Diagrams of Bode

• The frequency response function $F(\omega) = G(j\omega)$ can be represented graphically in three different ways: the *Bode diagrams*, the *Nyquist diagrams* and the *Nichols diagrams*.

• We have two **Bode diagrams**:

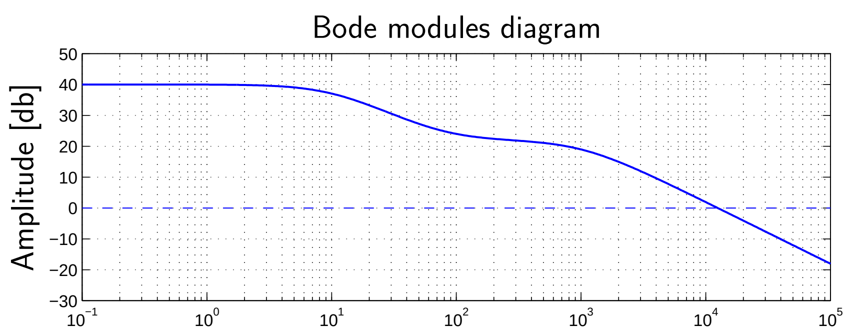
1) *Module (or Amplitude) diagram*: shows the module $|G(j\omega)|$ of the frequency response as a function of frequency ω. Both the module $|G(j\omega)|$ and the frequency ω are expressed in logarithmic scale. Usually, for the module $|G(j\omega)|$ the "db" scale ($A_{\text{db}} = 20\,\log_{10} A$) is used, while for frequency ω the 10-base logarithmic scale is used.

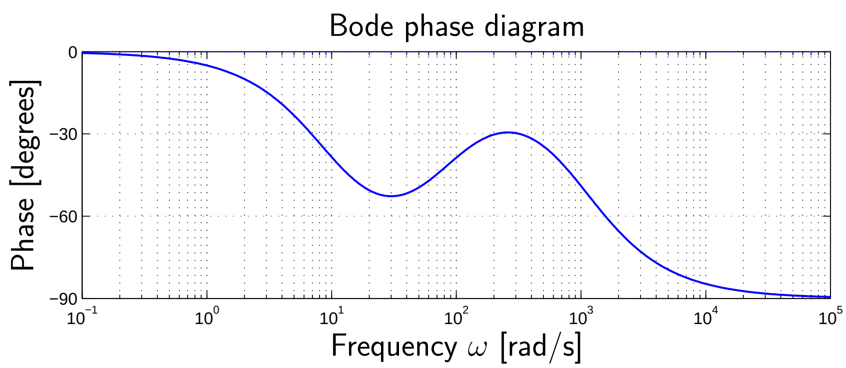
2) *Phase diagram*: shows the phase $\arg G(j\omega)$ of the frequency response as a function of frequency ω. The phase $\arg G(j\omega)$ is usually expressed in linear scale, while frequency ω is expressed using the 10-base logarithmic scale.

• Example:

$$G(s) = \frac{100\left(1+\frac{s}{80}\right)}{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{1000}\right)} \quad \rightarrow \quad G(j\omega) = \frac{100\left(1+j\frac{\omega}{80}\right)}{\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{1000}\right)}$$

• Bode diagrams





Module to db conversion

- The *decibel* is a conventional logarithmic unit that is normally used to express positive variables, typically the gain of amplifiers:

$$\boxed{A_{\text{db}} = 20 \log_{10} A}$$

- The conversion from module to db can be done using the following diagram:

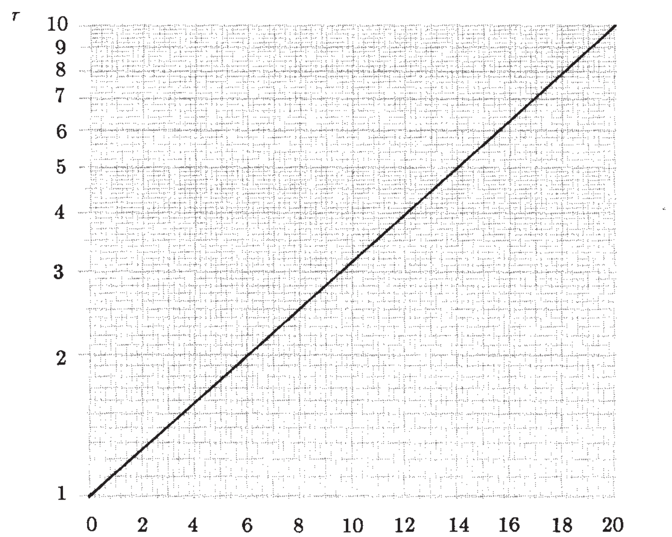
Written A in the following form:

$$A = r \cdot 10^n \quad \text{con} \quad 1 \leq r < 10,$$

the value of A in decibel is:

$$A_{\text{db}} = 20\,n + s \text{ db}$$

where s is obtained from the aside diagram: $0 \leq s < 20$.



- Some frequently used conversions:

The decades	
$\vdots$	$\vdots$
10000	80 db
1000	60 db
100	40 db
10	20 db
1	0 db
0.1	−20 db
0.01	−40 db
0.001	−60 db
0.0001	−80 db
$\vdots$	$\vdots$

$A > 1$	
$\sqrt{2}$	3 db
2	6 db
5	14 db
20	26 db
50	34 db
$A < 1$	
$1/\sqrt{2}$	−3 db
$1/2$	−6 db
$1/5$	−14 db
$1/20$	−26 db
$1/50$	−34 db

– Example 1:

$$A = 24$$
$$A = 2.4 \cdot 10^1$$
$$A_{\text{db}} \simeq 20 + 8 = 28$$

– Example 2:

$$A = 0.56$$
$$A = 5.6 \cdot 10^{-1}$$
$$A_{\text{db}} \simeq -20 + 15 = -5$$

– Every 6 db the value of A doubles;

– Every 20 db the value of A is multiplied by 10;

- For theoretical calculations, natural logarithm scale is typically used:

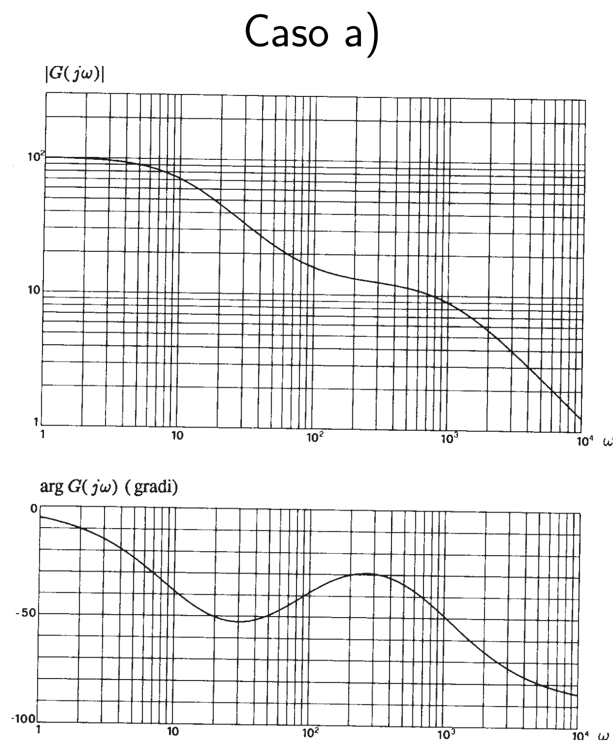
$$\ln G(j\omega) = \ln\left[|G(j\omega)|e^{j\arg G(j\omega)}\right] = \underbrace{\ln|G(j\omega)|}_{\alpha} + j\underbrace{\arg G(j\omega)}_{\beta}$$

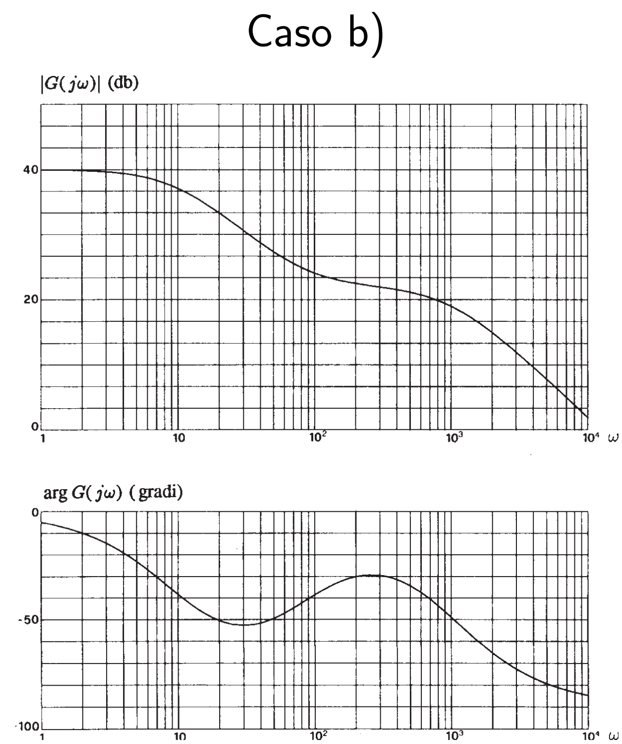
  A change of base is equivalent to a proportional change of scale.

- One can use two different types of Bode diagrams:

  a) diagrams with a double logarithmic scale for the modules and semi-logarithmic scale for the phases;

  b) diagrams with semi-logarithmic scale for both the modules and the phases. In this case, the module scale is in decibel: $A_{\text{db}} = 20\,\log_{10} A$.





- The advantages of using a logarithmic scale are:

  1) one obtains a detailed representation of the considered function over a quite large range of the variable ω;

  2) the Bode diagram of cascaded systems is the sum of the Bode diagrams of all the subsystems;

  3) the Bode diagram of a function $G(s)$ given in the factorial form is the sum of the Bode diagrams of all its factors.

- Function $G(s)$ given in the *polynomial form*:

$$G(s) = K_1 \, \frac{s^m + b_{m-1}\, s^{m-1} + \ldots + b_1\, s + b_0}{s^h\,(s^{n-h} + a_{n-1}\, s^{n-h-1} + \ldots + a_{h+1}\, s + a_h)}$$

- The factor s^h corresponds to a pole in the origin with multiplicity h. If $h\!=\!0$, function $G(s)$ does not have poles in the origin.

- Function $G(s)$ given in the *poles and zeros factorized form*:

$$G(s) = K_1 \, \frac{(s - z_1)\,(s - z_2)\;\ldots\;(s - z_m)}{s^h\,(s - p_1)\,(s - p_2)\;\ldots\;(s - p_{n-h})}$$

- Function $G(s)$ given in the *time constants factorized form*:

$$G(s) = K \, \frac{\left(1\!+\!\tau_1' s\right)\,\left(1\!+\!\tau_2' s\right)\ldots\left(1\!+\!2\delta_1' \frac{s}{\omega_{n1}'} + \frac{s^2}{\omega_{n1}'^2}\right)\;\ldots}{s^h\,\left(1\!+\!\tau_1 s\right)\,\left(1\!+\!\tau_2 s\right)\ldots\left(1\!+\!2\delta_1 \frac{s}{\omega_{n1}} + \frac{s^2}{\omega_{n1}^2}\right)\;\ldots}$$

- The ***log of the module*** of function $G(s)$ is:

$$\begin{aligned}\log|G(s)| \;=\; & \log|K| + \log\left|1\!+\!\tau_1' s\right| + \ldots + \log\left|1\!+\!2\delta_1' \frac{s}{\omega_{n1}'} + \frac{s^2}{\omega_{n1}'^2}\right| \\ & - \log|s^h| - \log\left|1\!+\!\tau_1 s\right| - \ldots - \log\left|1\!+\!2\delta_1 \frac{s}{\omega_{n1}} + \frac{s^2}{\omega_{n1}^2}\right|\end{aligned}$$

- The ***phase*** of the function $G(s)$ is:

$$\begin{aligned}\arg G(s) \;=\; & \arg K + \arg\left(1\!+\!\tau_1' s\right) + \ldots + \arg\left(1\!+\!2\delta_1' \frac{s}{\omega_{n1}'} + \frac{s^2}{\omega_{n1}'^2}\right) \\ & - \arg(s^h) - \arg\left(1\!+\!\tau_1 s\right) - \ldots - \arg\left(1\!+\!2\delta_1 \frac{s}{\omega_{n1}} + \frac{s^2}{\omega_{n1}^2}\right)\end{aligned}$$

- The Bode diagram of a function $G(s)$ is equal to the sum of the Bode diagrams of its basic elements:

$$K, \qquad s^{\pm 1}, \qquad \left(1 + s\,\tau\right)^{\pm 1}, \qquad \left(1 + 2\,\delta\,\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}\right)^{\pm 1}$$

- **Constant Gain**:

$$G(s) = K$$

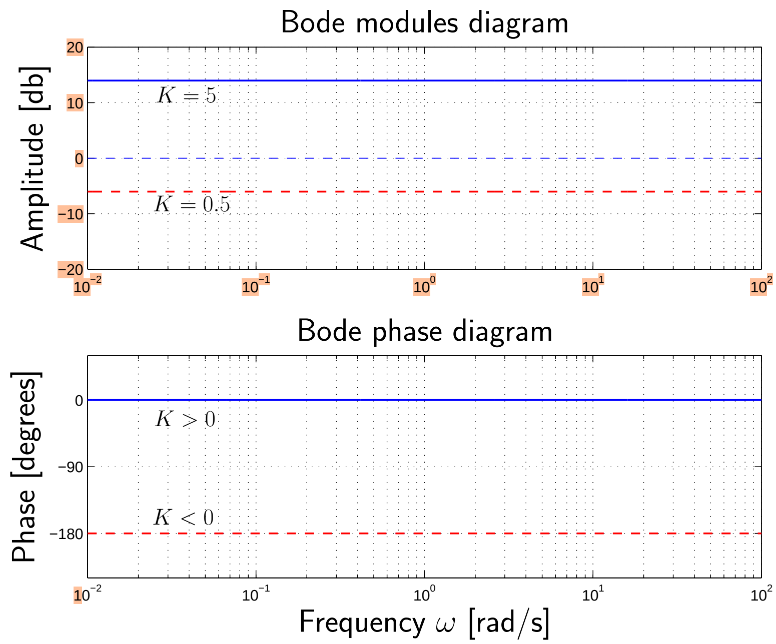
Frequency response function:

$$G(j\omega) = |K|\, e^{j\varphi}$$

Module: $|G(j\omega)| = |K|$.

Phase: $\begin{cases} 0 & \text{se} \quad K > 0 \\ -\pi & \text{se} \quad K < 0 \end{cases}$

The diagrams of the modules and phases are constant and independent of ω.



- **A pole in the origin**:

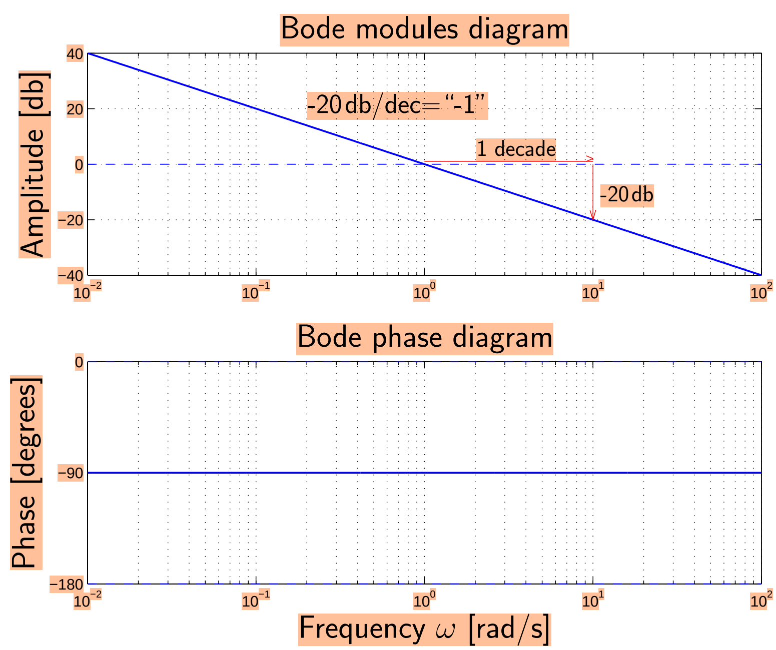
$$G(s) = \frac{1}{s}$$

Frequency response function:

$$G(j\omega) = \frac{1}{j\omega}$$

Module: $|G(j\omega)| = \frac{1}{\omega}$

Constant phase: $\varphi = -\frac{\pi}{2}$



The module diagram has a slope equal to "-1" (i.e. 20 db/dec) and it has a unitary gain at frequency $\omega = 1$.

- **A real pole**:

$$G(s) = \frac{1}{1+\tau s} \qquad \rightarrow \qquad G(j\omega) = \frac{1}{1+j\omega\tau}$$

- Modules and phases Bode diagrams:

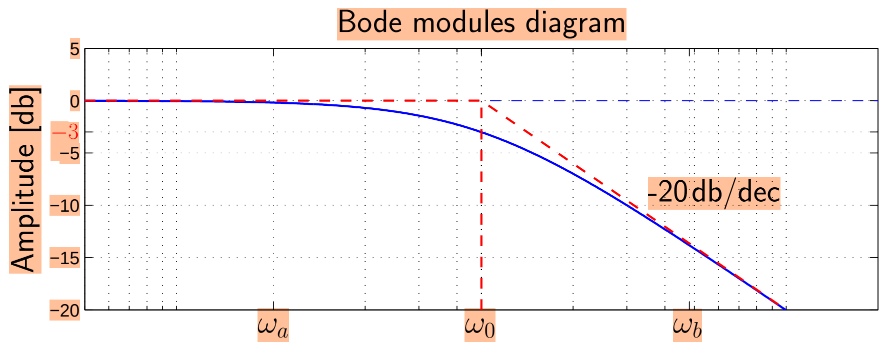
$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}, \qquad \arg G(j\omega) = -\arctan \omega\tau$$

a) At low frequencies:

$$G(j\omega)\Big|_{\omega \simeq 0} \simeq 1$$

Initial module: $G_0 = 1$.

Initial phase: $\varphi_0 = 0$.

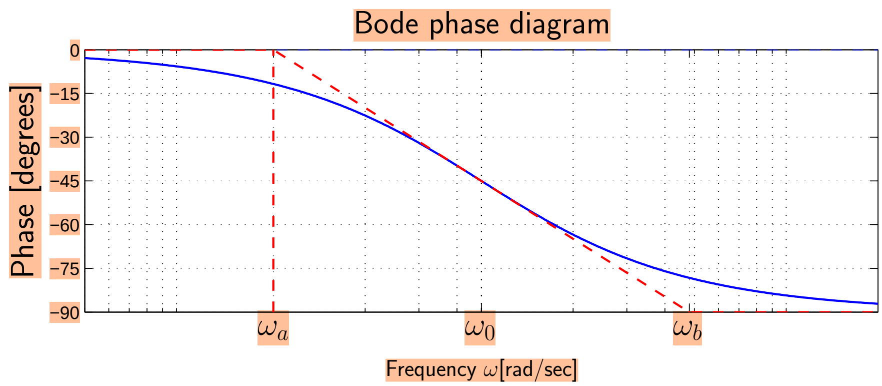


b) At high frequencies:

$$G(j\omega)\Big|_{\omega \simeq \infty} \simeq \frac{1}{j\omega\tau}$$

Final module: $G_\infty = 0$.

Final phase: $\varphi_\infty = -\frac{\pi}{2}$.



- One can easily prove that the following relation holds:

$$\frac{\omega_0}{\omega_a} = \frac{\omega_b}{\omega_0} = e^{\frac{\pi}{2}} = 4,81$$

- The slope changes of the phases asymptotic diagram occur at the frequencies:

$$\omega_a = \frac{\omega_0}{4.81} = \frac{1}{4.81\tau}, \qquad \omega_b = 4.81\omega_0 = \frac{4.81}{\tau}, \qquad \text{where} \qquad \omega_0 = \frac{1}{\tau}.$$

- The maximum distance between the asymptotic and real diagrams happens for $\omega = \omega_0 = 1/\tau$ and it is equal to $1/\sqrt{2} \simeq -3$ db.

- The -20 db/decade slope is usually denoted with the "-1" symbol.

- The Bode diagrams of function $G(s) = (1+\tau s)$ can be obtained by flipping upsidedown around the abscissa axis those of function $G(s) = (1+\tau s)^{-1}$.

- When τ is negative, the module diagram remains unchanged, while the phases diagram is flipped upsidedown around the abscissa axis.

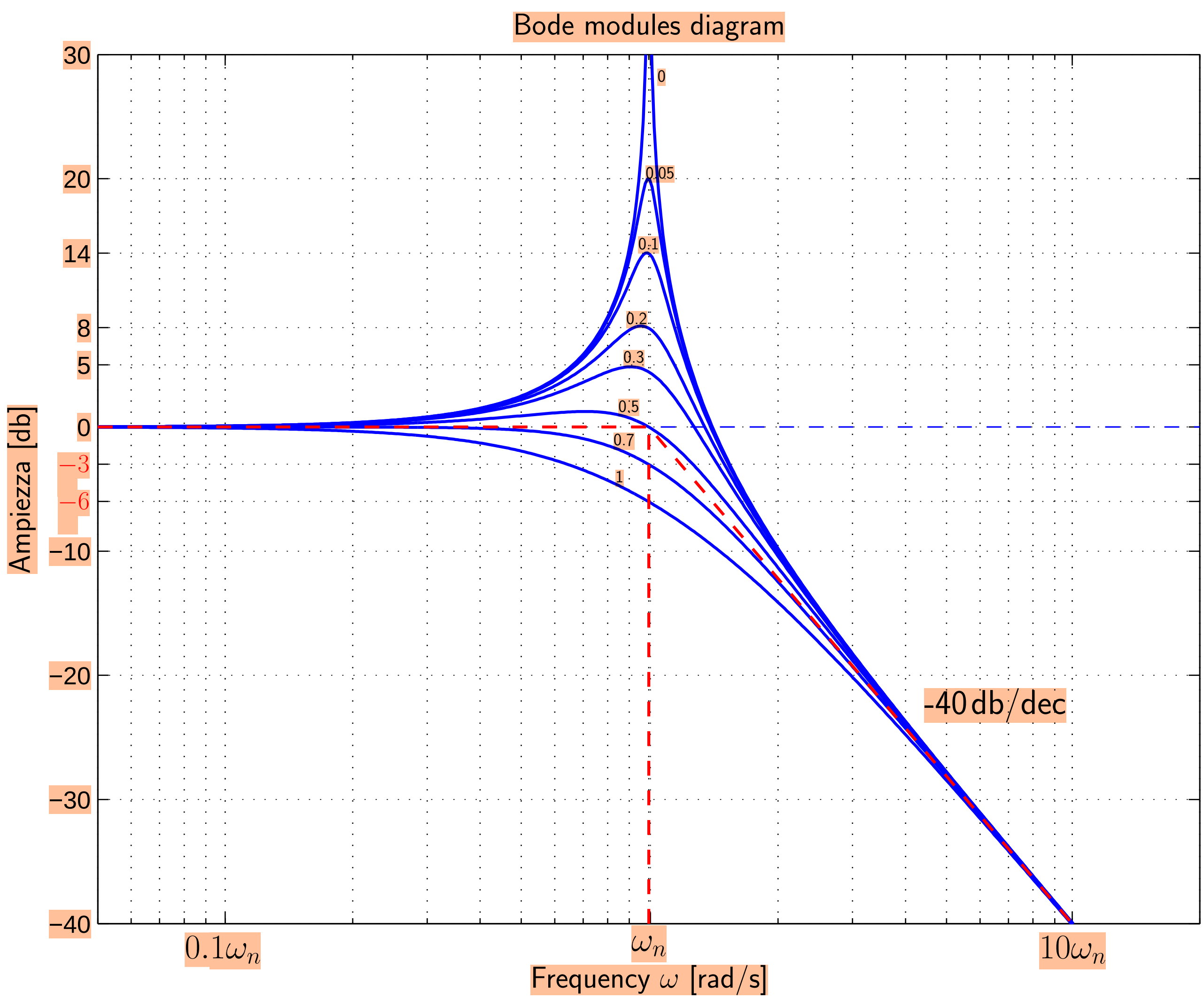
- **Complex conjugated poles** $(0 \le \delta < 1)$:

$$G(s) = \frac{1}{1 + \frac{2\delta}{\omega_n} s + \frac{s^2}{\omega_n^2}} \qquad \to \qquad G(j\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + j\,2\,\delta\,\frac{\omega}{\omega_n}}$$

- Module and phase Bode diagrams:

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\,\delta^2\,\frac{\omega^2}{\omega_n^2}}}, \qquad \arg G(j\omega) = -\arctan \frac{2\,\delta\,\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

- Bode module diagram for $\delta \in \{0,\ 0.05,\ 0.1,\ 0.2,\ 0.3,\ 0.5,\ 0.7,\ 1\}$:



- The -40 db/decade slope is usually denoted with the "-2" symbol.

- For small values of δ the real diagram deviates significantly from the asymptotic one. In particular, for $\delta = 0$ the deviation is infinite.

- The amplitude diagram has the following properties:

  1) for $0 \leq \delta \leq 1/\sqrt{2}$ the diagram has a maximum;
  2) for $0 \leq \delta \leq 1/2$ the diagram intersects the axis to the right of point $\omega = \omega_n$;
  3) for $1/2 \leq \delta \leq 1/\sqrt{2}$ the diagram intersects the axis to the left of point $\omega = \omega_n$;
  4) for $1/\sqrt{2} \leq \delta \leq 1$ the diagram does not intersect the abscissa axis and therefore it is completely located below its asymptotic approximation.

- **Resonance frequency** ω_R. Let us denote $u = \omega/\omega_n$: The maximum of the module function corresponds to a minimum of the following function

$$(1-u^2)^2 + 4\,\delta^2\,u^2$$

Computing the derivative and setting it to zero, one obtains:

$$-4\,(1-u^2)\,u + 8\,\delta^2\,u = 0$$

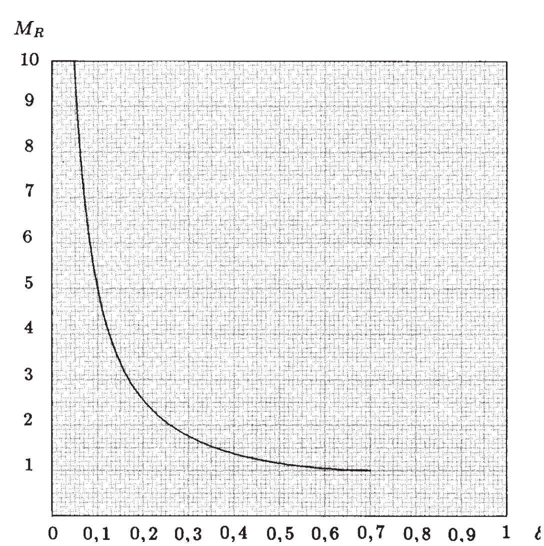
Neglecting the zero solution one obtains:

$$u_R = \sqrt{1-2\,\delta^2} \qquad \rightarrow \qquad \boxed{\omega_R = \omega_n\sqrt{1-2\,\delta^2}}$$

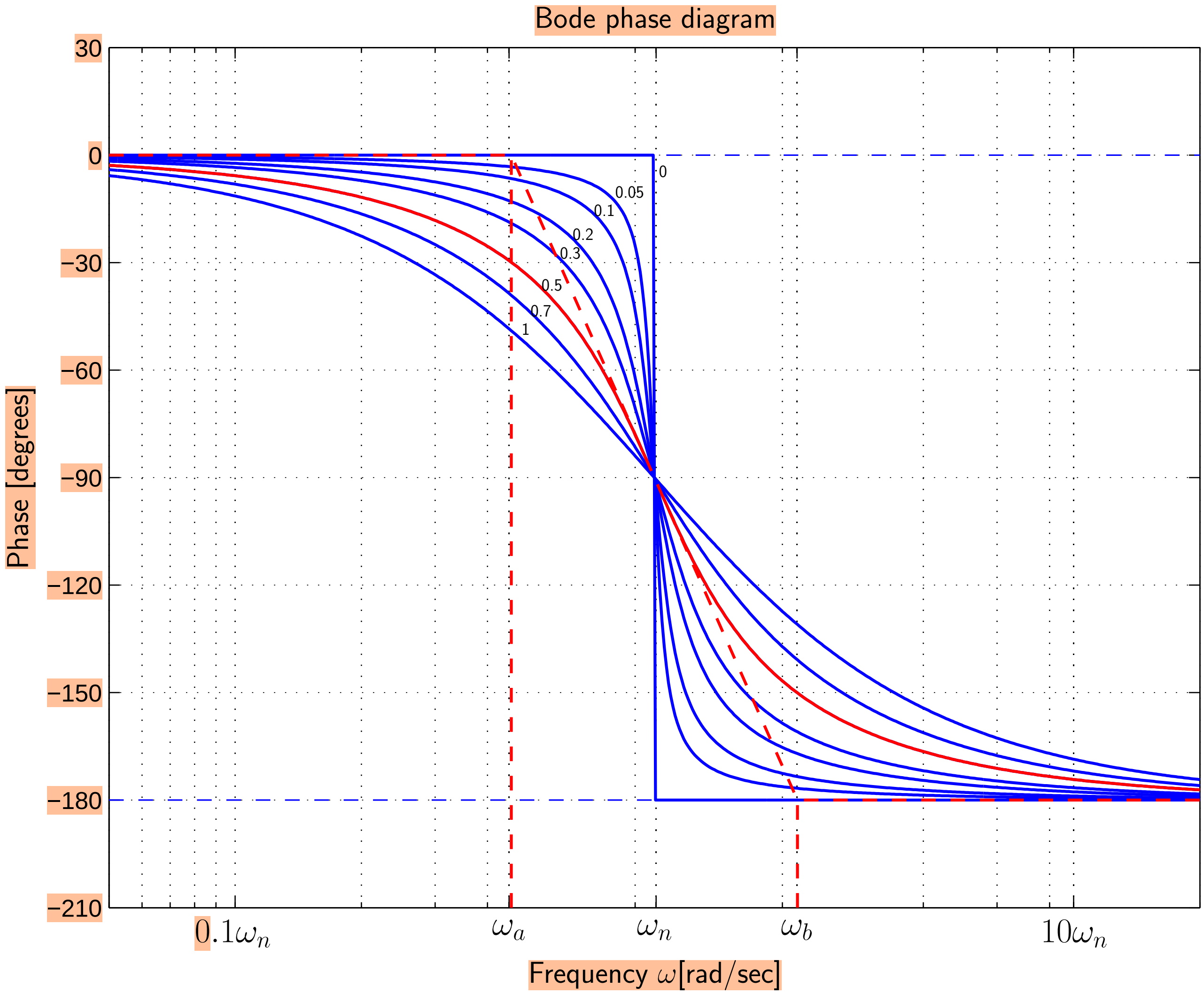
- **Resonance peak** M_R: it is the module of the frequency response function at the frequency ω_R:

$$M_R = \frac{1}{\sqrt{(1-1-2\,\delta^2)^2 + 4\,\delta^2\,(1-2\,\delta^2)}}$$

$$\boxed{M_R = \frac{1}{2\,\delta\sqrt{1-\delta^2}}}$$



- Bode phase diagram:



- The frequencies ω_a, ω_b and ω_n are related as follows:

$$\boxed{\frac{\omega_n}{\omega_a} = \frac{\omega_b}{\omega_n} = e^{\frac{\pi}{2}\delta} = 4,81^\delta}$$

- The asymptotic phase diagram changes its slope at frequencies:

$$\boxed{\omega_a = \frac{\omega_n}{4.81^\delta}, \qquad \omega_b = 4.81^\delta \omega_n}$$

• The damping coefficient δ can be calculated as follows:

1) from the resonance peak M_R:

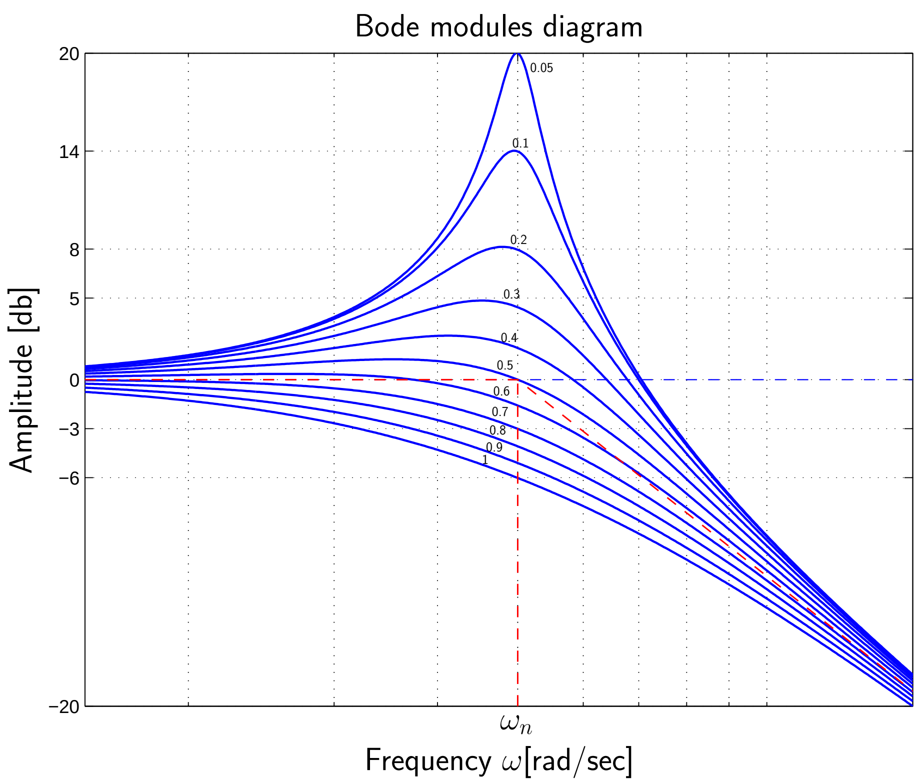
$$M_R = \frac{1}{2\delta\sqrt{1-\delta^2}} \qquad \rightarrow \qquad \boxed{\delta = \sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{1}{M_R^2}}\right)}}$$

2) from the critical frequencies ω_a and ω_b:

$$\frac{\omega_b}{\omega_a} = e^{\pi\delta} \qquad \rightarrow \qquad \boxed{\delta = \frac{1}{\pi}\ln\frac{\omega_b}{\omega_a}}$$

3) from the gain M_{ω_n} of function $G(j\omega)$ at frequency $\omega = \omega_n$:

$$M_{\omega_n} = G(j\omega)\big|_{\omega=\omega_n} = \frac{1}{2\delta} \qquad \rightarrow \qquad \boxed{\delta = \frac{1}{2M_{\omega_n}}}$$

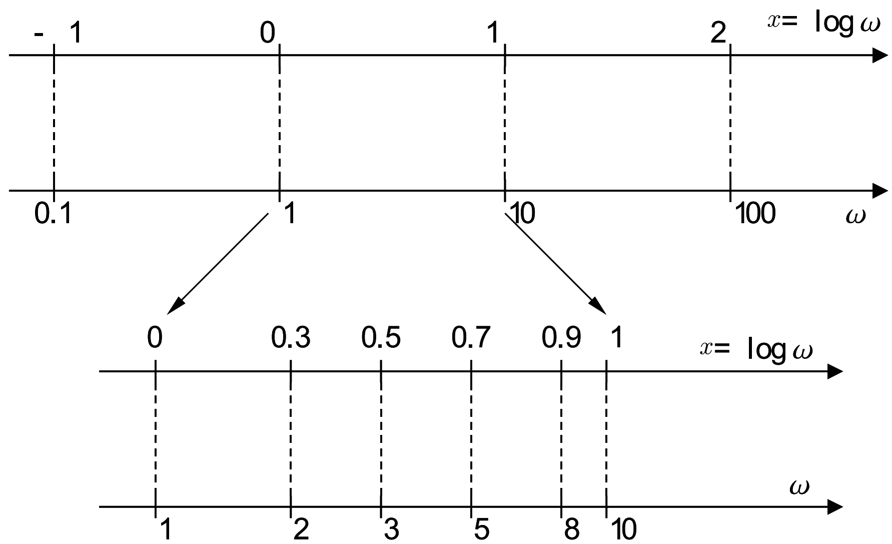


• If the static gain $G(0)$ is not unitary, the resonance peak M_R is defined as the ratio between the maximum value M_{max} and the static gain $M_0 = G(0)$:

$$M_R = \frac{M_{max}}{M_0}$$

Axes in the Bode diagrams

- The Bode diagrams use a 10-base logarithmic scale for the horizontal axis. Along the horizontal axis are usually reported the real values of the considered frequencies ω, not their logarithmic values.



- For the qualitative drawing of the diagrams it is worth to remember the following values: $\log_{10} 2 \simeq 0.3$, $\log_{10} 3 \simeq 0.5$, $\log_{10} 5 \simeq 0.7$, $\log_{10} 8 \simeq 0.9$

- The vertical axis in the amplitude diagram is graduated in decibels (db):

$$A|_{\text{db}} \overset{def}{=} 20 \log_{10} A$$

  Using this scale, the slopes of the asymptotic Bode module diagrams are ± 20 db/decade, ± 40 db/decade, etc. For convenience, these slopes are indicated with the symbols ± 1, ± 2, etc., respectively.

- In the Bode phase diagrams the vertical axis can be given both in radians or in degrees. Note: the phase diagrams can be shifted up or down by integer multiples of 2π, or 360^o, without changing the meaning of the given Bode phase diagram.

"Qualitative" drawing of the asymptotic Bode diagrams

First method: sum of the Bode diagrams of the components.

a) Function $G(s)$ is put in the "time constants factorized form":

$$G(s) = \frac{10(s-1)}{s(s+1)(s^2+8s+25)} \qquad \to \qquad G(s) = -\frac{10}{25}\frac{(1-s)}{s(1+s)(1+\frac{8s}{25}+\frac{s^2}{25})}$$

b) The asymptotic Bode diagrams of all the components are plotted:

$$K = -\frac{10}{25}, \qquad G_1(s) = (1-s), \qquad G_2(s) = \frac{1}{s}, \qquad G_3(s) = \frac{1}{(1+s)}, \qquad G_4(s) = \frac{1}{(1+\frac{8s}{25}+\frac{s^2}{25})}$$

c) The Bode diagrams of all the components are added.

The contribution of the term K is constant: $|K| = -7.96$ db and $\arg K = -\pi$.

The unstable zero $(1-s)$ and the stable pole $(1+s)^{-1}$ change slope at the same frequency $\omega = 1$ and therefore they provide two equal and opposite terms in the mobule diagram. Their contributions in the phase diagram are added: the overall width for $\omega \to \infty$ is $-\pi$.

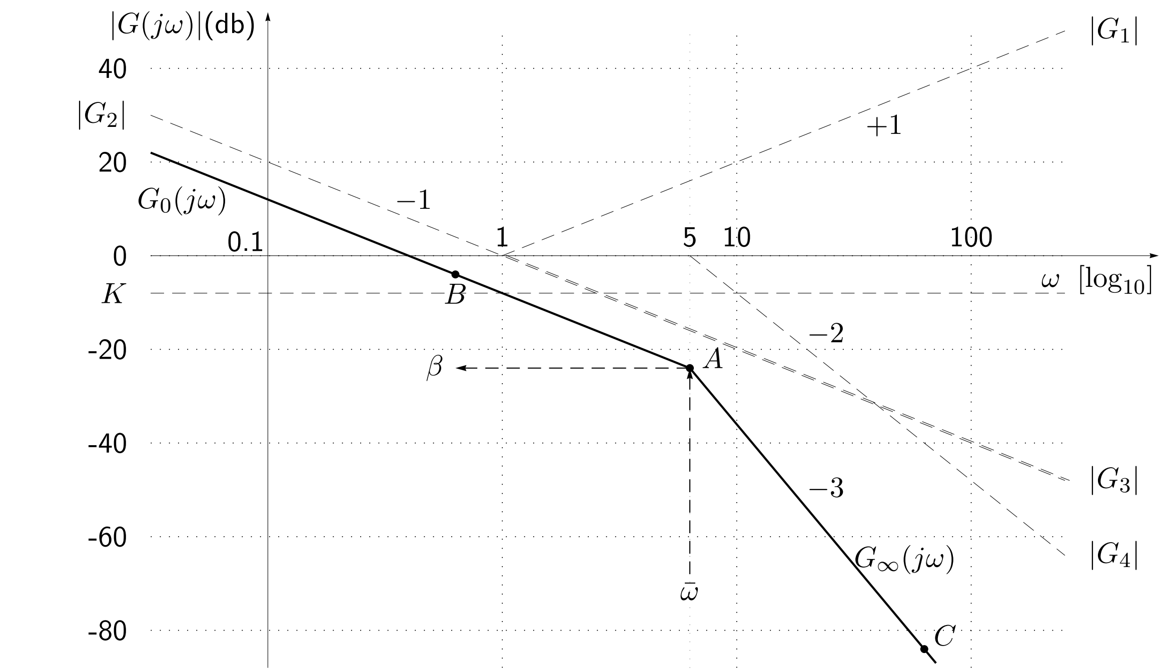
The two complex conjugate poles $(1+\frac{8s}{25}+\frac{s^2}{25})^{-1}$ show on the asymptotic module diagram an attenuation of -40db/dec starting from the frequency $\omega_n = 5$. The contribution to the phase diagram is negative and equal to $-\pi$ when ω tends to infinity. The asymptotic phase diagram change its slope at the following frequencies:

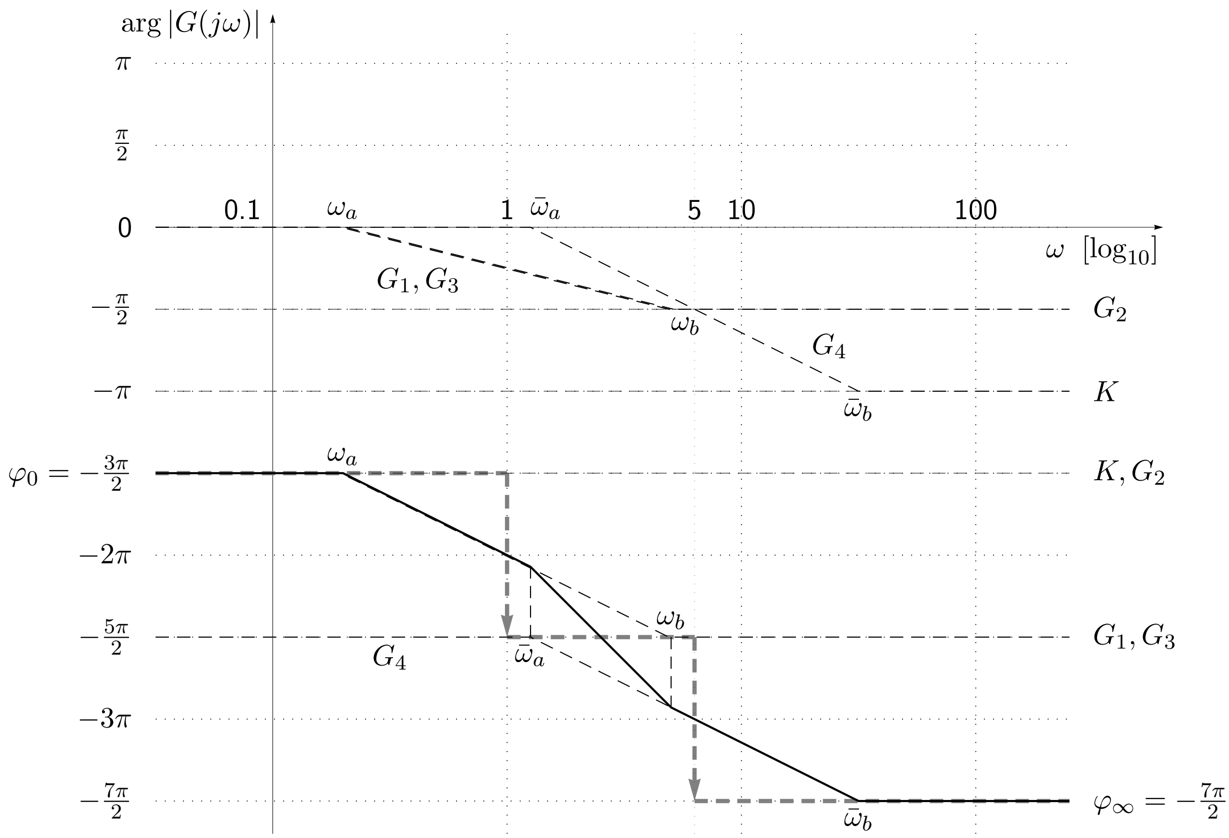
$$\omega_a = \frac{1}{4.81}, \qquad \omega_b = 4.81, \qquad \bar{\omega}_a = \frac{\omega_n}{4.81^\delta}, \qquad \bar{\omega}_b = \omega_n 4.81^\delta$$

where $\delta = 0.8$ is the damping coefficient of the two complex conjugate poles.

The difficulty of using this method lies on the fact that the sum of the individual contributions is not always easy.

Asymptotic module and phase Bode diagrams of function $G(s)$:





Approximate functions

- In the Bode and Nyquist diagrams the frequency behavior of a generic function $G(s)$ for $s \to 0^+$ and for $s \to \infty$ can be studied referring to the approximate functions $G_0(s)$ and $G_\infty(s)$.
- Consider, for example, the following function:

$$G(s) = \frac{10(s-1)}{s(s+1)(s^2+8s+25)}$$

- The *approximate function* $G_0(s)$ is obtained from $G(s)$ when $s \simeq 0$, that is neglecting all the terms in s except the poles or the zeros in the origin:

$$G_0(s) = G(s)|_{s\simeq 0} = \left.\frac{10(\cancel{s}-1)}{s(\cancel{s}+1)(\cancel{s}^2+8\cancel{s}+25)}\right|_{s\simeq 0} = \frac{-10}{25s}$$

- The approximate function $G_0(s)$ always has the structure $G_0(s) = \frac{K_0}{s^h}$, where h is the "type" of system $G(s)$, that is the number of poles of $G(s)$ in the origin.
- The *approximate function* $G_\infty(s)$ is obtained from $G(s)$ when $s \simeq \infty$, that is considering for each element of function $G(s)$ only the term with the highest degree in s:

$$G_\infty(s) = G(s)|_{s\simeq\infty} = \left.\frac{10(s-\cancel{1})}{s(s+\cancel{1})(s^2+8\cancel{s}+\cancel{25})}\right|_{s\simeq\infty} = \frac{10}{s^3}.$$

- The approximate function $G_\infty(s)$ always has the structure $G_\infty(s) = \frac{K_p}{s^r}$, where $r = n - m$ is the relative degree of function $G(s)$.
- Using the approximate functions $G_0(s)$ and $G_\infty(s)$ it is easy to calculate the module and the phase of the frequency response function when $\omega = 0$ and $\omega = \infty$:

$$G_0(s) = \frac{-10}{25s} \quad \Rightarrow \quad G_0(j\omega) = \begin{cases} |G_0(j\omega)| = \frac{10}{25\omega} \\ \varphi_0 = -\frac{3\pi}{2} \end{cases}$$

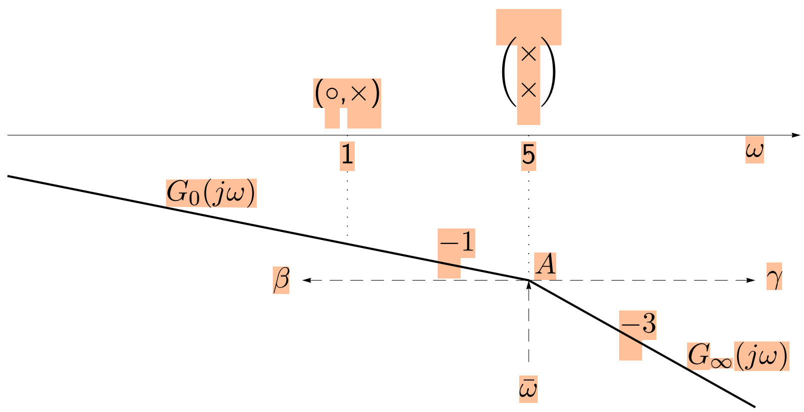
$$G_\infty(s) = \frac{10}{s^3} \quad \Rightarrow \quad G_\infty(j\omega) = \begin{cases} |G_\infty(j\omega)| = \frac{10}{\omega^3} \\ \varphi_\infty = -\frac{3\pi}{2} \end{cases}$$

Second method: "direct drawing"

Bode module diagram

a) A factorized function $G(s)$ clearly shows the frequencies at which the Bode module diagram changes its slope. These frequencies coincide with the modules of the real zeros and the real poles, and the natural frequencies ω_n of the complex conjugate poles and zeros of function $G(s)$. For the considered example we have $\omega = 1$ and $\omega = 5$.

b) Knowing that all the zeros (real or complex conjugates) determine an increase in the slope of the module diagram (+1 and +2, respectively) and that, conversely, all the poles (real or complex conjugates) determine a decrease in the slope of the module diagram (-1 and -2,m respectively), it is clear that the "shape" of the asymptotic module diagram is known "a priori". For the considered example we have:



At frequency $\omega = 1$ there is no change in the slope because at that frequency the pole and the zero eliminates each other.

c) The "vertical" position of the diagram can be determined as follows:

1) If function $G(s)$ is of type 0, the vertical position can be determined computing the static gain $G(0)$.

2) If function $G(s)$ is of type 1, or generally of type h, the vertical position of the asymptotic diagram can be determined computing the position of a particular point A belonging to the asymptotic diagram. This calculation can easily done by using the approximate functions $G_0(s)$ and $G_\infty(s)$.

To calculate the module β of point A at frequency $\bar{\omega}=5$, for example, one can use the approximate function $G_0(s)$:

$$\beta = |G_0(s)|_{s=j\bar{\omega}} = \left|\frac{-10}{25s}\right|_{s=j5} = \frac{2}{25} = -21.94 \text{ db}.$$

In this case, the same value can also be obtained using the approximate function $G_\infty(s)$:

$$\gamma = |G_\infty(s)|_{s=j\bar{\omega}} = \left|\frac{10}{s^3}\right|_{s=j5} = \frac{2}{25} = -21.94 \text{ db}.$$

d) Then, starting from point A and knowing the "general shape" of the asymptotic diagram, it is easy to plot the overall asymptotic diagram by plotting each section of the diagram with its own slope.

Let us consider, for example, the first segment of the asymptotic diagram of the considered system $G(s)$. Starting from point A, the slope of the first segment can be determined plotting a B point obtained, starting from A, decreasing a decade in frequency and increasing 20 db in amplitude: $B=(0.5,\ \beta+20)$. Then plot a straight line through points A and B.

In the same way one can plot the slopes of the sections that follow the point A. For the considered system, the slope of the second segment is -3. The slope of this segment can be determined plotting a point C obtained, starting from A, increasing a decade in frequency and decreasing 60 db in amplitude: $C=(50,\ \beta-60)$.

Diagram of the phases

The asymptotic phase diagram can be plotted "directly" acting as follows.

1) Find the starting phase φ_0 of the asymptotic diagram by using the approximate function $G_0(s)$:

$$\varphi_0 = \arg G_0(j\omega) = \arg\left(\frac{-10}{25s}\right)_{s=j\omega} = -\frac{3\pi}{2}.$$

The initial phase φ_0 is inclusive of the negative sign of the constant K and of the constant phase $-\frac{\pi}{2}$ introduced by the pole in the origin.

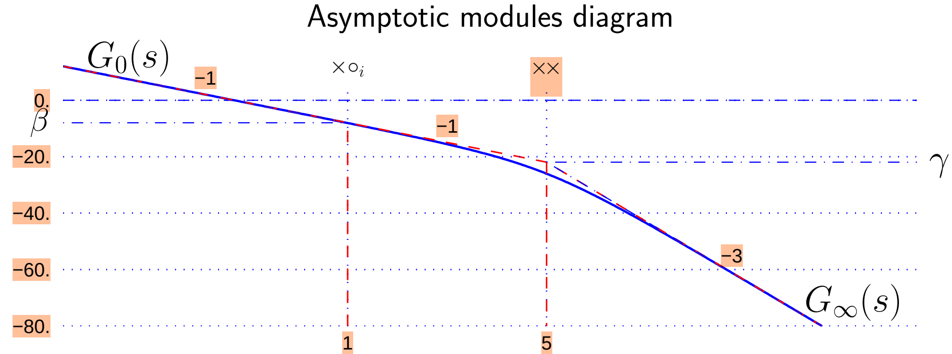
2) Starting from phase φ_0 one can plot a ***stepped diagram*** whose discontinuity points coincide with the critical frequencies of function $G(s)$ function. The amplitude of each discontinuity is equal to the phase variation $\Delta\varphi_i$ introduced by the dynamic term that generates the discontinuity.

The phase variations $\Delta\varphi_i$ are always a multiple of $\frac{\pi}{2}$ and can be both positive and negative depending on the type (pole or zero) and the stability (stable or unstable) of the considered dynamic term.

For the considered system $G(s)$ the first two terms are the stable pole $(s+1)^{-1}$ and the unstable zero $(s-1)$, both at frequency $\omega = 1$. The two terms introduce two negative phase variation $-\frac{\pi}{2}$ which are plotted starting from the initial phase $\varphi = -\frac{3\pi}{2}$. The two stable complex conjugate poles $(1+\frac{8s}{25}+\frac{s^2}{25})^{-1}$ introduce a phase variation $-\pi$ at frequency $\omega_n = 5$. This phase variation must be plotted in the range $[-\frac{5\pi}{2}, -\frac{7\pi}{2}]$.

3) The asymptotic phase diagram is finally obtained from the *stepped diagram* substituting for each discontinuity the specific asymptotic interpolation of the dynamic element acting at that frequency: a) for real poles and real zeros the values $\omega_a = \bar{\omega}4.81$ and $\omega_b = 4.81\bar{\omega}$ are used; b) for complex conjugate poles or zeros, the values $\omega_a = \bar{\omega}4.81^{\delta}$ and $\omega_b = 4.81^{\delta}\bar{\omega}$ are used.

Asymptotic modules diagram



Step phase diagram