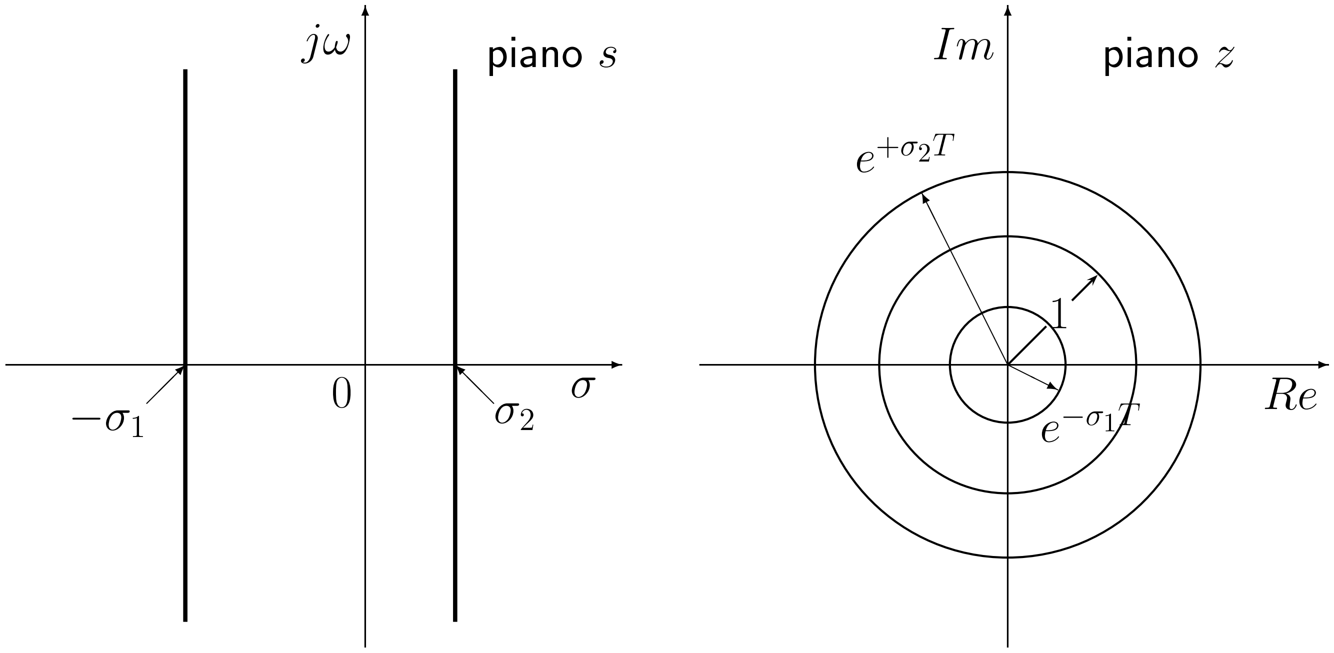
where

$$0 \leq \omega \leq \frac{\omega_s}{2} = \frac{\pi}{T}$$

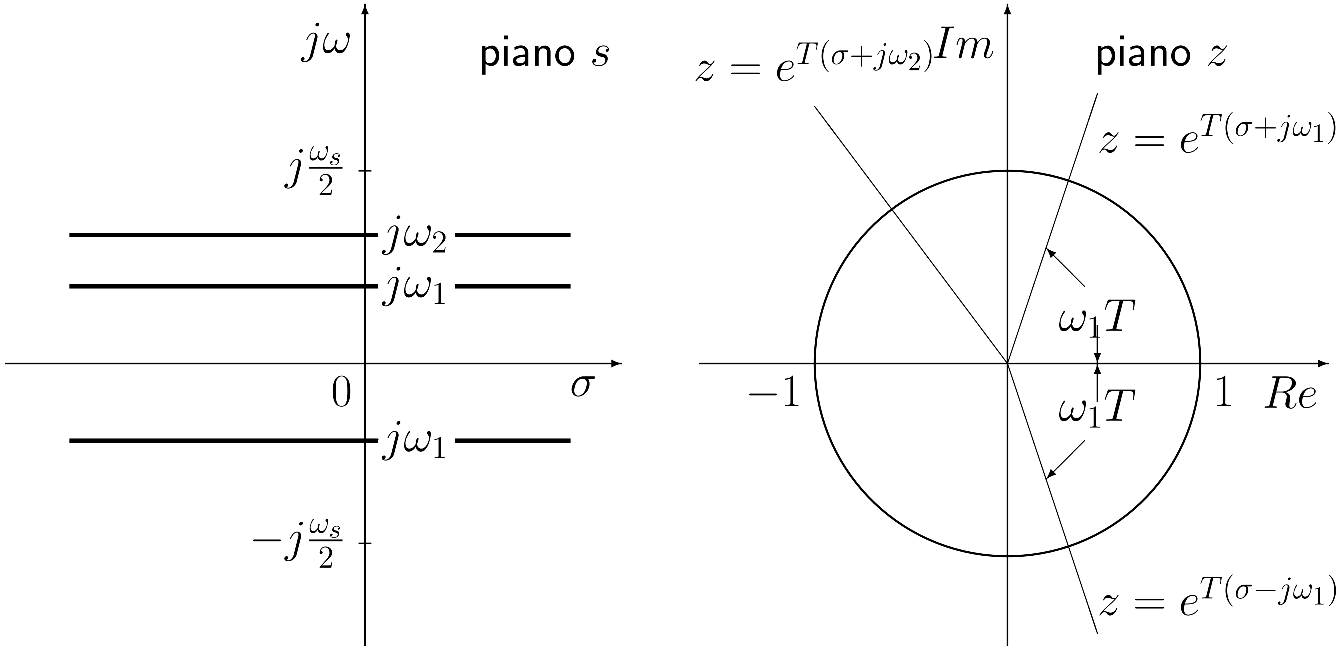
- Mapping between primary strip and z plane:



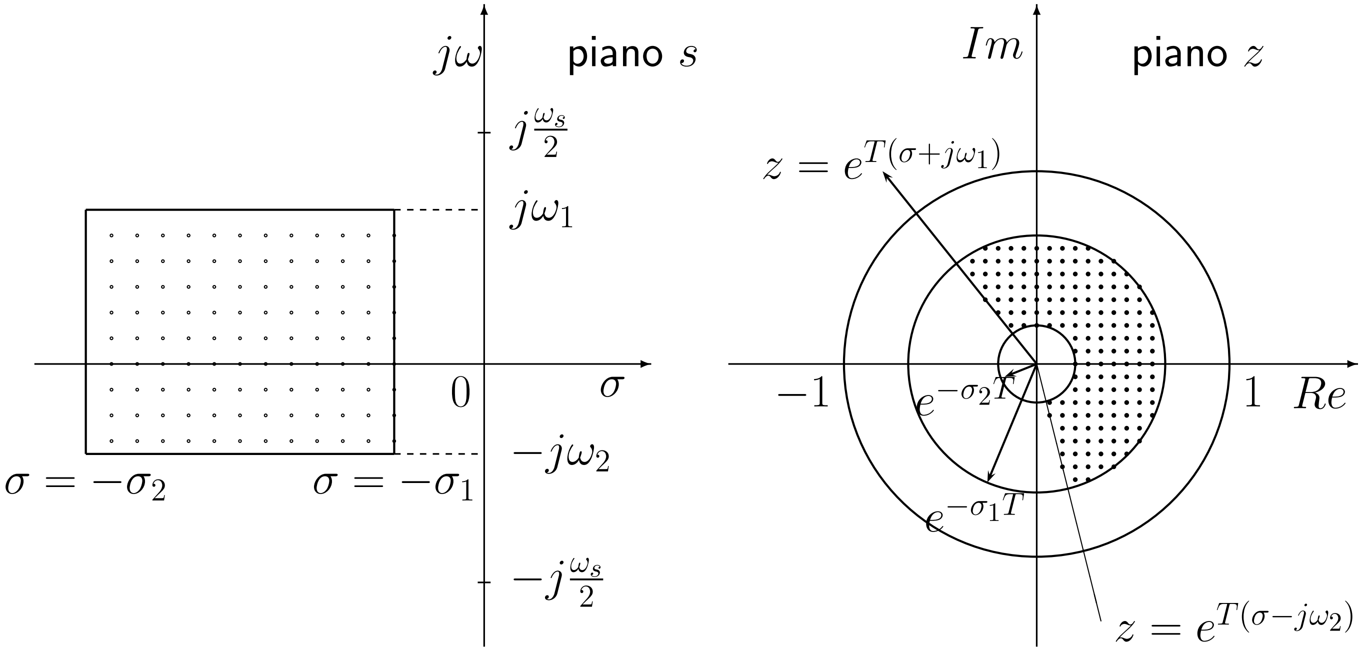
- Places with constant exponential decay:



- Places with constant pulsation:



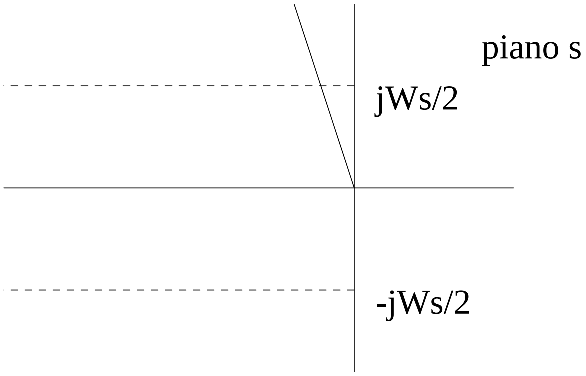
- An example of a correspondence between two regions of the s plan and the z plan:

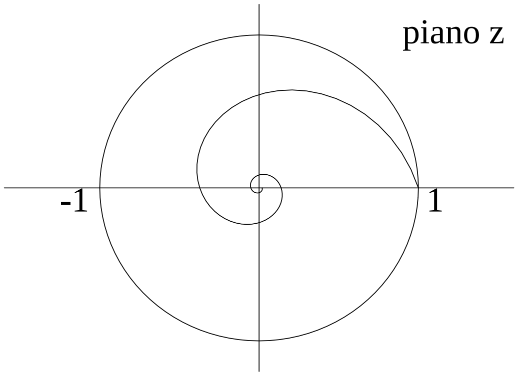


- Place of constant damping coefficient points $\delta = \delta_1$:

$$s = -\omega \tan\beta + j\omega = -\omega \frac{\delta}{\sqrt{1-\delta^2}} + j\omega$$

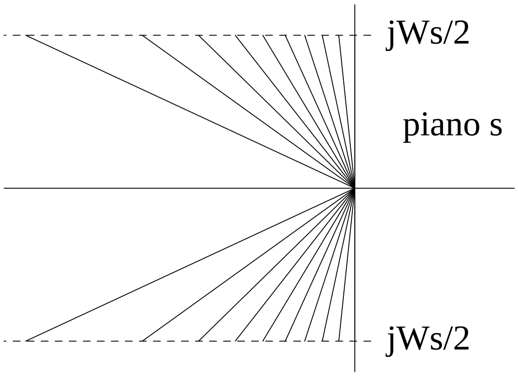
$\beta = \arcsin \delta_1$

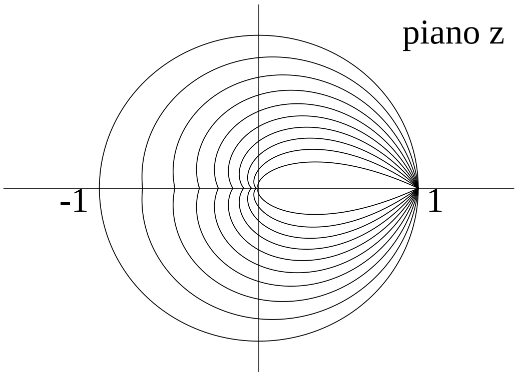




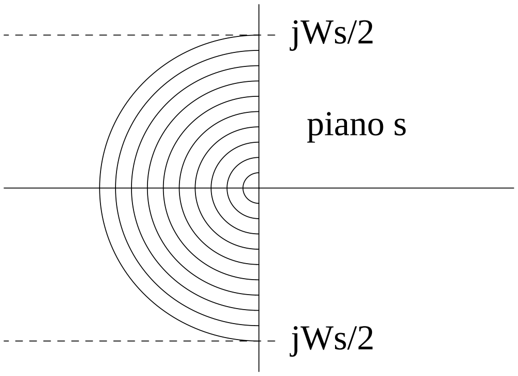
$$z = e^{sT} = e^{(-\omega \tan\beta + j\omega)T} = e^{-\varphi \tan\beta} e^{j\varphi} \qquad \varphi = \omega T$$

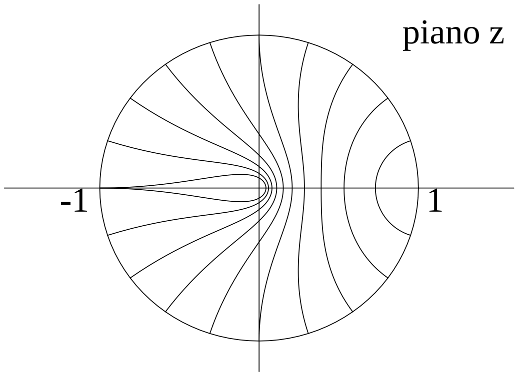
- Places with a constant damping coefficient δ:



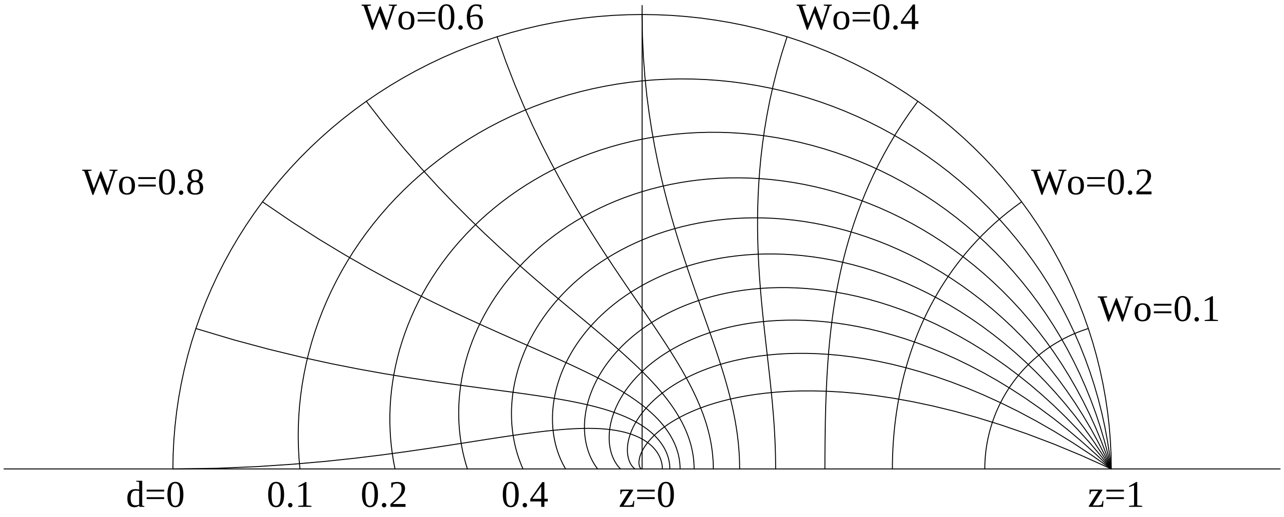


- Places with natural pulsation ω_n constant:

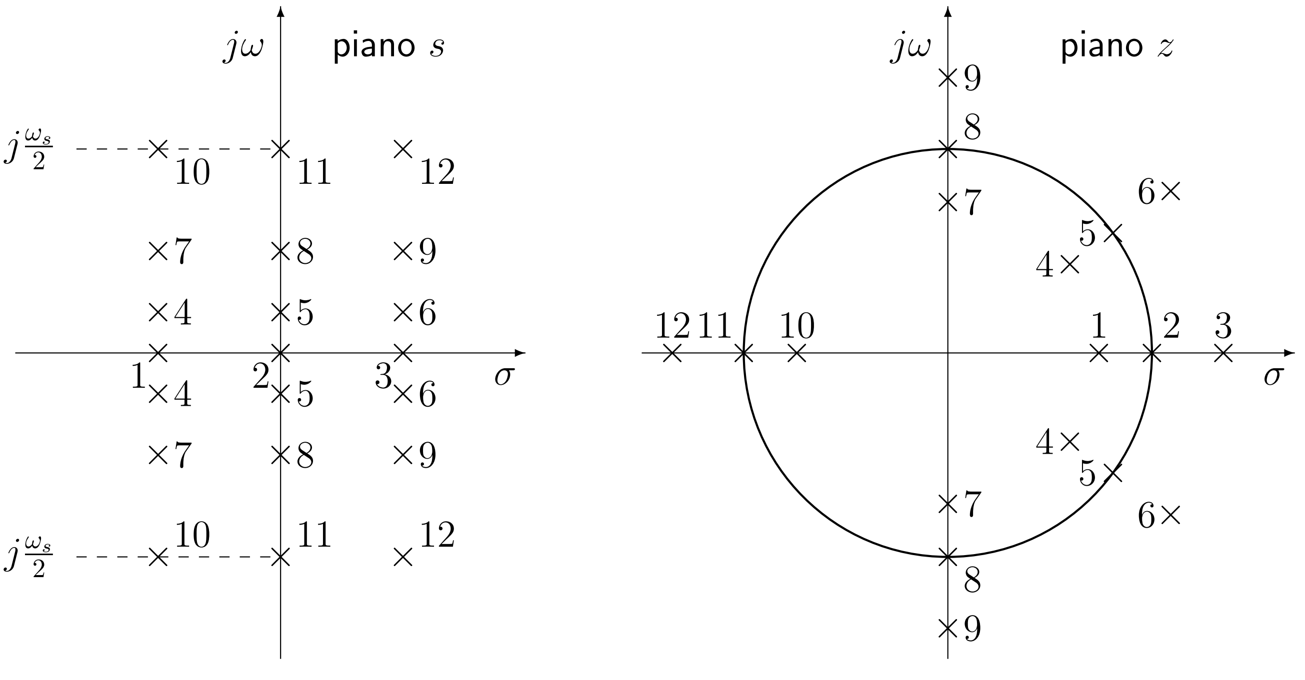




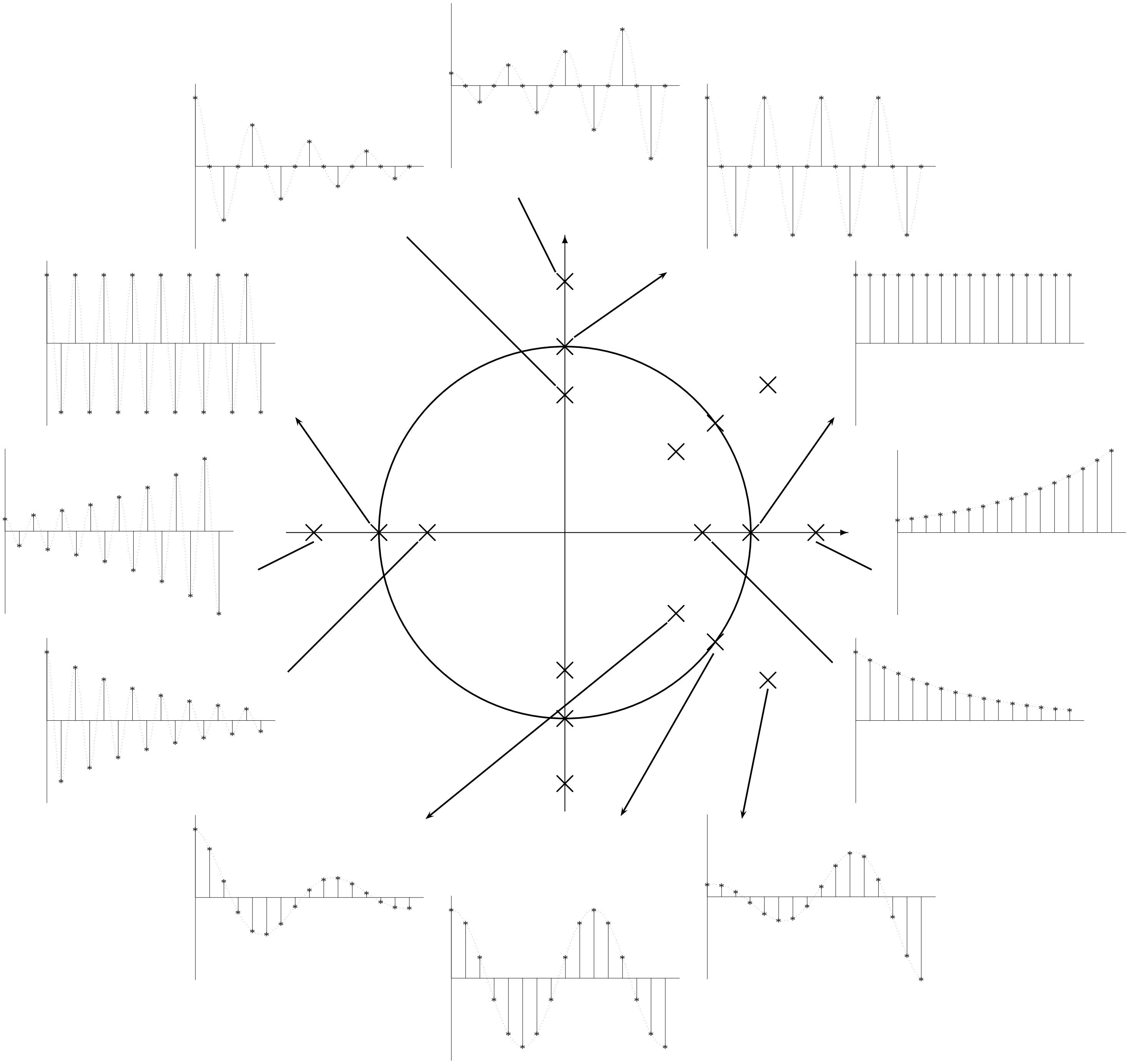
- Places of the plan z a δ and ω_n constants:



- The points of the plan s and of the plan z, placed in correspondence by means of the relation $z = e^{sT}$, can be considered as corresponding poles of transforms $F(s)$ and $F(z)$, with $F(z)$ calculated by sampling $F(s)$



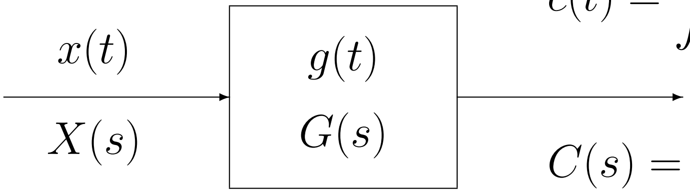
POSITION OF POLES AND TRANSITORY REPLIES



DISCRETE TIME SYSTEMS

- Continuous time systems:

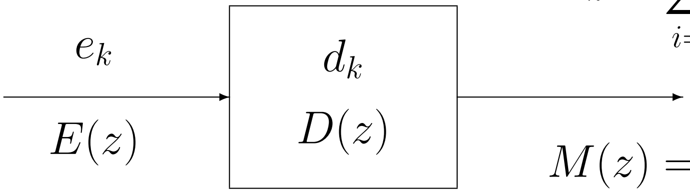
$$c(t) = \int_0^t g(\tau)x(t-\tau)\,d\tau$$



$$C(s) = G(s)\,X(s)$$

- Discrete-time systems:

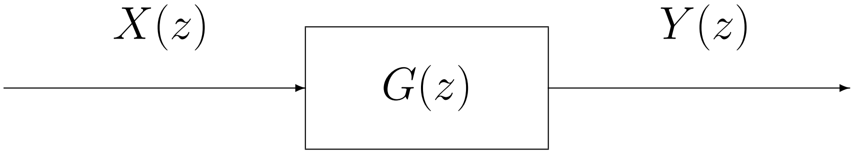
$$m_k = \sum_{i=0}^{k} d_i\,e_{k-i} = \sum_{i=0}^{k} e_i\,d_{k-i}$$



$$M(z) = D(z)\,E(z)$$

- Discrete transfer function:

$$y(kT) = \sum_{h=0}^{\infty} g(kT-hT)x(hT)$$



$$X(z) = \mathcal{Z}[x(kT)] = 1 \qquad \to \qquad Y(z) = G(z)$$

- Discrete harmonic response function:

$$G(e^{j\omega T}), \qquad\qquad 0 \le \omega \le \frac{\pi}{T}$$

$$G(e^{j(\omega+k\omega_s)T}) = G(e^{j\omega T}), \qquad G(e^{j(-\omega)T}) = G^*(e^{j\omega T})$$

- The response of an asymptotically stable $G(z)$ system to a sinusoidal sinusoidal $\sin(\omega kT)$ unit width 'and, when fully operational, a sinusoid $A\sin(\omega kT+\varphi)$ whose amplitude A is given by the mo -du -lo of the vector $G(e^{j\omega T})$, and whose phase φ is given by the phase of the vector $G(e^{j\omega T})$:

$$A = |G(e^{j\omega T})|, \qquad\qquad \varphi = Arg[G(e^{j\omega T})]$$

STABILITY OF DISCRETE SYSTEMS

- Stability of discrete systems:

$$\frac{Y(z)}{U(z)} = G(z) = \frac{B(z)}{A(z)}$$

- The dynamic behavior of the system:

$$G(z) = \frac{B(z)}{A(z)}$$

  it depends on the poles of $G(z)$, that is from the roots of the polynomial $A(z)$.

- Asymptotic Stability: all the p_i poles of the $G(z)$ must be inside the unit circle: $|p_i| < 1$.

- Stability simple: all poles p_i of the $G(z)$ must belong to the unit disk ($|p_i| \leq 1$) and those on the unit circle ($|p_i| = 1$) must have multiplicity ‘a unitary.

- Example. Be given the system:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{4z^{-1}}{1 + az^{-1}} = \frac{4}{z + a}$$

  It corresponds to the following equation to differences:

$$Y(z)(1 + az^{-1}) = 4z^{-1}U(z)$$

$$y(k) = -ay(k-1) + 4u(k-1)$$

  The response of this system to the unit impulse

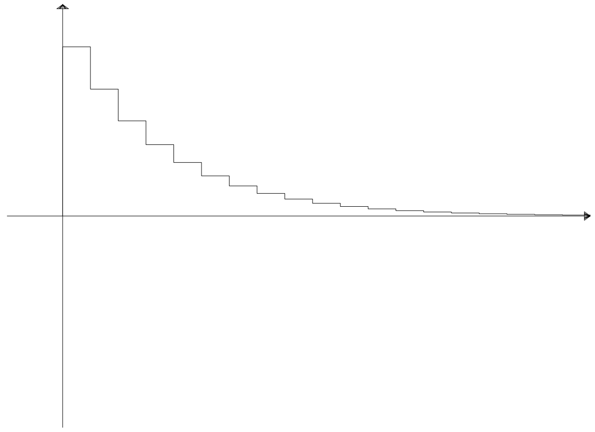
$$u(0) = 1, \qquad u(k) = 0, \;\; k > 0;$$

  is the following:

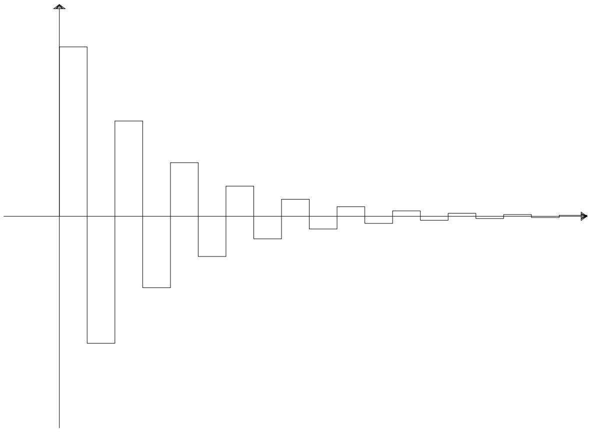
$$y(k) == 4(-a)^{k-1}$$

- Time trends that are obtained at the values $a = 0.75$, $a = -0.75$, $a = 1.25$, $a = -1.25$, $a = 1$, and $a = -1$:

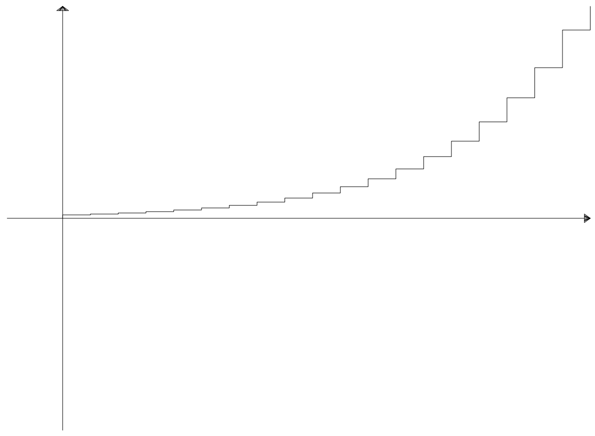
polo in z=0.75



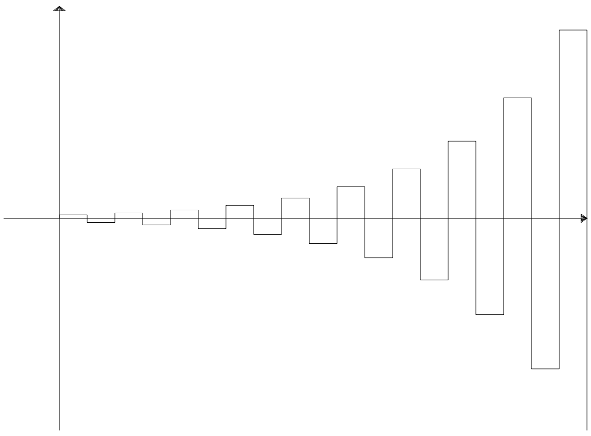
polo in z=-0.75



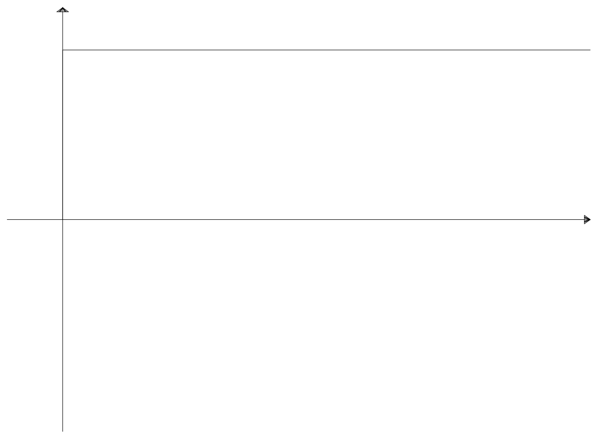
polo in z=1.25



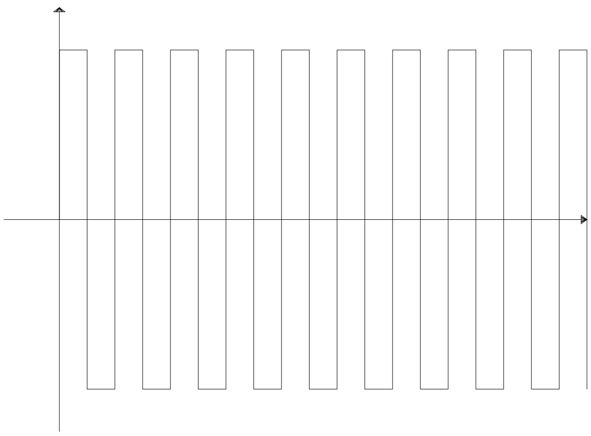
polo in z=-1.25



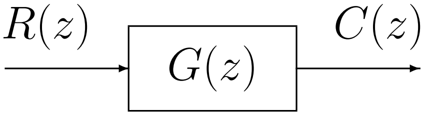
polo in z=1



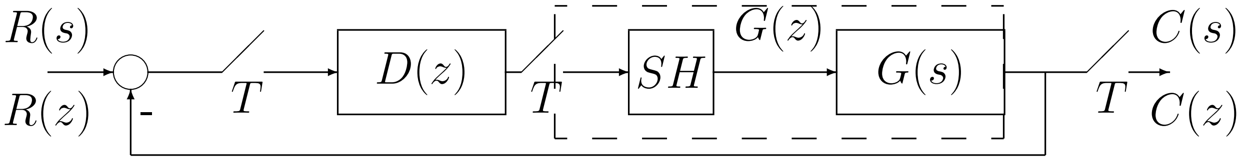
polo in z=-1



- Stability of discrete systems:



(a)



(b)

$$G_0(z) = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

- Be given a system described by

$$G(z) = \frac{B(z)}{A(z)} \quad \text{oppure} \quad G_0(z) = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

- The system is asymptotically stable if and only if all the roots of the polynomial $A(z)$ (or of the polynomial $1 + D(z)G(z)$), that is the poles of the system, are within the circle of unit radius with center in the origin of the plane z ie $|p_i| < 1, \forall i$.

- The system is stable if all poles with unit module $|p_i| = 1$ are simple poles (their multiplicity 'a is 1), while all the remaining poles are within the unit circle.

- A polynomial equation must be resolved:

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

whose solution is easy only for small values of n.

- Three methods for studying stability:
  1. use a bilinear transformation and apply the Routh-Hurwitz criterion;
  2. use the Jury criterion that directly processes the coefficients of $A(z)$, that is of the de -no -mi -na -to -re of $G(z)$;
  3. Nyquist criterion.
- Bilinear transformation and Routh-Hurwitz criterion:

$$z = \frac{1+w}{1-w} \qquad \leftrightarrow \qquad w = \frac{z-1}{z+1}$$

- The unit circle in z corresponds to the left half plane of the w plane.
- $|z| = \left|\frac{1+w}{1-w}\right| = \left|\frac{1+\sigma+j\omega}{1-\sigma-j\omega}\right| < 1$

$$\frac{(1+\sigma)^2+\omega^2}{(1-\sigma)^2+\omega^2} < 1$$

$$(1+\sigma)^2+\omega^2 < (1-\sigma)^2+\omega^2 \quad \Rightarrow \quad \sigma < 0$$

- $|z| = 1 \quad \Rightarrow \quad (1+\sigma)^2+\omega^2 = (1-\sigma)^2+\omega^2 \quad \Rightarrow \quad \sigma = 0$
- For the analysis of the stability of $G(z)$ $\big(G_0(z)\big)$ we proceed as follows:
  1. the characteristic equation is considered

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

  2. the transformation is carried out

$$\left(\frac{1+w}{1-w}\right)^n + a_1\left(\frac{1+w}{1-w}\right)^{n-1} + \ldots a_{n-1}\frac{1+w}{1-w} + a_n = 0$$

  from which it is obtained

$$Q(w) = q_0 w^n + q_1 w^{n-1} + \cdots + q_{n-1} w + q_n = 0$$

  3. by applying the Routh-Hurwitz criterion, the root signs of $Q(w)$ are then studied.

- Example:

$$G(z) = \frac{z+1}{z^3 + 2z^2 + z + 1}$$

$$\left(\frac{1+w}{1-w}\right)^3 + 2\left(\frac{1+w}{1-w}\right)^2 + \frac{1+w}{1-w} + 1 = 0$$

$$-w^3 + 3w^2 + w + 5 = 0$$

$$\begin{array}{c|cc} 3 & -1 & 1 \\ 2 & 3 & 5 \\ 1 & 8/3 & \\ 0 & 5 & \end{array}$$

- The system has an unstable pole

- The criterion of Nyquist allows to decide about the stability of feedback systems by analyzing the frequency behavior of the harmonic ring response in relation to the critical point $(-1 + j0)$

$$G(e^{j\omega T}), \qquad -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T}$$

- If the $G(z)$ is of type 0, then the relative diagram is a closed curve; if it is of type 1 or 2, then you have an open curve, which is closed with a circumference or semi-circumference at infinity traveled clockwise

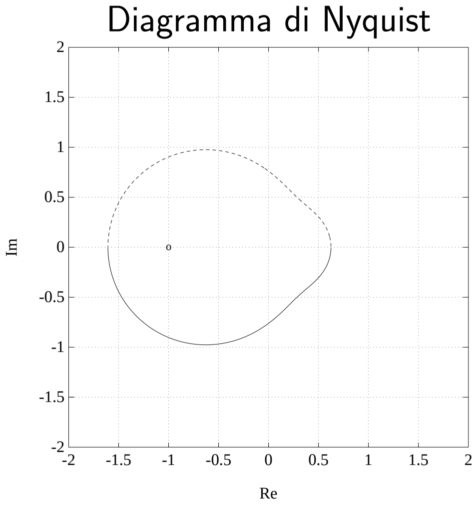


Diagramma di Nyquist

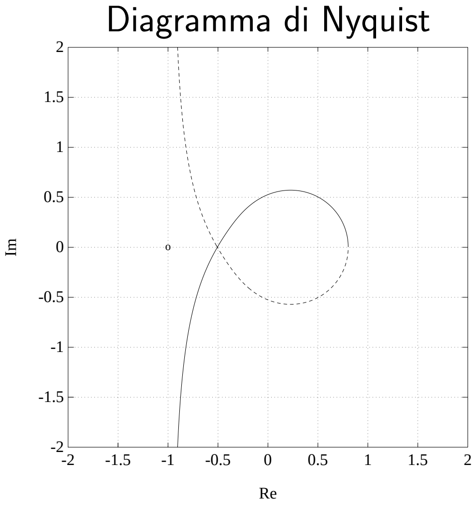


Diagramma di Nyquist

- Nyquist criterion I

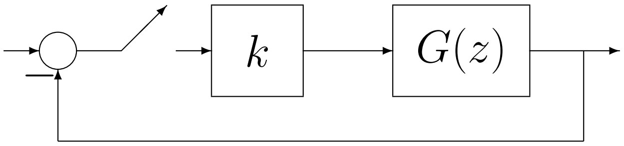
  Let be given a ring gain function $G(z)$ with all stable poles (a module smaller than one), with the possible exception of a simple or double pole in $z = 1$. A necessary and sufficient condition for the feedback system to be -sin -to -statefully stable is that the complete polar diagram of the function $G(e^{j\omega T})$ traced for $-\pi/T \leq \omega \leq \pi/T$ does not surround n is touches the critical point $-1 + j0$.

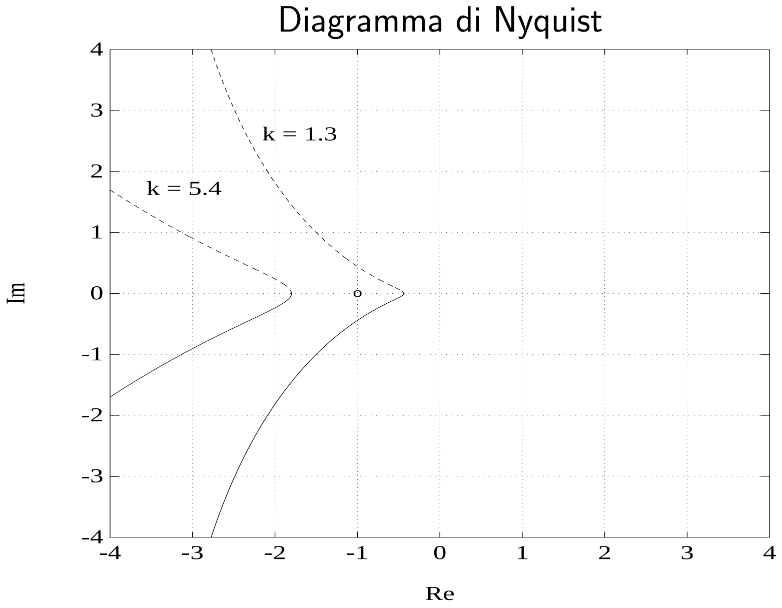
- Nyquist criterion II

  Both given a function of loop gain $G(z)$ without poles in unitary form, with the possible exception of a simple or double pole in $z = 1$. A necessary and sufficient condition for the feedback system to be asymptotically stable is that the complete polar diagram of the function $G(e^{j\omega T})$ traced for $-\pi/T \leq \omega \leq \pi/T$ surround the critical point $-1 + j0$ as many times in a counterclockwise direction as the poles of $G(z)$ with module greater than one. Each revolution in a counterclockwise direction, or each turn in a more clockwise direction, corresponds to the presence of a module pole greater than one in the feedback system.

- Example:

$$G(z) = \frac{z}{(z-1)(z-0.5)}$$

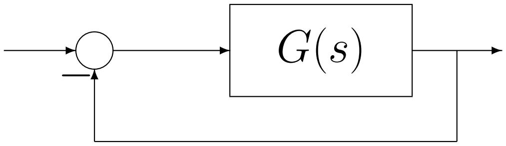


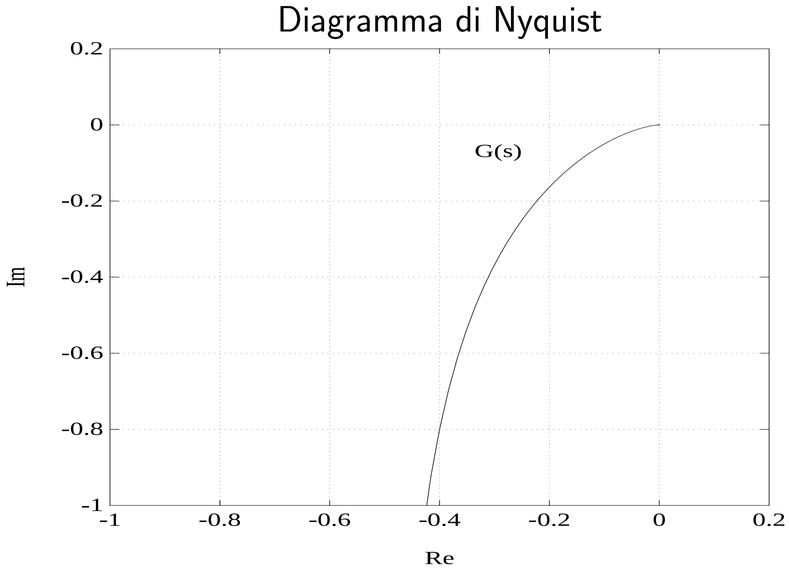


The feedback system is stable for $k = 1.3$ and unstable for $k = 5.4$

- Esempio:

$$G(s) = \frac{2}{s(s+2)}$$





The system is stable

- Place of the roots. ‘It is the place described by the zeros of a function

  $$F(s) = 1 + k\ G(s) = 1 + k\ \frac{B(s)}{A(s)}$$

  when the parameter k varies in the interval $[0, +\infty]$.

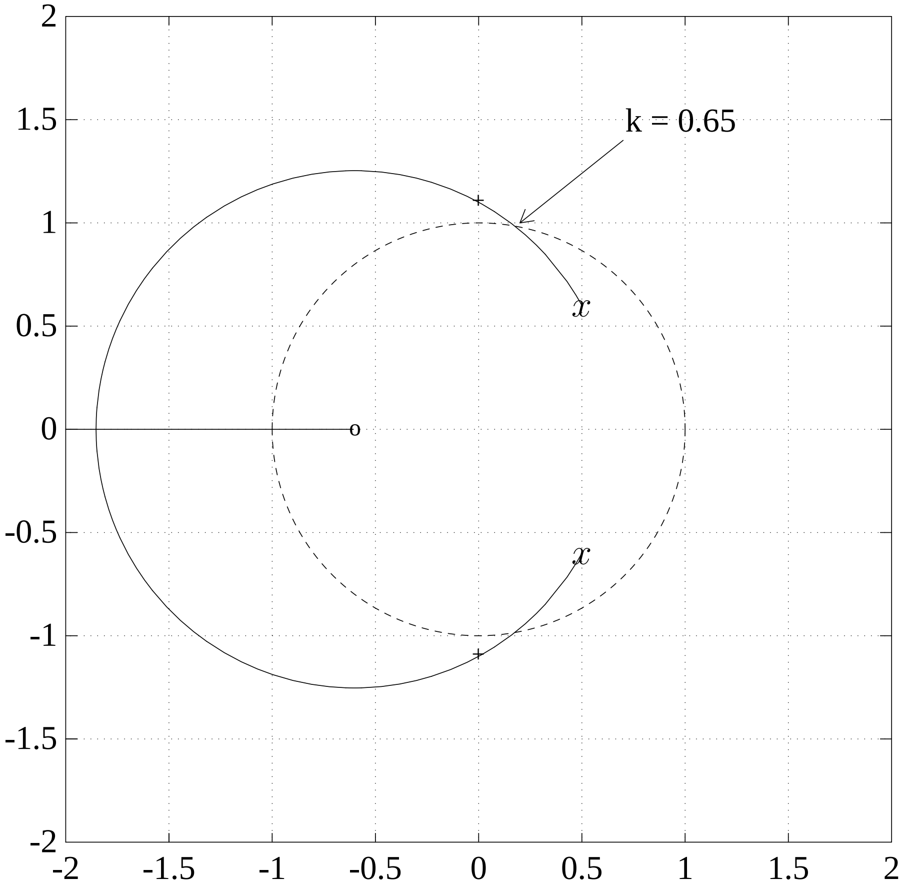
- For the tracing of the place the same rules as the continuous case apply.

- Change the interpretation of the results obtained.

- Example. Given the following open chain system with two poles in $z_{1,2} = 0.5 \pm j0.6$:

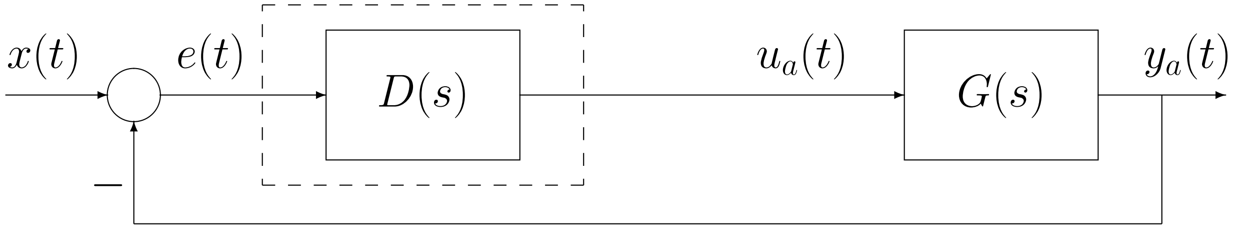
  $$G(z) = k\frac{z + 0.6}{z^2 - z + 0.61}$$

  For the unit feedback system we have:

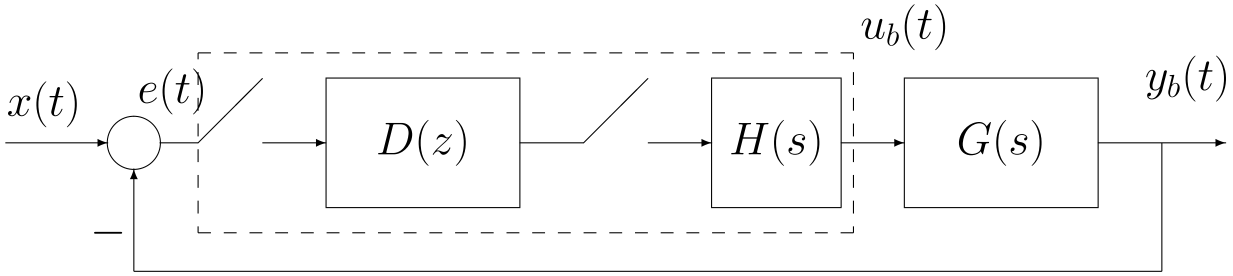


PROJECT FOR DISCRETIZATION

- The $D(s)$ controller designed in the "continuous time" (case a) is "discretized" by obtaining a $D(z)$ function that will be inserted into the control loop discrete (case b):



(a)



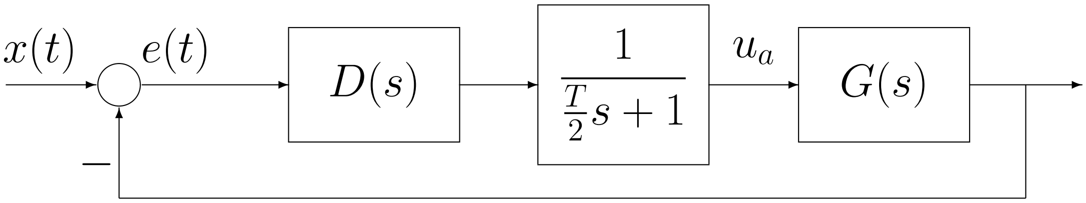
(b)

- All the discretization methods that will be presented are "approximate", that is, they provide a discrete $D(z)$ system that reproduces well, but not exactly, the dynamic behavior of the $D(s)$ system.
- The smaller the T sampling period, the more the $D(z)$ system has a dynamic behavior similar to that of the $D(s)$ system.

- The project for discretization proceeds following three conceptual steps:

  1) Choice of the T sampling period and verification of system stability margins:

$$H_0(s) \approx \frac{1}{\frac{T}{2}s+1} \approx e^{-sT/2}$$



  2) Discretization of the $D(s)$;
  3) Ex post verification (simulative and/or experimental) of the dynamic behavior of the feedback system.

- DISCRETIZATION TECHNIQUES:

  1. Backward difference method
  2. Forward difference method
  3. Bilinear transformation
  4. Bilinear transformation with precompensation
  5. Method of $\mathcal{Z}$-transformed
  6. $CalZ$ method -transformed with order rebuilder 0
  7. Poles/zeros correspondence method

1. METHOD OF BACKWARD DIFFERENCES

- The method consists in the following replacement:

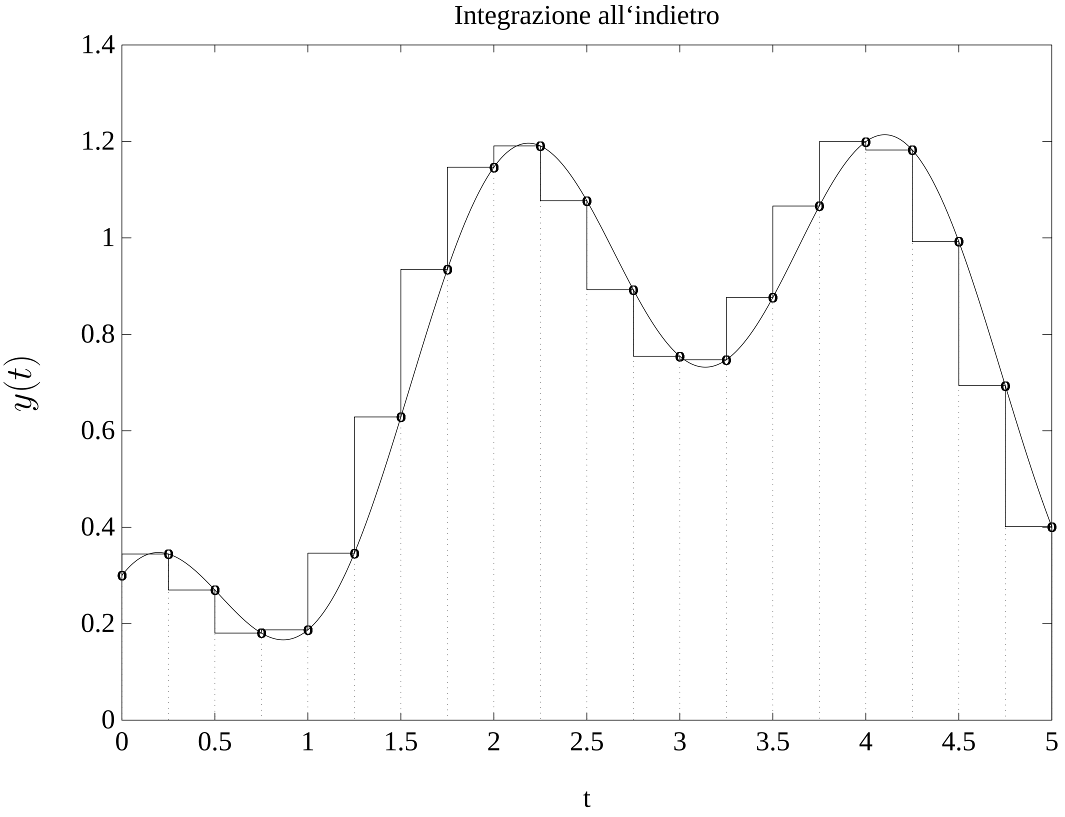
$$\boxed{D(z) = D(s)\Big|_{s = \frac{1 - z^{-1}}{T}}}$$

- The difference equation describing the operation of **backward rectangular integration** is the following:

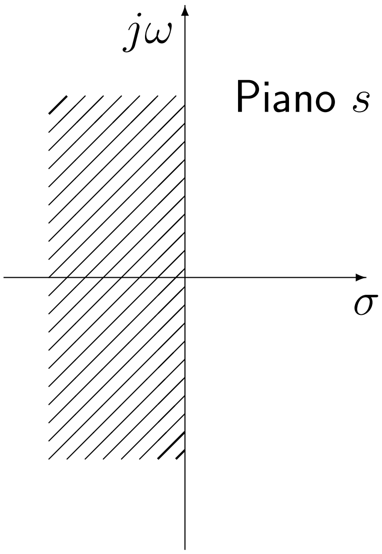
$$y(n) = y(n-1) + T\,x(n) \qquad \leftrightarrow \qquad Y(z) = \underbrace{\frac{T}{1 - z^{-1}}}_{G_1(z)} X(z)$$

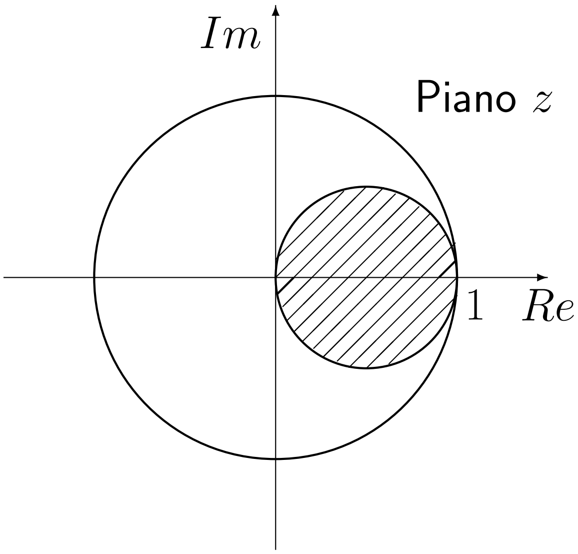
  where with $x(n)$ is indicated the input sequence and with $y(n)$ the corresponding integral sequence of output.

- The binding $s = \frac{1-z^{-1}}{T}$ arises from matching the continuous time integrator $\frac{1}{s}$ with the corresponding discrete integrator $G_1(z)$ obtained by backward rectangular integration.



- Link between the s plan and the z plan:





- Continuous time regulators $D(s)$ are transformed into discrete time regulators $D(z)$.

Example. Using the backward differences method, discretizing the following anticipatory network:

$$D(s) = \frac{M(s)}{E(s)} = \frac{1+s}{1+0.2s}$$

also reaching the determination of the corresponding difference equation. Use the sampling period $T = 0.1$.

[Solution.] Using the backward difference method you get:

$$D(s) = \frac{1+s}{1+0.2s} = 5\frac{s+1}{s+5} \quad \rightarrow \quad D(z) = D(s)\big|_{s=\frac{1-z^{-1}}{T}} = \frac{5(1+T-z^{-1})}{1+5T-z^{-1}}$$

The corresponding difference equation is derived from the relation:

$$D(z) = \frac{M(z)}{E(z)} = \frac{5(1+T-z^{-1})}{1+5T-z^{-1}}$$

obtaining

$$M(z)(1+5T-z^{-1}) = 5E(z)(1+T-z^{-1}) \quad \leftrightarrow \quad M(z)(1.5-z^{-1}) = E(z)(5.5-5z^{-1})$$

that is

$$m(k) = \frac{1}{1.5}[m(k-1) + 5.5e(k) - 5e(k-1)]$$

from which

$$m(k) = 0.666\, m(k-1) + 3.666\, e(k) - 3.333\, e(k-1)]$$

Example. Using the backward difference method, discretize the following function:

$$D(s) = \frac{M(s)}{E(s)} = 2\,\frac{s+2}{s+5}$$

also reaching the determination of the corresponding difference equation. Use the sampling period $T = 0.1$.

[Solution.] Using the backward difference method is obtained

$$D(z) = D(s)|_{s=\frac{1-z^{-1}}{T}} = 2\,\frac{1+2T-z^{-1}}{1+5T-z^{-1}} = 2\,\frac{1.2-z^{-1}}{1.5-z^{-1}}$$

The calculation of the corresponding difference equation is immediate:

$$m(n) = \frac{1}{1.5}\,[m(n-1) + 2.4e(n) - 2e(n-1)]$$

2. METHOD OF DIFFERENCES FORWARD

$$\boxed{D(z) = D(s)|_{s = \frac{z-1}{T}}}$$

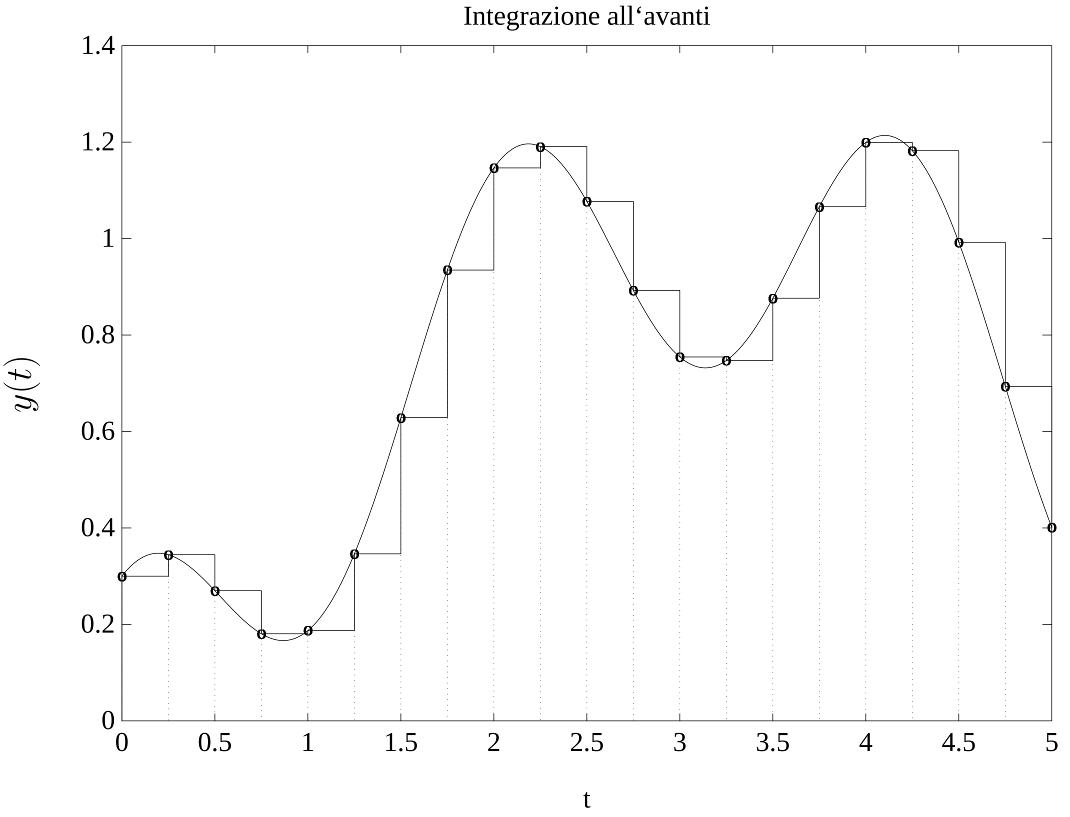
- The transfer function $G_2(z)$ which describes the operation of **rectangular integration to the ancestor** is obtained as follows:

$$y(n) = y(n-1) + T\,x(n-1) \qquad \leftrightarrow \qquad Y(z) = \underbrace{\frac{Tz^{-1}}{1-z^{-1}}}_{G_2(z)} X(z)$$

  where $x(n)$ and $y(n)$ represent the input and output sequences. It is therefore obtained that:

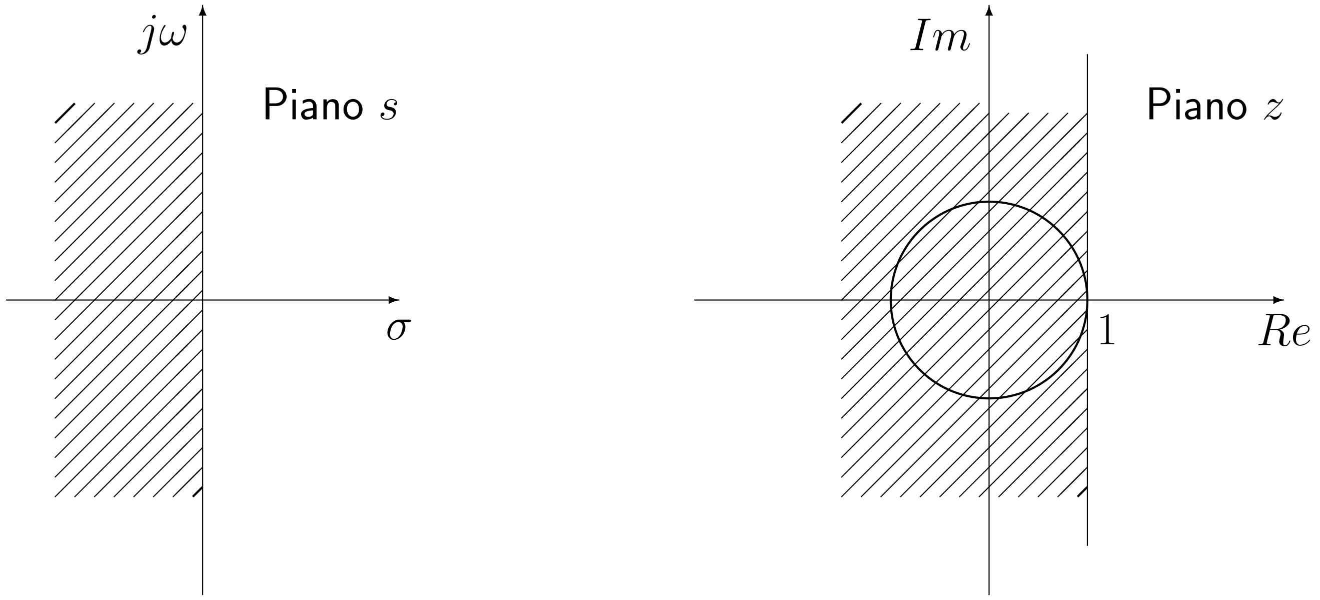
$$\frac{1}{s} \;\leftrightarrow\; G_2(z) = \frac{Tz^{-1}}{1-z^{-1}} = \frac{T}{z-1} \qquad \Rightarrow \qquad \boxed{s \;\leftrightarrow\; \frac{z-1}{T}}$$

- Approximation of the integral with the differences to the forward:



- Analyzing the plan-s plan-z correspondence we have that:

$$Re(s) = Re\left(\frac{z-1}{T}\right) < 0 \qquad \to \qquad Re(z) < 1$$



- A stable regulator $D(s)$ can turn into an unstable regulator $D(z)$.

3. BILINARY TRANSFORMATION (or TUSTIN)

$$\boxed{D(z) = D(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}}$$

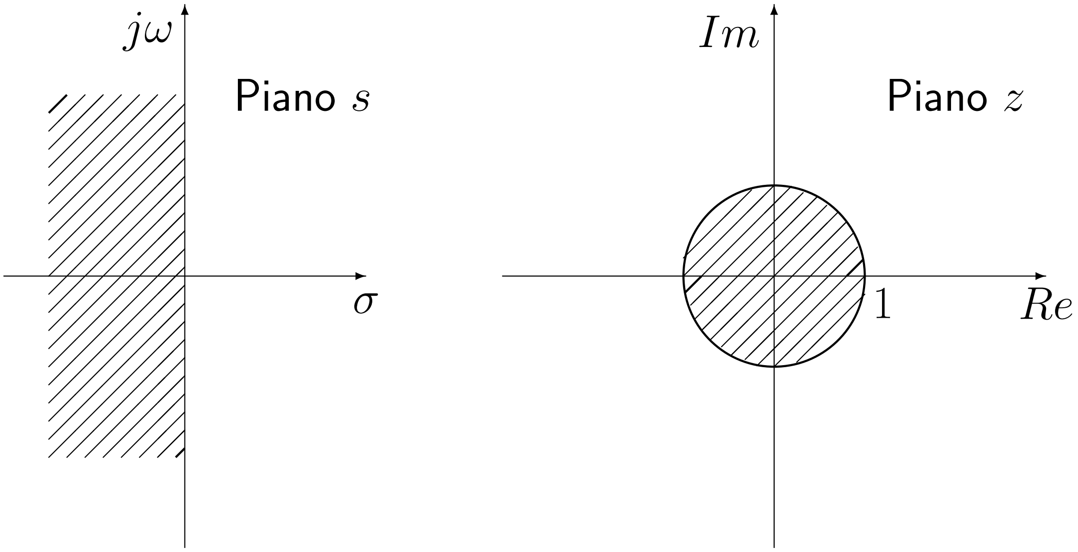
- The transfer function $G_3(z)$ describing the operation of **trapezoidal integration** is obtained as follows:

$$y(n) = y(n-1) + \frac{T}{2}[\,x(n) + x(n-1)] \qquad \leftrightarrow \qquad Y(z) = \underbrace{\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}}_{G_3(z)} X(z)$$

Being $G_3(z)$ a discrete version of the $1/s$ operator, we have that:

$$\frac{1}{s} \;\leftrightarrow\; G_3(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \qquad \Rightarrow \qquad \boxed{s \;\leftrightarrow\; \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

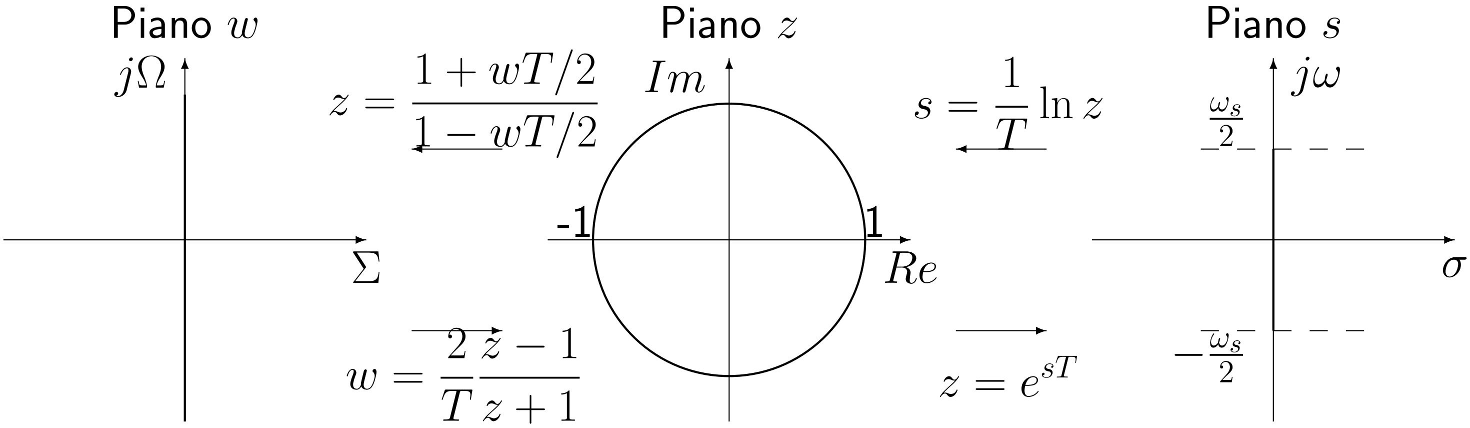
The analysis of the legane s-z shows that the negative half-plane in s is placed in a biunivocal correspondence with the points z of the unit circle:



$$Re\left(\frac{z-1}{z+1}\right) = Re\left(\frac{\sigma + j\omega - 1}{\sigma + j\omega + 1}\right) = Re\left[\frac{\sigma^2 - 1 + \omega^2 + j2\omega}{(\sigma+1)^2 + \omega^2}\right] < 0$$

$$\sigma^2 + \omega^2 < 1$$

- Frequency relationship between the w plan, the z plan and the s plan:



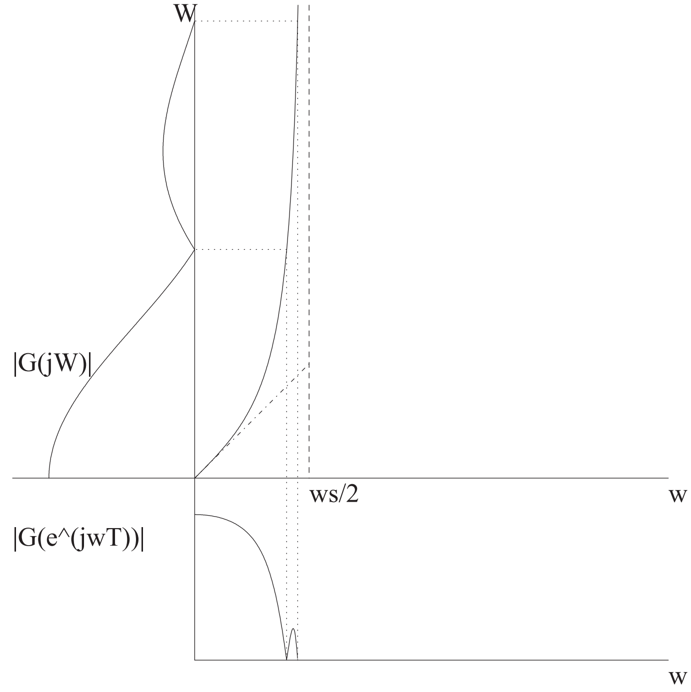
- Transformation does not generate frequency overlap, but introduces distortions:

$$
\begin{aligned}
j\Omega &= \frac{2}{T}\frac{1-e^{-j\omega T}}{1+e^{-j\omega T}} \\
&= \frac{2}{T}\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{e^{j\omega T/2}+e^{-j\omega T/2}} = \frac{2}{T}\frac{2j\sin\omega T/2}{2\cos\omega T/2} \\
&= j\frac{2}{T}\tan\frac{\omega T}{2}
\end{aligned}
$$

$$D_c(j\Omega) = D_d(e^{j\omega T})$$

per

$$\Omega = \frac{2}{T}\tan\frac{\omega T}{2}$$



Example. Using the bilinear transformation method, discretize the following PI regulator:

$$D(s) = \frac{M(s)}{E(s)} = \frac{s+1}{s}$$

also reaching the determination of the corresponding difference equation. Use the sampling period $T = 0.2$.

[Solution.] Using the bilinear transformation method is obtained

$$D(z) = D(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{10(1-z^{-1})+(1+z^{-1})}{10(1-z^{-1})} = \frac{11-9\,z^{-1}}{10(1-z^{-1})}$$

The corresponding difference equation is derived from the relation

$$M(z)(1-z^{-1}) = \frac{E(z)}{10}(11-9\,z^{-1})$$

from which it is obtained

$$m(k) = m(k-1) + 1.1\,e(k) - 0.9\,e(k-1)$$

Example. Using the bilinear transformation method, discretize the following transfer function:

$$D(s) = \frac{M(s)}{E(s)} = 2\frac{(1+0.25s)}{(1+0.1s)}$$

Use the sampling period $T = 0.05$.

[Solution] The transfer function to be discretized is the following:

$$D(s) = 2\frac{(1+0.25\,s)}{(1+0.1\,s)} = 5\frac{(s+4)}{(s+10)}$$

Using the bilinear transformation method we obtain $(T = 0.05)$

$$D(z) = 5\frac{(s+4)}{(s+10)}\Bigg|_{s\,=\,\dfrac{2}{T}\dfrac{z-1}{z+1}} = 4.4\frac{z-\frac{9}{11}}{z-\frac{3}{5}} = 4.4\,\frac{1-0.8182\,z^{-1}}{1-0.6\,z^{-1}}$$

The corresponding difference equation is derived from the relation

$$(1-0.6\,z^{-1})M(z) = 4.4(1-0.8182\,z^{-1})E(z)$$

from which it is obtained

$$m(k) = 0.6\,m(k-1) + 4.4\,e(k) - 3.6\,e(k-1)$$

Unified treatment

- The first 3 methods of discretization can be described in a unified way by referring to the following discrete equation:

$$y(n) = y(n-1) + \frac{T}{2}[(1+\alpha)\,x(n) + (1-\alpha)\,x(n-1)]$$

where $-1 < \alpha < 1$. The corresponding transfer function $G(z)$ is:

$$Y(z) = \underbrace{\frac{T}{2}\frac{(1+\alpha) + (1-\alpha)z^{-1}}{1-z^{-1}}}_{G(z)}\,X(z)$$

to which the following substitution is associated:

$$\boxed{s \quad \leftrightarrow \quad \frac{2}{T}\,\frac{1-z^{-1}}{(1+\alpha) + (1-\alpha)\,z^{-1}}}$$

- For $\alpha = 0$ we get the bilinear transformation method:

$$y(n) = y(n-1) + \frac{T}{2}[\,x(n) + x(n-1)] \qquad \rightarrow \qquad \boxed{s \quad \leftrightarrow \quad \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}}$$

- For $\alpha = 1$ we get the backwards differences method:

$$y(n) = y(n-1) + T\,x(n) \qquad \rightarrow \qquad \boxed{s \quad \leftrightarrow \quad \frac{1-z^{-1}}{T}}$$

- For $\alpha = -1$ we get the difference to the forward method:

$$y(n) = y(n-1) + T\,x(n-1) \qquad \rightarrow \qquad \boxed{s \quad \leftrightarrow \quad \frac{1-z^{-1}}{T\,z^{-1}} = \frac{z-1}{T}}$$

- For other values of α, other possible methods of discretization are obtained.

To determine the points of the z plane corresponding to the negative real s semiplan, place $z = x+j\,y$ inside the function $s = f(z)$ and impose $\mathrm{Re}(s) \leq 0$:

$$\begin{aligned}
\mathrm{Re}[s] &= \mathrm{Re}\left[\frac{2}{T}\,\frac{1-z^{-1}}{(1+\alpha)+(1-\alpha)\,z^{-1}}\right]_{z=x+j\,y} = \frac{2}{T}\,\mathrm{Re}\left[\frac{x+j\,y-1}{(1+\alpha)(x+j\,y)+(1-\alpha)}\right] \leq 0\\
&= \frac{2}{T}\,\mathrm{Re}\left[\frac{x-1+j\,y}{(1+\alpha)x+1-\alpha+j\,(1+\alpha)y}\right] \leq 0\\
&= \frac{2}{T}\,\mathrm{Re}\left[\frac{(x-1+j\,y)[(1+\alpha)x+1-\alpha-j\,(1+\alpha)y]}{[(1+\alpha)x+1-\alpha]^2+(1+\alpha)^2y^2}\right] \leq 0\\
&= \frac{2}{T}\,\mathrm{Re}\left[\frac{(x-1+j\,y)[(1+\alpha)x+1-\alpha-j\,(1+\alpha)y]}{[(1+\alpha)x+1-\alpha]^2+(1+\alpha)^2y^2}\right] \leq 0
\end{aligned}$$

This relationship is satisfied if and only if:

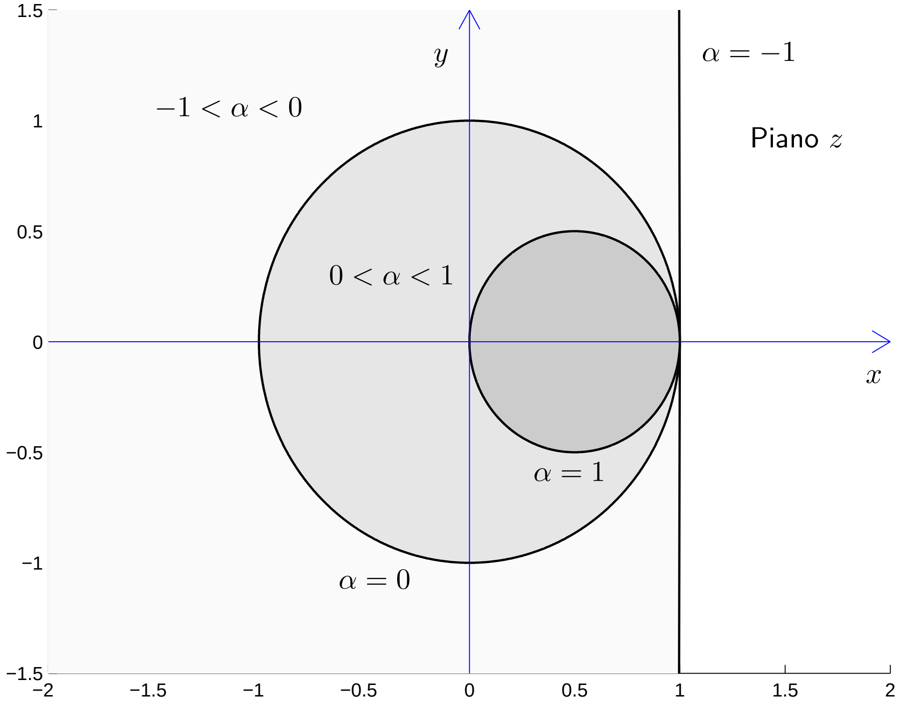
$$(x-1)[(1+\alpha)x+1-\alpha]+\;(1+\alpha)y^2 \leq 0$$

from which the report is derived:

$$(1+\alpha)x^2+\;(1+\alpha)y^2-2\,\alpha\,x+\alpha-1 \leq 0$$

The points that satisfy this relation are all the only ones inside a circle having the center in $(x_0,\;0)$ and radius r:

$$x_0 = \frac{\alpha}{1+\alpha}, \qquad\qquad r = \frac{1}{1+\alpha}$$



4. BILINEARY TRANSFORMATION WITH PRE -COM -PEN -SA -ZIO -NE

$$\boxed{s = \frac{\omega_1}{\tan\frac{\omega_1 T}{2}}\,\frac{1-z^{-1}}{1+z^{-1}} = \frac{\omega_1}{\tan\frac{\omega_1 T}{2}}\,\frac{z-1}{z+1}}$$

- For $\omega = \omega_1$ you have $\omega = \omega_1$
- Example

$$G(s) = \frac{a}{s+a}$$

- Precompensation at frequency $\omega = a$

$$s = \frac{a}{\tan\frac{aT}{2}}\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$G_d(z) = \frac{\tan\frac{aT}{2}(1+z^{-1})}{(\tan\frac{aT}{2}-1)z^{-1} + (\tan\frac{aT}{2}+1)}$$

- Example. Design a discrete low pass filter that approximates the frequency behavior in the band $[0,\ 10] rad/s$ of the analogue filter

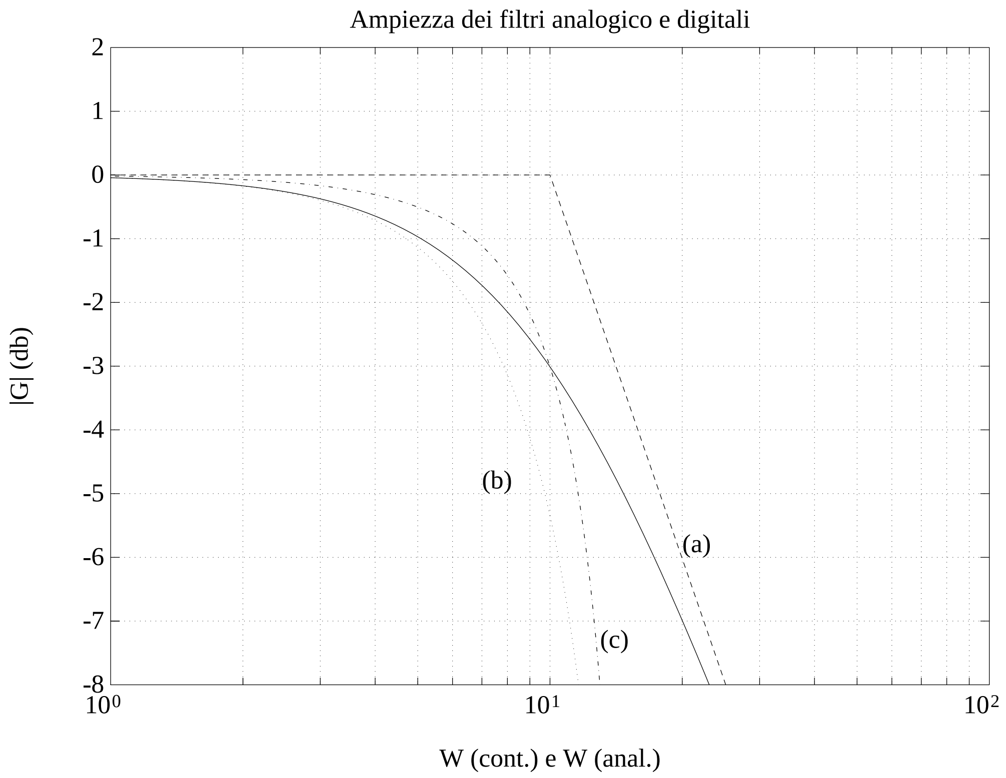
$$G(s) = \frac{10}{s+10} \qquad \text{con} \qquad T = 0.2\ s$$

$$G_d(z) = \frac{10}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + 10} = \frac{1+z^{-1}}{2}$$

$$G_d(e^{j\omega T}) = \frac{10}{j\frac{2}{T}\tan\frac{\omega T}{2} + 10} = \frac{1}{j\tan 0.1\omega + 1}$$

- Using frequency precompensation for $\omega = 10\ rad/s$, is obtained

$$G_d(z) = \frac{10}{\frac{10}{\tan\frac{10T}{2}}\frac{1-z^{-1}}{1+z^{-1}} + 10} = \frac{0.609(1+z^{-1})}{1+0.218z^{-1}}$$



5. METHOD OF $\mathcal{Z}$-TRANSFORMED

$$\boxed{D(z) = \mathcal{Z}\left[\mathcal{L}^{-1}[D(s)]\right]}$$

- Invariance of impulse response
- Possibility of aliasing
- From $D(s)$ stable to $D(z)$ stable

6. METHOD OF $\mathcal{Z}$-TRANSFORMED WITH ORDER RECONSTRUCTION

0 or step response invariance

$$\mathcal{Z}^{-1}\left[D(z)\frac{1}{1-z^{-1}}\right] = \mathcal{L}^{-1}\left[D(s)\frac{1}{s}\right]\bigg|_{t=kT}$$

$$D(z) = (1-z^{-1})\mathcal{Z}\left[\frac{D(s)}{s}\right] = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}D(s)\right]$$

- Possibility of aliasing
- From $D(s)$ stable to $D(z)$ stable

Example. Using the method of the $\mathcal{Z}$ transformed with zero order rebuilder, discretize the function:

$$D(s) = \frac{M(s)}{E(s)} = \frac{s+2}{s+5}$$

also reaching the determination of the corresponding difference equation. Use the sampling period $T = 0.1$.

[Solution.] Using the $\mathcal{Z}$ method transformed with zero order rebuilder, the discretization of the $D(s)$ regulator proceeds as follow

$$D(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\frac{s+2}{s+5}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{s+2}{s(s+5)}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{2}{5\,s} + \frac{3}{5(s+5)}\right]$$

da cui si ricava

$$\begin{aligned} D(z) &= (1-z^{-1})\left[\frac{2}{5(1-z^{-1})} + \frac{3}{5(1-e^{-5T}z^{-1})}\right] \\ &= \frac{2(1-e^{-5T}z^{-1}) + 3(1-z^{-1})}{5(1-e^{-5T}z^{-1})} \\ &= \frac{5-(3+2e^{-5T})z^{-1}}{5(1-e^{-5T}z^{-1})} = \frac{5-4.213\,z^{-1}}{5(1-0.6065\,z^{-1})} \end{aligned}$$

The corresponding difference equation is derived from the relation

$$M(z)(5-3.0325\,z^{-1}) = E(z)(5-4.213\,z^{-1})$$

from which

$$m(k) = \frac{1}{5}[3.0325\,m(k-1) + 5e(k) - 4.213e(k-1)]$$

that is

$$m(k) = 0.6065\,m(k-1) + e(k) - 0.8426e(k-1)$$

7. HHugeb POLI/ZERI CORRESPONDENCE METHOD

- It is factorized numerator and denominator of $D(s)$
- Transformation of poles and zeros

$$(s+a) \;\rightarrow\; (1-e^{-aT}z^{-1})$$

$$(s+a\pm jb) \;\rightarrow\; (1-2e^{-aT}\cos bTz^{-1}+e^{-2aT}z^{-2})$$

- Zeroes are entered in $z=-1$ in number equal to the relative degree
- Adjust the gain at low ($z=1$) or at high ($z=-1$) frequencies
- Example

$$D(s)=\frac{s+b}{s+a} \qquad \rightarrow \qquad D(z)=k\frac{z-e^{-bT}}{z-e^{-aT}}$$

$$D(z=1)=k\frac{1-e^{-bT}}{1-e^{-aT}}=D(s=0)=\frac{b}{a}$$

$$k=\frac{b}{a}\frac{1-e^{-aT}}{1-e^{-bT}}$$

- Example. High pass filter

$$D(s)=\frac{s}{s+a}$$

$$D(z)=k\frac{z-1}{z-e^{-aT}} \qquad k=\frac{1+e^{-aT}}{2}$$

- Example

$$D(s)=\frac{1}{(s+a)^2+b^2}=\frac{1}{(s+a+jb)(s+a-jb)}$$

- Excess poly-zeros equal to 2

$$D(z)=k\frac{(z+1)^2}{z^2-2ze^{-aT}\cos bT+e^{-2aT}}$$

$$k=\frac{1-2e^{-aT}\cos bT+e^{-2aT}}{4(a^2+b^2)}$$

Example. Using the poly/zero match method, discretize the following PI regulator:

$$D(s) = \frac{M(s)}{E(s)} = \frac{s+1}{s}$$

also reaching the determination of the corresponding difference equation. Use the sampling period $T = 0.2$ and impose the equality of gains at high frequencies.

[Solution.] Using the poly/zero match method is obtained

$$D(s) = \frac{s+1}{s} \qquad \rightarrow \qquad D(z) = k\frac{1-e^{-T}z^{-1}}{1-z^{-1}}\bigg|_{T=0.2} = k\frac{1-0.8187z^{-1}}{1-z^{-1}}$$

The value of k is calculated by imposing the equality of gains at high frequencies

$$D(s)|_{s\to\infty} = D(z)|_{z=-1} \qquad \leftrightarrow \qquad 1 = k\frac{1+e^{-T}}{2} \qquad \rightarrow \qquad k = \frac{2}{1.8187} = 1.1$$

The corresponding difference equation is derived from the relation

$$M(z)(1-z^{-1}) = kE(z)(1-0.8187z^{-1})$$

obtaining

$$m(n) = m(n-1) + k\,e(n) - k\,0.8187e(n-1)$$

from which

$$m(n) = m(n-1) + 1.1\,e(n) - 0.9\,e(n-1)$$

Example. Using the poly/zero match method, discretize the following transfer function:

$$D(s) = \frac{M(s)}{E(s)} = \frac{s+1}{s+3}$$

also reaching the determination of the corresponding difference equation. Use the sampling period $T = 0.1$.

[Solution.] Using the poly/zero match method is obtained

$$D(s) = \frac{s+1}{s+3} \qquad \rightarrow \qquad D(z) = k\frac{1-e^{-T}z^{-1}}{1-e^{-3T}z^{-1}}\bigg|_{T=0.1} = k\frac{1-0.905z^{-1}}{1-0.741z^{-1}}$$

The value of k is calculated by imposing the equality of static gains

$$D(s)|_{s=0} = D(z)|_{z=1} \qquad \leftrightarrow \qquad \frac{1}{3} = k\frac{1-e^{-T}}{1-e^{-3T}} \qquad \rightarrow \qquad k = \frac{1-e^{-3T}}{3(1-e^{-T})} = 0.908$$

The corresponding difference equation is derived from the relation

$$M(z)(1-0.741z^{-1}) = kE(z)(1-0.905z^{-1})$$

obtaining

$$m(n) = 0.741m(n-1) + k\,e(n) - k\,0.905\,e(n-1)$$

from which

$$m(n) = 0.741\,m(n-1) + 0.908\,e(n) - 0.821\,e(n-1)$$

Example. Using the poly/zero correspondence method to discretize the anticipatory network

$$D(s) = \frac{M(s)}{E(s)} = \frac{1+\tau s}{1+\alpha\tau s}$$

also reaching the determination of the corresponding difference equation. Use the following parameters: $\tau = 1$, $\alpha = 0.2$ and $T = 0.1$.

[Solution.] Using the poly/zero match method is obtained

$$D(s) = \frac{1+s}{1+0.2s} = 5\frac{s+1}{s+5} \qquad \rightarrow \qquad D(z) = k\frac{1-e^{-T}z^{-1}}{1-e^{-5T}z^{-1}}\Bigg]_{T=0.1} = k\frac{1-0.905z^{-1}}{1-0.606z^{-1}}$$

The value of k is calculated by imposing the equality of static gains

$$D(s)|_{s=0} = D(z)|_{z=1} \qquad \leftrightarrow \qquad 1 = k\frac{1-e^{-T}}{1-e^{-5T}} \qquad \rightarrow \qquad k = \frac{1-e^{-5T}}{1-e^{-T}} = 4.135$$

The corresponding difference equation is derived from the relation

$$D(z) = \frac{M(z)}{E(z)} = k\frac{1-0.905z^{-1}}{1-0.606z^{-1}}$$

obtaining

$$M(z)(1-0.606z^{-1}) = kE(z)(1-0.905z^{-1})$$

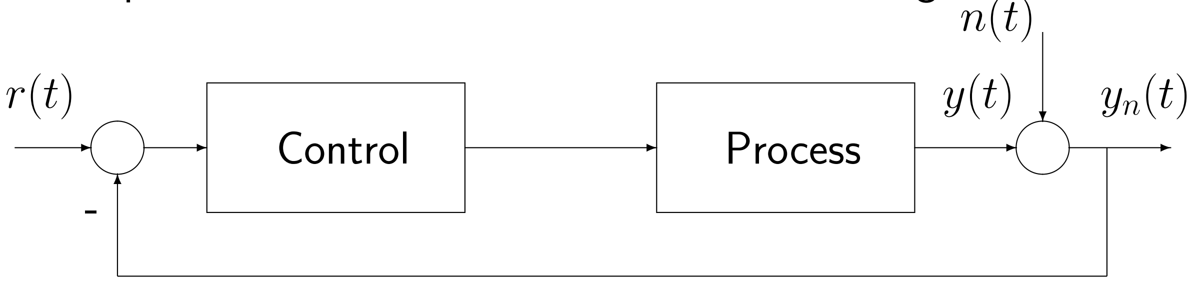
that is

$$m(n) = 0.606\,m(n-1) + k\,e(n) - k\,0.905\,e(n-1)$$

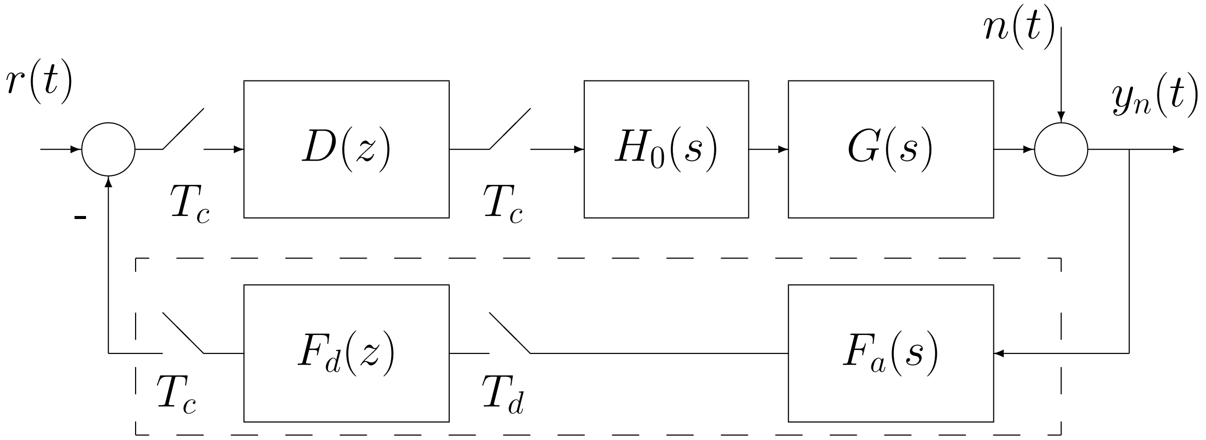
from which

$$m(n) = 0.606\,m(n-1) + 4.135\,e(n) - 3.742\,e(n-1)$$

- Antialiasing Filtering
- The aliasing produced by sampling introduces unwanted signal components at low frequencies, that is, in the band of the useful signal in closed loop

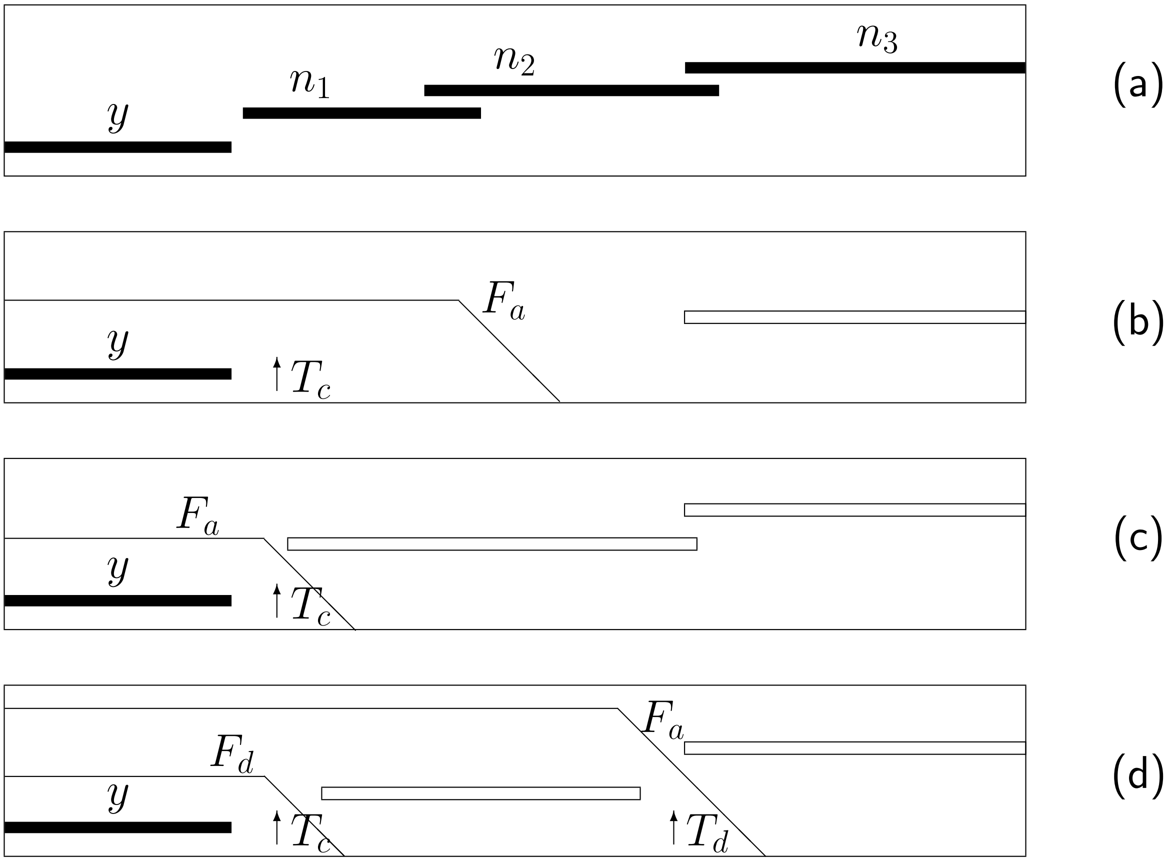


- It is necessary to introduce suitable filters that allow -mi -na -no as much as possible, before sampling, the noise signal
- Noise frequency band
- Complexity
- Calculation power available



- Analog type filters: passive or active

- Several cases of filtering and noise in the system



- Analog filtering
- RC filters (of the first order)
- Slope of $20\ db$ per decade
- It is good that the filtering action does not affect the useful signal area, to maintain the readiness of the system

- Digital filtering
- Sampling period smaller than the control period
- Media filter or filter for discretization
- Media filter:

$$y(k) = \frac{1}{N}\sum_{i=0}^{N-1} u(k-i)$$

with acquisition period

$$T_d = \frac{T_c}{N}$$

- Discretization of an analogue filter
- Example. First order filter:

$$F(s) = \frac{1}{1+\tau s}$$

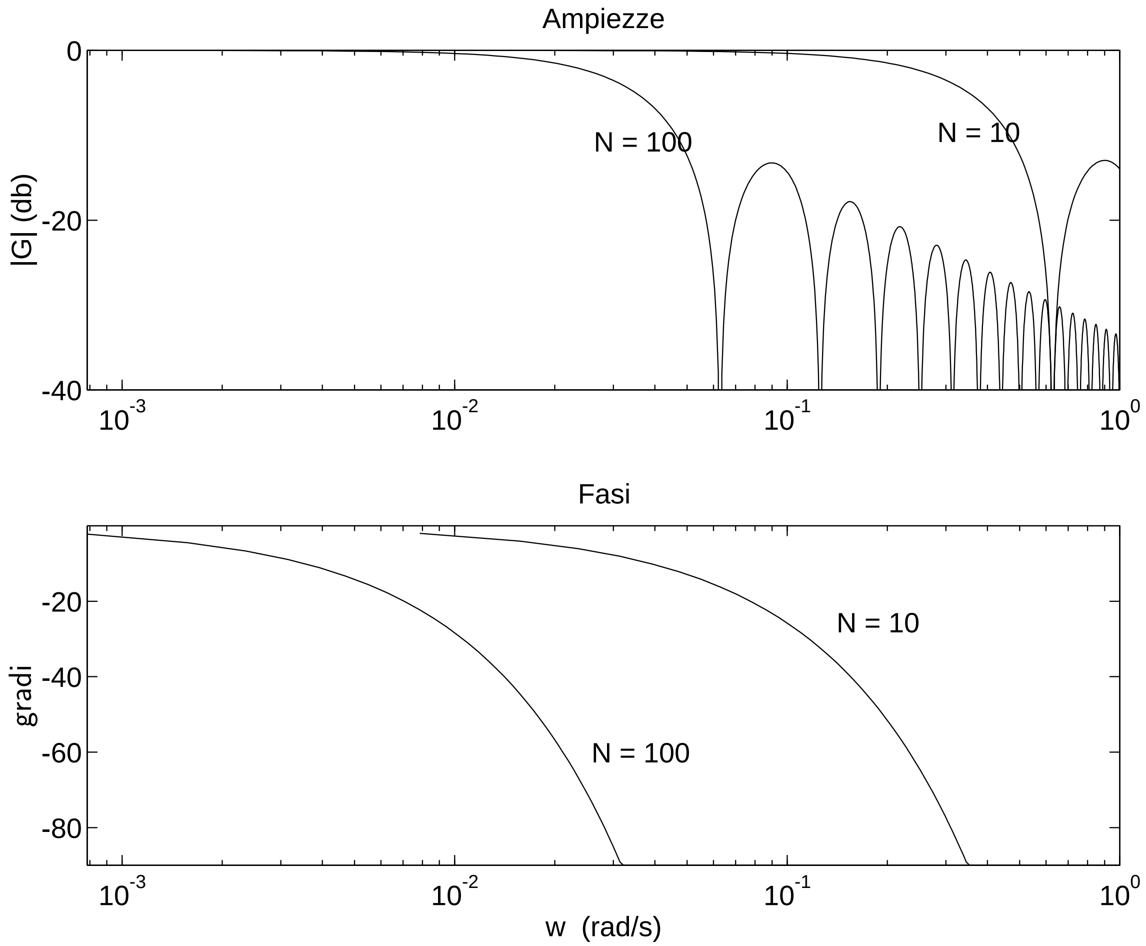
\- With the backward differences method:

$$F(z) = \frac{T_d/\tau}{(T_d/\tau) - z^{-1}}$$

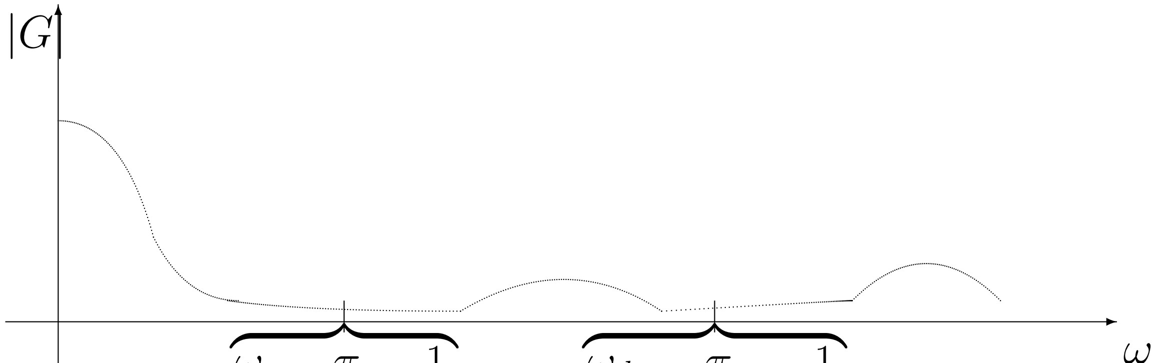
\- With the poles/zeros correspondence method:

$$F(z) = \frac{1-e^{-T_d/\tau}}{2}\frac{1+z^{-1}}{1-e^{-T_d/\tau}z^{-1}}$$

- Bode diagrams of two media filters



- Choice of time constants of the analogue filter, and of any digital filter

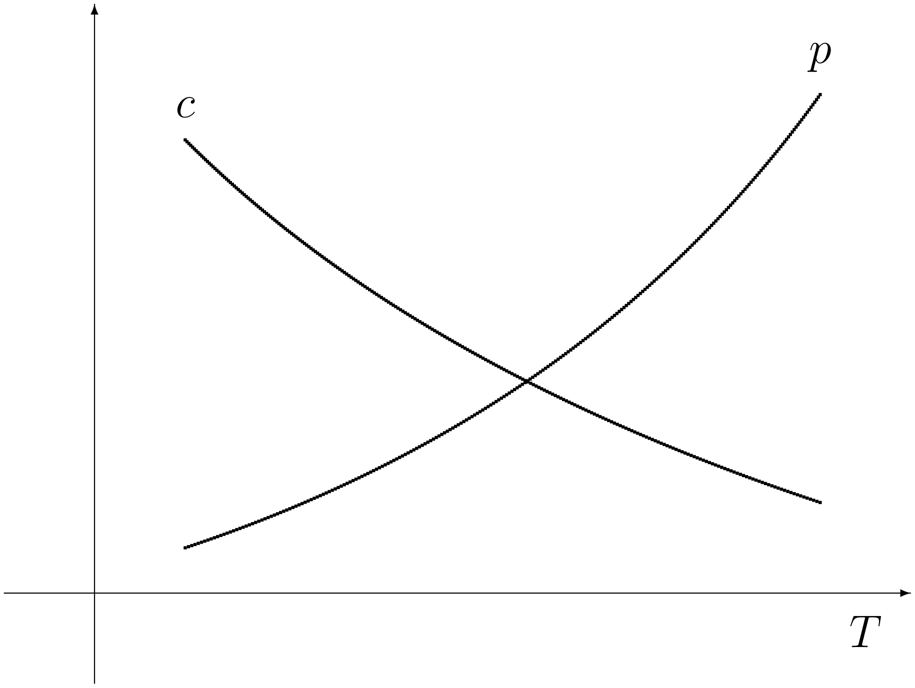


$$\frac{\omega_c}{2}=\frac{\pi}{T_c}=\frac{1}{\tau_d} \qquad \frac{\omega_d}{2}=\frac{\pi}{T_d}=\frac{1}{\tau_a}$$

$$\omega_a=\frac{\omega_c}{2}=\frac{\pi}{T_c}, \qquad \omega_a \approx 2/T_c, \qquad \tau_a=1/\omega_a \approx T_c/2$$

- $\tau_d \approx T_c/2$, while the time constant of the analogue filter τ_a must be related to the sampling period of the filter, like $\tau_a \approx T_d/2$

- Summary Considerations on Choosing the Sample -rio -do



- Prestazioni
  - rejection of disorders
  - tracking of the set-point
  - control energy
  - delays and stability
  - robustness to parametric variations
- Cost
  - exploitation of the processing capacity
  - conversion speed
  - processing speed
  - accuracy in storing parameters and variables