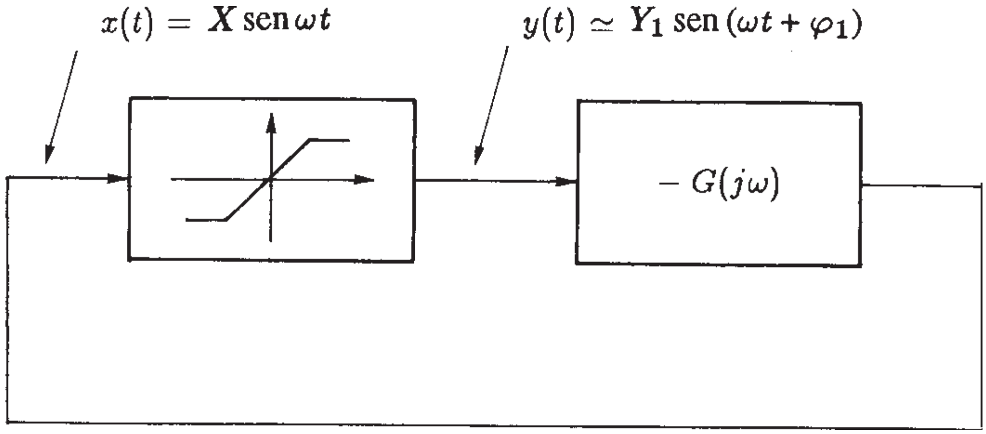
Describing function

- The method applies to systems having the following structure:



- The following hypotheses are considered:

  i) the input signal r is zero: $r = 0$;

  ii) the nonlinear function $y = f(x)$ is purely algebraic and independent of the frequency ω of the input signal $x(t) = X\sin(\omega t)$;

  iii) the nonlinear function $y = f(x)$ is symmetric with respect to the origin.

- Let $x(t) = X\,\sin(\omega t)$ be the sinusoidal input of the nonlinear function $y = f(x)$. The corresponding output $y = f(x) = f(X\,\sin(\omega t))$ is a periodic signal which can be developed in Fourier series:

$$y(t) = \sum_{n=1}^{\infty} Y_n\,\sin\left(n\omega t + \varphi_n\right)$$

- The periodic signal $y(t)$ can be approximated as follows:

$$y(t) \simeq Y_1(X)\ \sin\big(\omega t + \varphi_1(X)\big)$$

- The **describing function** $F(X)$ associated to the nonlinear function $y = f(x)$ is defined as follows:

$$F(X) = \frac{1}{X}\,Y_1(X)\,e^{j\varphi_1(X)}$$

- The structure of the describing function $F(X)$ is "similar" to the frequency response function $G(j\omega)$. The main difference is that function $F(X)$ depends on the amplitude X, while $G(j\omega)$ depends on the frequency ω.

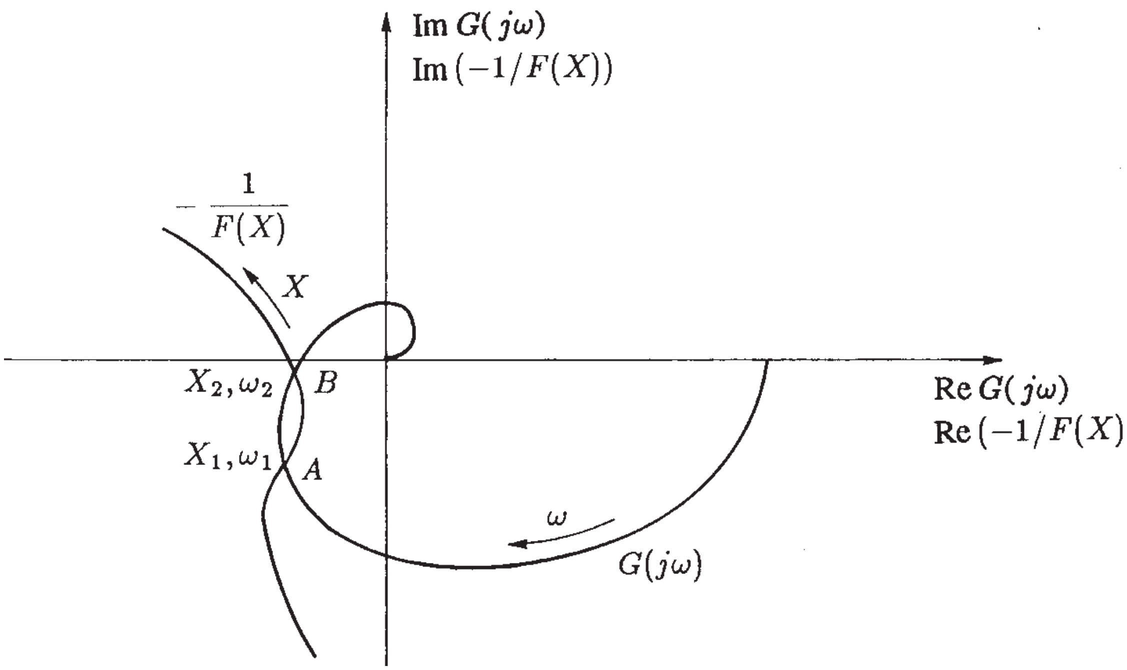
- The high order harmonics can be neglected because:
  - *i)* their amplitude is usually smaller than that of the fundamental;
  - *ii)* system $G(s)$ usually acts as a low pass filter and therefore it tends to reduce the amplitude of the higher order harmonics.

- A persistent oscillation is present in the system if the following complex nonlinear equation is satisfied:

$$\boxed{F(X)\,G(j\omega) = -1} \qquad \Rightarrow \qquad G(j\omega) = -\frac{1}{F(X)}$$

- This equation (in the two unknowns X and ω) cannot be easily solved "mathematically", it can be solved "numerically" or "graphically".

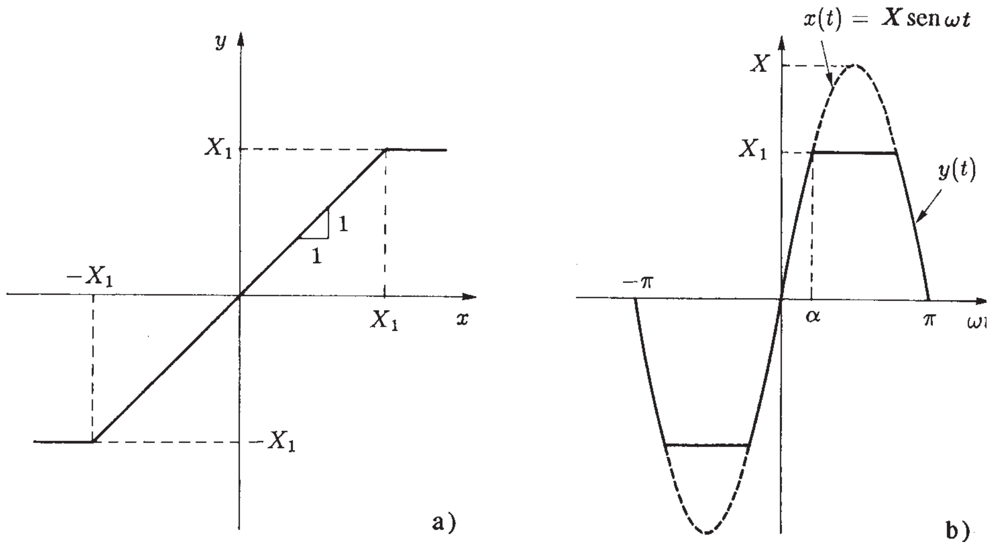
- The graphical solutions can be obtained plotting both the functions $G(j\omega)$ and $-1/F(X)$ on the complex plane:



- The intersection points (see points A and B) of the two functions $G(j\omega)$ and $-1/F(X)$ provide the frequencies (ω_1 and ω_2) and the amplitudes (ω_1 and ω_2) of the persistent oscillations present in the feedback system.

Describimg functions of the main non-linearities

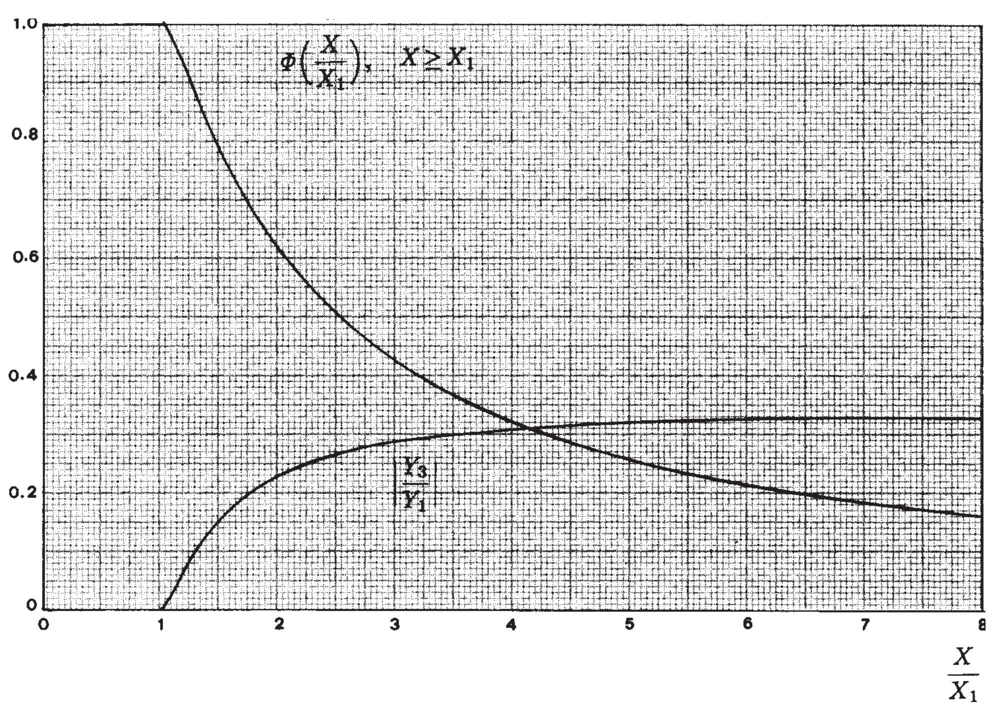
• **Saturation**. It is a nonlinear function quite frequent in the physical systems. The function $y = y(x)$ of a saturation with unitary slope is:



• Let us define $\Phi\left(\frac{X}{X_1}\right) := \frac{2}{\pi}\left(\operatorname{arcsen}\frac{X_1}{X} + \frac{X_1}{X}\sqrt{1-\left(\frac{X_1}{X}\right)^2}\right)$. The describing function $F(X)$ of the saturation is:

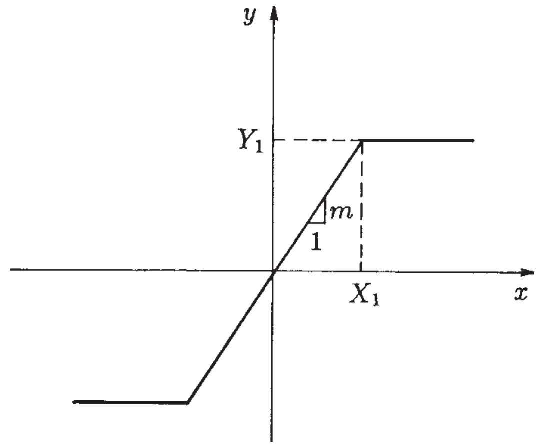
$$F(X) = \begin{cases} 1 & \text{if } X \leq X_1 \\ \Phi\left(\dfrac{X}{X_1}\right) & \text{if } X \geq X_1\ . \end{cases}$$

• Describing function $F(X)$:

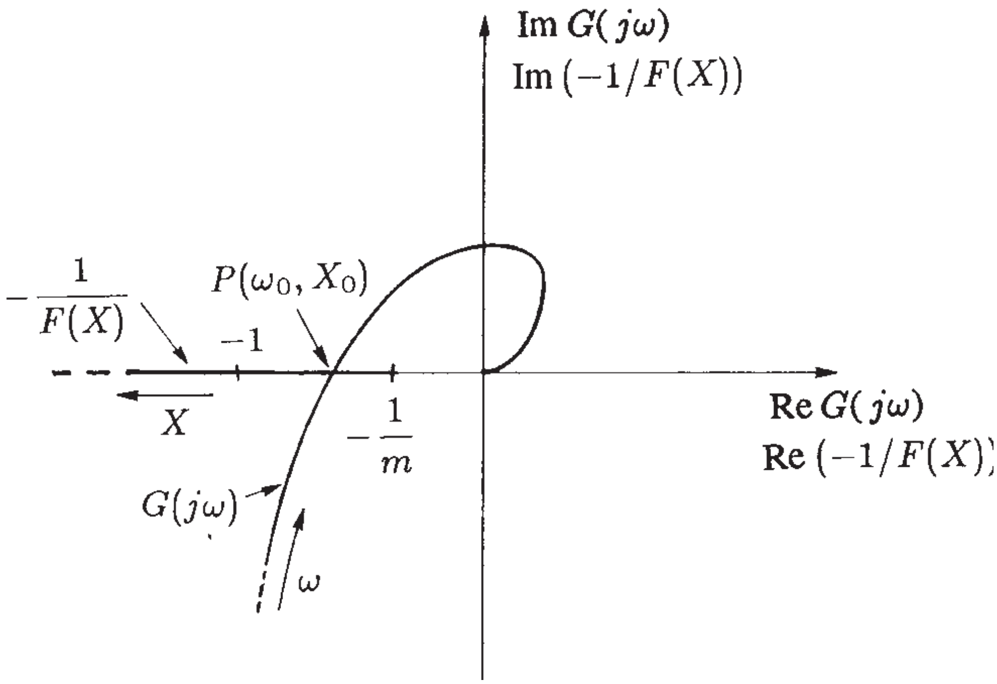


- Describing function $F(X)$ of a saturation with slope $m = Y_1/X_1$:

$$F(X) = \begin{cases} m & \text{if } X \le X_1 \\ m\,\Phi\!\left(\frac{X}{X_1}\right) & \text{if } X \ge X_1 \end{cases}$$

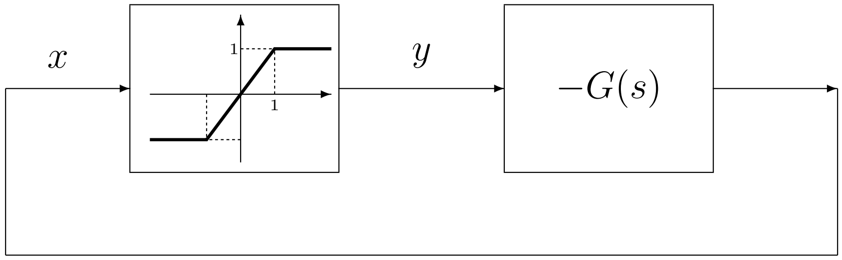


- Graphical solution of function $F(X)\,G(j\omega) = -1$ (system $G(s)$ is supposed to be stable in open loop):



- *An intersection point $P(\omega_0, X_0)$ of the two functions $G(j\omega)$ and $-1/F(X)$ corresponds a "stable limit cycle" if, for increasing values of parameter X, the point $-1/F(X)$ tends to exit from the Nyquist diagram of function $G(j\omega)$. The point $P(\omega_0, X_0)$ corresponds an "unstable limit cycle" in the opposite case.*

Example. Consider the following feedback system:



where

$$G(s) = \frac{20}{s(s+1)(s+3)}$$

- Characteristic equation: $1 + K\,G(s) = 0 \qquad \leftrightarrow \qquad s^3 + 4\,s^2 + 3\,s + 20\,K = 0$
- Routh table:

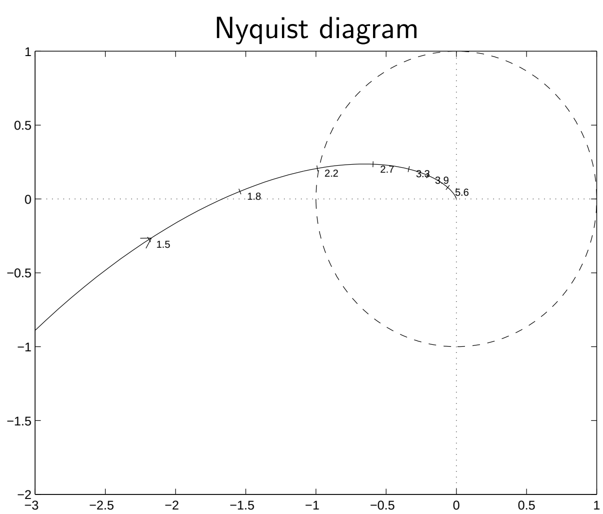
3	1	3
2	4	$20K$
1	$12 - 20K$	
0	$20K$	

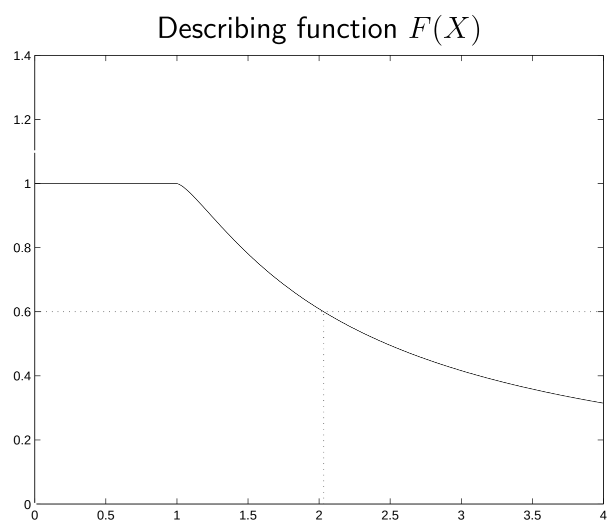
Gain margin:

$$K^* = \frac{12}{20} = 0.6$$

Intersection σ^* with the real negative semiaxis and frequency ω_0 of the limit cycle:

$$\sigma^* = -\frac{1}{K^*} = -\frac{20}{12}, \qquad \omega_0 = \sqrt{3} = 1.7321.$$





The amplitude X_0 of the limit cycle is found using the describing function $F(X)$.

$$F(X_0)\,G(j\omega_0) = -1 \qquad \rightarrow \qquad F(X_0) = -\frac{1}{G(j\omega_0)} = K^*$$

From function $F(X)$ one obtains $X_0 = 2.06$.

Describing function $F(X)$: qualitative drawing

If $y = f(x)$ is a continuous piecewise function, the "qualitative" behavior of the corresponding describing function $F(X)$ can be easily determined using the following rules:

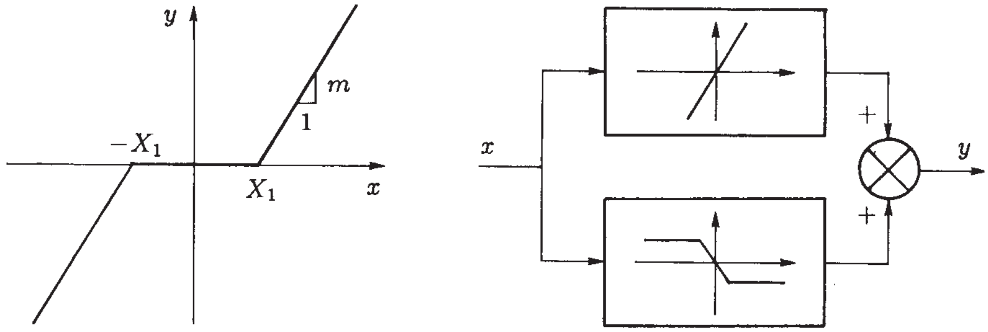
1) The describing function $F(X)$ is always a continuous function of the parameter X. If $y = f(x)$ is linear, the corresponding describing function $F(X)$ is constant and equal to the slope: if $y = K\,x$ then $F(X) = K$.

2) The value of the describing function $F(X)$ for $X \simeq 0^+$ is equal to the "slope of the first segment" of the piecewise function $y = f(x)$. If function $y = f(x)$ is discontinuous for $x = 0$, the corresponding value of the describing function $F(X)$ for $X \to 0^+$ is infinite.

3) When function $y = f(x)$ changes its slope or has a discontinuity, the corresponding describing function $F(X)$ changes its slope according to the slope change of function $y = f(x)$.

4) The value of the describing function $F(X)$ for $X \to \infty$ is equal to the "slope of the last segment" of the piecewise function $y = f(x)$.

• **Threshold**. The nonlinear function $y(x)$ of a threshold is:

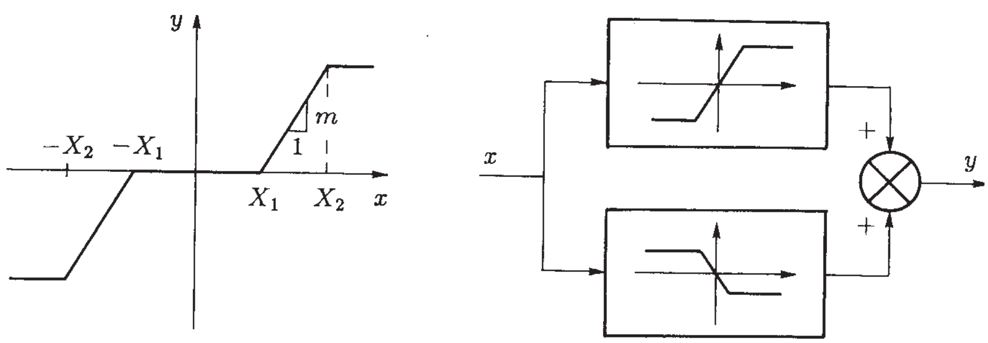


a) b)

The describing function of the threshold is equal to the sum of the describing functions of a linear element and a saturation element:

$$F(X) = \begin{cases} 0 & \text{if } X \leq X_1 \\ m\left(1 - \Phi\left(\frac{X}{X_1}\right)\right) & \text{if } X \geq X_1 \;. \end{cases}$$

• **Threshold with saturation**. The nonlinear function $y(x)$ of a threshold with saturation is:

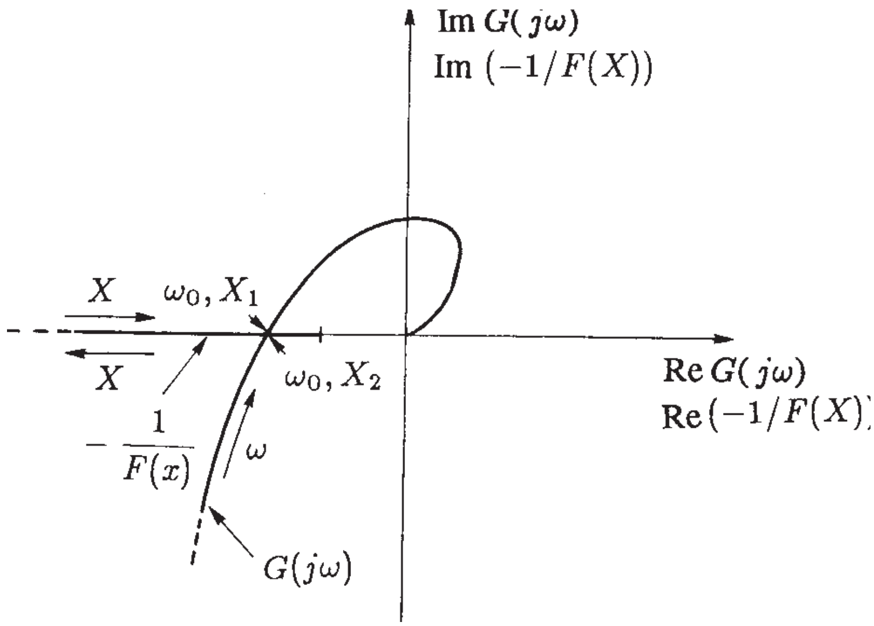


a) b)

The describing function of the threshold with saturation is:

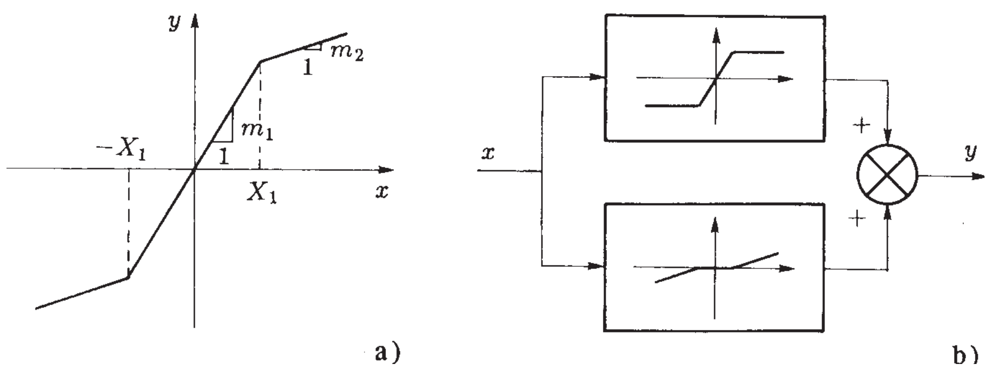
$$F(X) = \begin{cases} 0 & \text{if } X \leq X_1 \\ m\left(1 - \Phi\left(\frac{X}{X_1}\right)\right) & \text{if } X_1 \leq X \leq X_2 \\ m\left(\Phi\left(\frac{X}{X_2}\right) - \Phi\left(\frac{X}{X_1}\right)\right) & \text{if } X \geq X_2 \;. \end{cases}$$

- Graphical solution of function $F(X)\,G(j\omega) = -1$:



The two functions $G(j\omega)$ and $-1/F(X)$ intersect in two points: (X_1, ω_0) and (X_2, ω_0). The two points correspond to two different limit cycles characterized by the same frequency ω_0 and by two different amplitudes X_1 and x_2. The first limit cycle is unstable, the second is stable.

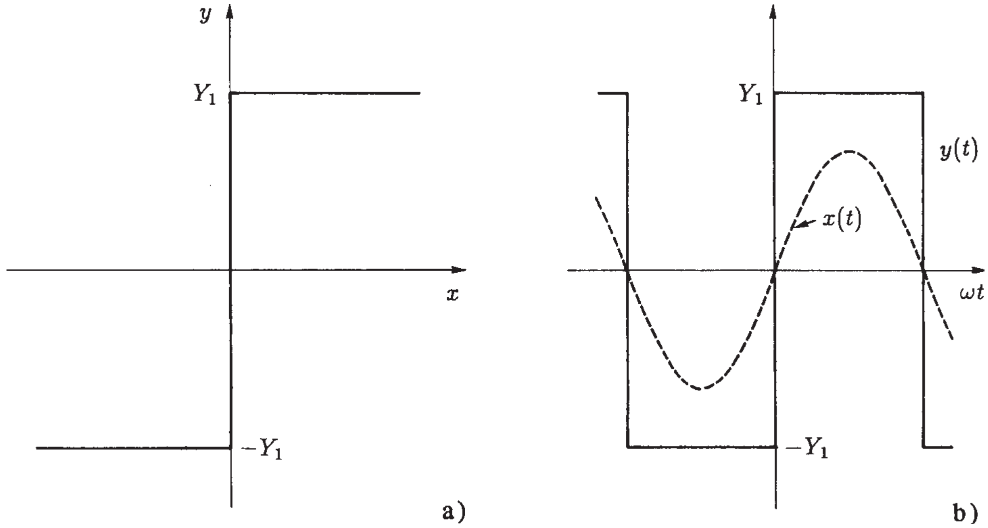
- **Partial saturation**. The nonlinear function $y = f(x)$ of a partial saturation is:



The describing function $F(X)$ of a partial saturation is the sum of the describing functions of a saturation and a threshold:

$$F(X) = \begin{cases} m_1 & \text{if } X \leq X_1 \\ (m_1 - m_2)\,\Phi\!\left(\dfrac{X}{X_1}\right) + m_2 & \text{if } X \geq X_1\,. \end{cases}$$

• **Ideal switch**. The nonlinear function $y = f(x)$ of an ideal switch is:



• This nonlinearity is widely used in nonlinear control systems. The describing function $F(X)$ of a ideal switch is:

$$F(X) = \frac{4\,Y_1}{\pi\,X}$$

• In this case the describing function $F(X)$ an hyperbole: