Contour Locus

- Let us refer to the following characteristic equation:

$$1+\frac{4(1+5\tau s)}{s(1+\tau s)(1+0.2s)}=0 \qquad \rightarrow \qquad 1+G_2(s,\tau)=0$$

  Qualitatively plot the root locus of the feedback system as a function of the parameter $\tau > 0$.

- The root locus is used only when the parameter K multiplies the transfer function $G_2(s)$. If the parameter τ, as in this case, is present within the transfer function $G_2(s,\tau)$, the problem is called **contour locus**.

- A contour locus problem can be solved by using the following procedure:

  1) Rewrite the characteristic equation in polynomial form:

$$s(1+\tau s)(1+0.2s)+4(1+5\tau s)=0$$

  2) Collect all the terms that "multiply" the parameter τ:

$$s(1+0.2s)+4+\tau[s^2(1+0.2s)+20s]=0$$

  3) Divide the characteristic equation for the group of terms that "do not multiply" the parameter τ:

$$1+\frac{\tau[s^2(1+0.2s)+20s]}{s(1+0.2s)+4}=0 \quad \leftrightarrow \quad \underbrace{1+\frac{\tau s[s^2+5s+100]}{s^2+5s+20}}_{1+\tau G_3(s)}=0$$

  4) Apply the root locus rules to transfer function $G_3(s)$ when parameter τ varies from 0 to ∞.

- This procedure highlights that *"a contour locus problem can be transformed back to a root locus problem only if the τ parameter enters linearly in the characteristic equation"*.

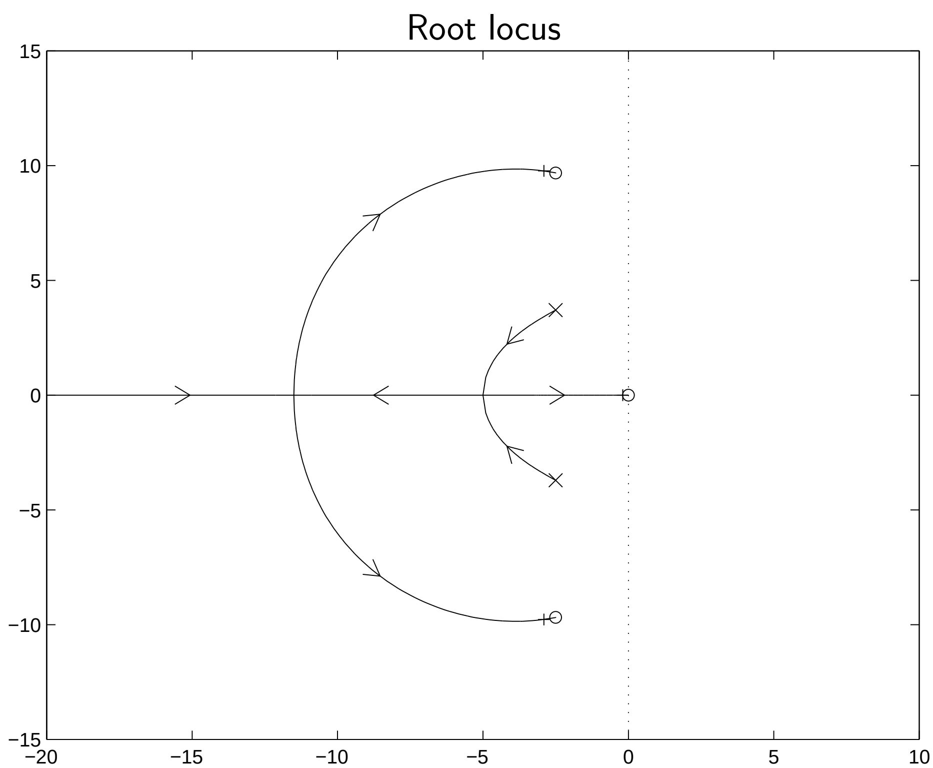
- Function $G_3(s)$ does not have a particular physical meaning, and therefore the relative degree of this function can also be negative: $n - m < 0$ (as for the considered example).

- Note: the polynomial that appears at the denominator of function $G_3(s)$ coincides with the characteristic equation of the feedback system when $\tau = 0$.

- In this case, the zeros and the poles of function $G_3(s)$ are:

$$z_1 = 0, \qquad z_{2,3} = -2.5 \pm j9.682, \qquad p_{1,2} = -2.5 \pm j3.708$$

- Qualitative plot of the contour locus as a function of parameter $\tau > 0$:



Example. Qualitatively plot the contour locus of the following system $G(s)$ as a function of parameter $\tau > 0$.

$$G(s) = \frac{K}{s\,(1 + \tau\, s)}$$

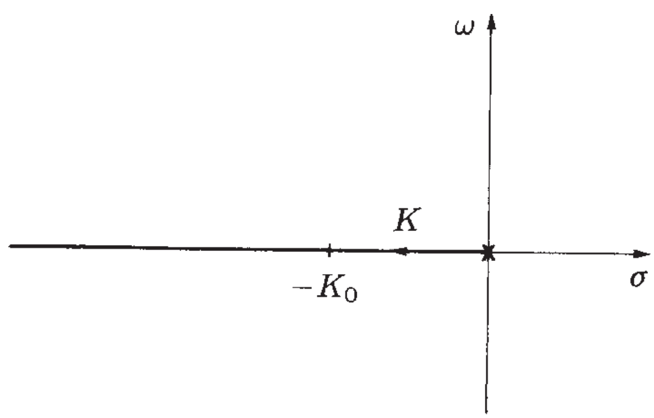
The characteristic equation of the feedback system is:

$$1 + \frac{K}{s\,(1 + \tau\, s)} = 0 \qquad \rightarrow \qquad \tau\, s^2 + s + K = 0$$

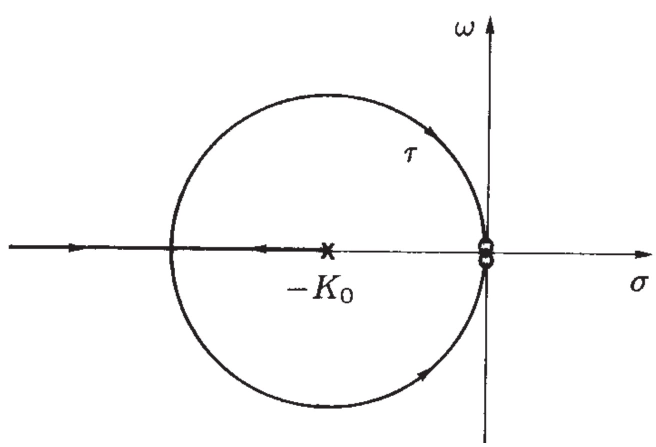
from which one obtains:

$$1 + \frac{\tau\, s^2}{s + K} = 0 \qquad \leftrightarrow \qquad 1 + \tau\, G_3(s) = 0$$

The root locus of function $G(s)$ when $K > 0$, and the root locus of function $G_3(s)$ when $\tau > 0$ (i.e. the contour locus of function $G(s)$ when $\tau > 0$) are the following:



a)



b)

By choosing a value K_0 for parameter K, one obtains that the starting point of the contour locus when $\tau = 0$ is $p = -K_0$.

Note: in this case function $G_3(s)$ is improper: the system has one pole and two zeroes. When the relative degree is negative, $n - m < 0$ the root locus has $|n - m|$ asymptotes which are traveled from infinity to finite.

Example. Plot the contour locus of the following system $G(s)$, for $\tau > 0$.

$$G(s) = \frac{K}{s\,(s+1)\,(1+\tau\,s)}$$

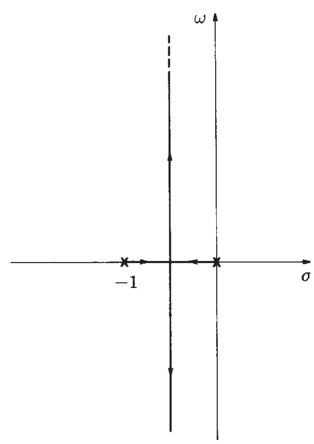
The characteristic equation of the feedback system is:

$$1 + \frac{K}{s\,(s+1)\,(1+\tau\,s)} = 0 \qquad \rightarrow \qquad s\,(s+1) + K_0 + \tau\,s^2\,(s+1) = 0$$

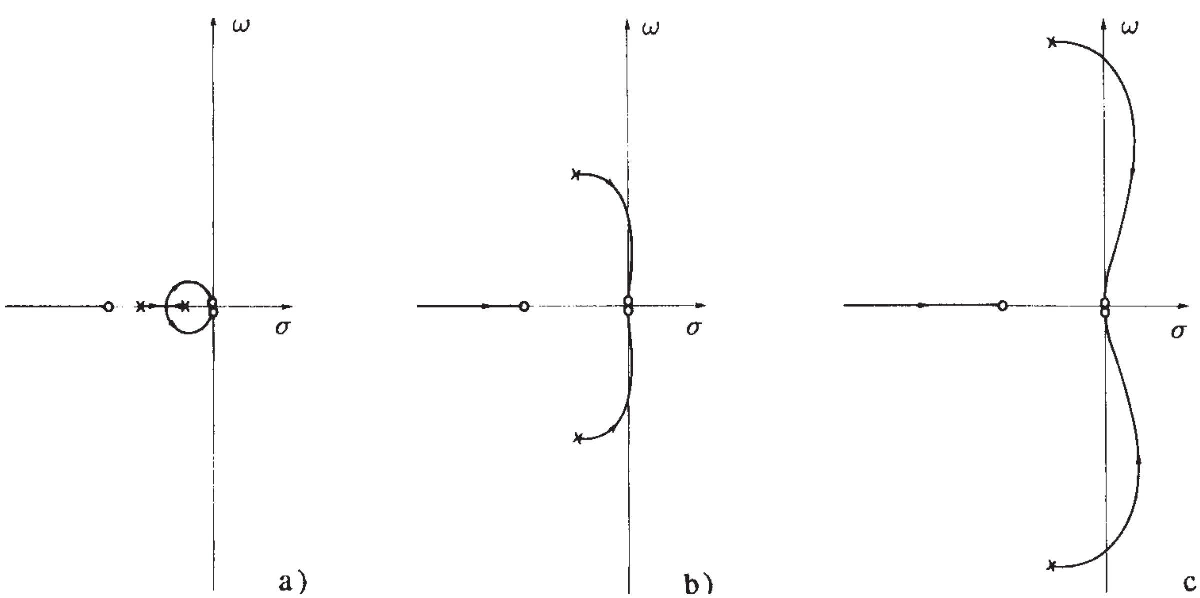
from which one obtains:

$$1 + \frac{\tau\,s^2\,(s+1)}{s\,(s+1) + K_0} = 0 \qquad \leftrightarrow \qquad 1 + \tau G_3(s) = 0$$

The root locus of system $G(s)$ when K varies from 0 to ∞ is:



The shape of the contour locus of system $G(s)$ when $\tau > 0$ is a function of the value of parameter K:



The contour locus of case a) corresponds to two real roots; the other two cases correspond to a couple of complex conjugate roots.

Example. Plot the contour locus of the following system $G(s)$ when $\tau > 0$:

$$G(s) = \frac{K_1\,(1+\tau\, s)}{s\,(s+1)\,(s+2)}$$

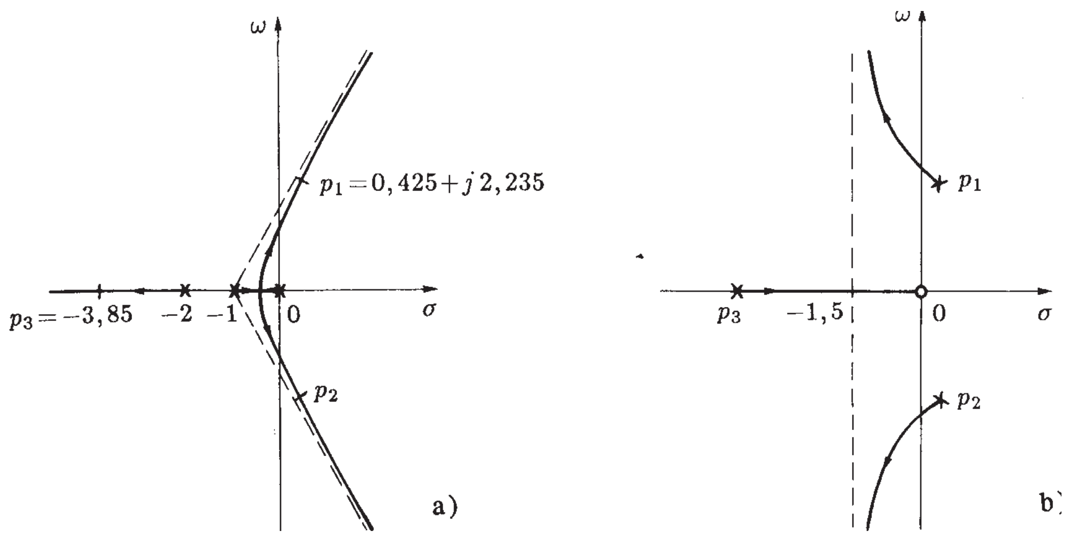
The characteristic equation of the feedback system is:

$$1 + \frac{K_1\,(1+\tau\, s)}{s\,(s+1)\,(s+2)} = 0 \qquad \rightarrow \qquad s\,(s+1)\,(s+2) + K_1 + \tau\, s\, K_1 = 0$$

from which one obtains:

$$1 + \frac{\tau\, s\, K_1}{s\,(s+1)\,(s+2) + K_1} = 0 \qquad \leftrightarrow \qquad 1 + \tau G_3(s) = 0$$

The root locus of system $G(s)$ when K varies from 0 to ∞, and the corresponding contour locus when $\tau > 0$ are the following:



The contour locus shown in b) corresponding to the case $K_1 := 20$. The poles p_1, p_2 and p_3 from which the contour locus starts when $\tau = 0^+$, are those shown in figure a).

The contour locus has two asymptotes. The intersection point of the asymptotes is on the real axis at the following abscissa:

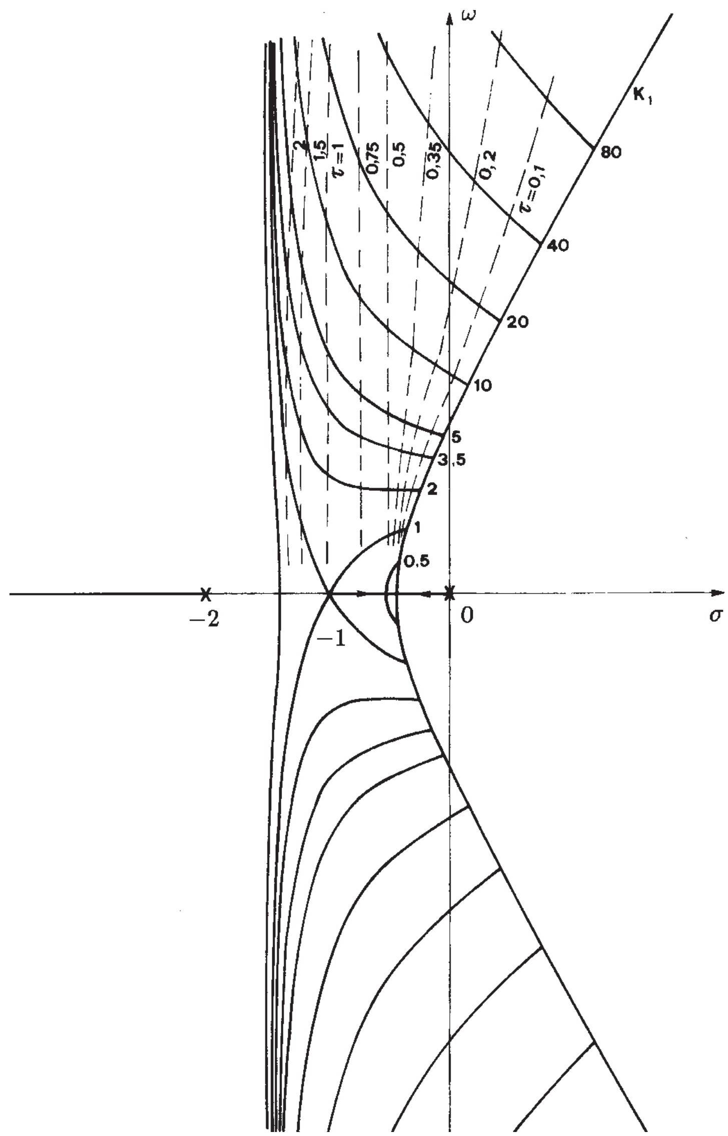
$$\sigma_a = \frac{p_1 + p_2 + p_3 - 0}{3-1} = -\frac{3}{2} = -1.5$$

Note: in this case, point σ_a is independent of the value of parameter K_1.

The Centroid Theorem

The Centroid Theorem. *Referring to the characteristic equation* $1 + K\,G(s) = 0$*, if the relative degree of function* $G(s)$ *is greater than or equal to 2, the sum of the poles of the feedback system is independent of the value of parameter* K*, and therefore it is equal the sum of the poles of the open loop system when* $K = 0$*.*

Contour loci of system $G(s) = \frac{K_1\,(1+\tau\,s)}{s\,(s+1)\,(s+2)}$ for different values of constant K_1, and for $\tau > 0$.



The main branches of the contour locus are plotted for different values of parameter K_1 providing a family of curves which fill a limited region of the complex plane.