Direct methods for the synthesis of PID compensators: analytical and graphical design

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***Abstract*—In this paper direct methods for the synthesis of continuous and discrete PID compensators for a robust control are proposed. Analytical and graphical techniques on the Nyquist plane to meet specifications on phase margin, gain margin and crossover frequency of the close-loop system are presented. Numerical examples demonstrate the effectiveness and the simplicity of the proposed approach, that can be useful both on educational and industrial applications.**

I. INTRODUCTION

Nowadays compensators are often implemented by microprocessors and calculations are done at discrete-time. The main reasons of this choice are that the digital control is less expensive to install and to maintain, more reliable, easy to manipulate and flexible than the analog control. However in control textbooks discrete-time design receives far less attention than the continuous-time control design [1]-[3]. Moreover knowledge on discrete-time control theory is not so widespread as continuous-time control theory. The discrete-time design methods are classified as indirect or direct. In the first case a continuous-time compensator is designed from a continuous-time model of the system and discretized for a discrete-time implementation. These methods require limited knowledge of discrete control and are widely used in educational and industrial environments. However the discretization of the continuous-time control system creates new phenomena not present in the original continuous-time control system [4]. In this paper a direct approach for the design of the proportional-integral-derivative (PID) regulators for robust control easy to teach and to apply is proposed. The tuning of regulator parameters leads to achieve steady-state requirements, phase margin, gain margin and gain or phase crossover specifications. In classical control design, the gain and the phase margins are important frequency-domain measures used to assess robustness and performance, while the gain crossover frequency affects the rise time and the bandwidth of the close-loop system [5]. In continuous-time domain several methods have been developed in order to meet these specifications. They can be divided in approximated and exact methods. The first ones are normally based on numerical or graphical trial-and-error solutions or fuzzy neural network (FNN). They are widely used in industrial environment because provide a quite good tuning of the compensator parameters using automated algorithms. However they are usually not easy for teaching and for searching the optimal solution.

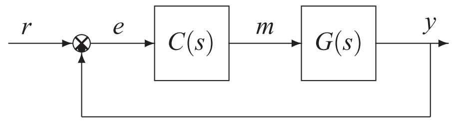

Fig. 1. Unity feedback control structure.

A unified closed-form solution to exactly satisfy gain and phase margins specifications by using a continuous-time PID controllers has been recently presented [6]. This paper presents a new graphical solution for continuous-time PID compensator design that can be done on the Nyquist diagram of the plant to be controlled. Both the numerical and the graphical solutions are also extended to the design of discrete-time PID compensators. The similarity of the continuous and discrete design methods and the simplicity of the graphical solutions allow an easy use of the discrete direct procedure.

In Section II, the graphical characteristics of the continuous-time PID compensator and the design method to meet the given frequency domain specifications are given. In Section III the general structure and the properties of the discrete-time PID compensator are described and the direct synthesis in discrete-time domain is presented. Numerical examples and the comparison with other methods end the paper.

II. THE CONTINUOUS-TIME CASE

Consider the block-diagram of the continuous-time system shown in Fig. 1, where $G(s)$ denotes the transfer function of the LTI plant to be controlled and $C(s)$ is the PID compensator to be design. The classical form of the transfer function $C(s)$ is the following

$$C(s) = K_P + sK_D + \frac{K_I}{s}, \tag{1}$$

where the proportional, derivative and integrative terms K_P, K_D and K_I are supposed to be real and positive. The frequency response of $C(s)$

$$C(j\omega) = K_P + j\left(\omega K_D - \frac{K_I}{\omega}\right), \tag{2}$$

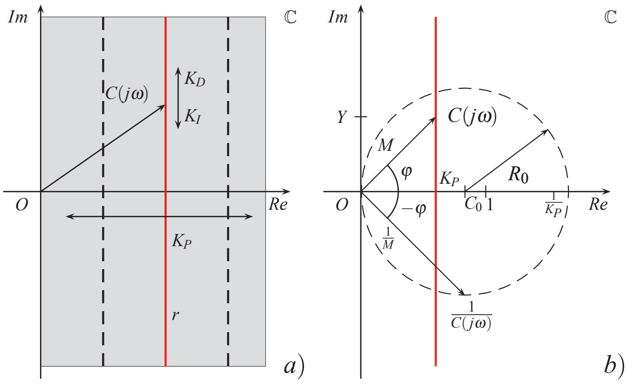

Fig. 2. Nyquist plot of functions $C(j\omega)$ and $C^{-1}(j\omega)$.

can be written as

$$C(j\omega) = X + jY(\omega),$$

where $X = K_P$ and $Y(\omega) = \omega K_D - \frac{K_I}{\omega}$. The parameter X is always positive, while the function $Y(\omega)$ is positive for $\omega > \sqrt{\frac{K_I}{K_D}}$ and negative for $0 < \omega < \sqrt{\frac{K_I}{K_D}}$.

Let $\mathscr{C}(K_P)$ and $\mathscr{C}^-(K_P)$ denote the sets of all the PID compensators $C(s)$ and $C(s)^{-1}$ having the same parameter K_P:

$$\mathscr{C}(K_P) = \left\{ C(s) \text{ as in (1)} \;\middle|\; K_I > 0, K_D > 0 \right\}, \tag{3}$$

$$\mathscr{C}^-(K_P) = \left\{ \frac{1}{C(s)} \;\middle|\; C(s) \in \mathscr{C}(K_P) \right\}. \tag{4}$$

It can be easily shown that the graphical representation of each element of $\mathscr{C}(K_P)$ on the Nyquist plane is a vertical straight line r which passes through point $(K_P, 0)$, see Fig. 2.

Property 1: *The shape of the frequency response of each element of set $\mathscr{C}^-(K_P)$ is a circle with center $C_0 = \frac{1}{2K_P}$ and radius $R_0 = \frac{1}{2K_P}$ which intersects the real axis at points 0 and $\frac{1}{K_P}$.*

Proof: The frequency response (2) can be written in polar form as follows

$$C(j\omega) = M(\omega) e^{j\varphi(\omega)},$$

where $M(\omega) = \frac{X}{\cos\varphi(\omega)}$ and $\varphi(\omega) = \arctan\frac{Y(\omega)}{X}$. It follows that $C^{-1}(j\omega)$ can be expressed in the form

$$\begin{aligned} C^{-1}(j\omega) &= \frac{1}{C(j\omega)} = \frac{\cos\varphi(\omega)}{X} e^{-j\varphi(\omega)} \\ &= \frac{1}{2X}[1+\cos(2\varphi(\omega))] - j\frac{1}{2X}\sin(2\varphi(\omega)) \\ &= \frac{1}{2X} + \frac{1}{2X} e^{-j2\varphi(\omega)}, \end{aligned}$$

for $Y(\omega) \in [-\infty, +\infty]$. The last relation clearly shows that the shape of $C^{-1}(j\omega)$ in the complex plane is a circle with center $C_0 = \frac{1}{2X}$ and radius $R_0 = \frac{1}{2X}$.

The aim of the design method proposed in this section is to chosen the parameters K_P, K_I and K_D to exactly satisfy the steady-state requirements on the tracking error and on the design specifications on the phase margin ϕ_m and the gain margin G_m at a given crossover frequency ω_0. The solution can be obtained using the modified Zigler-Nichols method, see [7] pp. $140-142$. This method finds the controller $C(s)$ that moves the point $A = M_A e^{j\varphi_A}$ of plant $G(s)$ at frequency ω_0 to a suitable point $B = M_B e^{j\varphi_B}$ of the the complex plane. Let $C(j\omega_0) = M_0 e^{j\varphi_0}$ denote the value of the frequency response $C(j\omega) = M(\omega)e^{j\varphi(\omega)}$ at frequency ω_0, where $M_0 = M(\omega_0)$ and $\varphi_0 = \varphi(\omega_0)$. Referring to Fig. 3, we say that *point A can be moved to point B* if a value $C(j\omega_0)$ exists such that $B = C(j\omega_0) \cdot A$, that is if and only if the following conditions hold:

$$M_B = M_A M_0, \qquad \varphi_B = \varphi_A + \varphi_0. \tag{5}$$

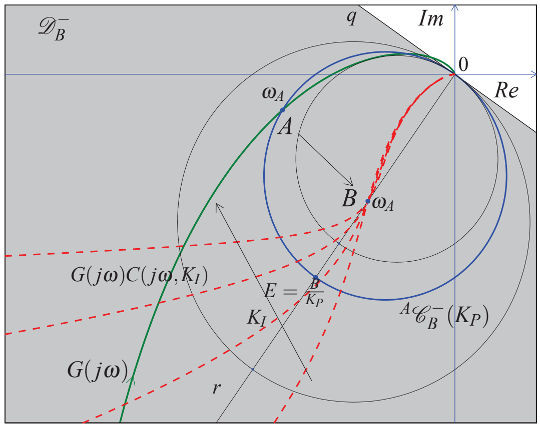

Fig. 3. Admissible domain $\mathscr{D}_B^-$ and graphical design of compensators $C(j\omega, K_I)$ moving point A to B.

Definition 1: Given a point $B \in \mathbb{C}$, let us define "*admissible domain of PID compensator $C(s)$ for reaching point B*" the set $\mathscr{D}_B^-$ defined as follows:

$$\mathscr{D}_B^- = \left\{ A \in \mathbb{C} \,\middle|\, \exists K_P, K_I, K_D > 0, \exists\, \omega \geq 0 : C(j\omega) \cdot A = B \right\}.$$

$\triangle$

It can be easily shown that the domain $\mathscr{D}_B^-$ on the Nyquist plane is the half-plane including point B delimited by the straight line q passing through point O and perpendicular to segment $\overline{B0}$, see the gray region in Fig. 3. In order to easily apply the modified Zigler-Nichols method to a large class of compensators using a generalize procedure, let us introduce the following *PID Inversion Formulae*.

Definition 2: ***(PID Inversion Formulae)*** Given two points $A = M_A e^{j\varphi_A}$ and $B = M_B e^{j\varphi_B}$ of the complex plane $\mathbb{C}$, the PID Inversion Formulae are defined as follows:

$$\left\{ \begin{aligned} X(A,B) &= \frac{M_B}{M_A}\cos(\varphi_B - \varphi_A), \\ Y(A,B) &= \frac{M_B}{M_A}\sin(\varphi_B - \varphi_A). \end{aligned} \right. \tag{6}$$

These formulae are similar to the ones used in [8] and to the *Inversion Formulae* introduced and used in [9] and [10] for the design of continuous and discrete lead and lag compensators.

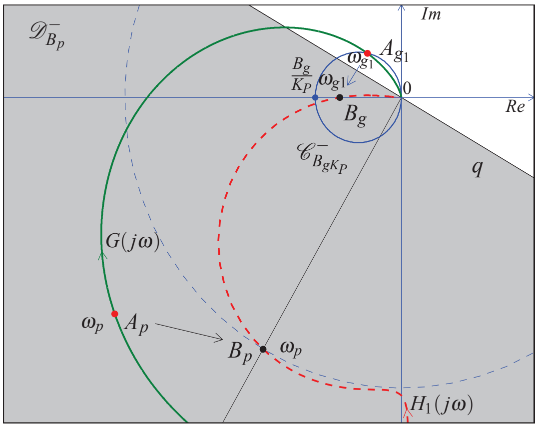

Fig. 4. Graphical solution of Design Problem B.

The following property holds.

Property 2: ***(From** A **to** B)* *Given a point $B \in \mathbb{C}$ and chosen a point $A = G(j\omega_A) \in \mathscr{D}^-_B$, the sets $C(s, K_D)$ and $C(s, K_I)$ of all the PID compensators $C(s)$ that move point A to point B is obtained from (1) using, respectively, the parameters*

$$K_P = X(A,B), \qquad K_I = K_D\omega_A^2 - Y(A,B)\omega_A, \tag{7}$$

for all $K_D > \frac{Y(A,B)}{\omega_A}$, or the parameters

$$K_P = X(A,B), \qquad K_D = \frac{Y(A,B)}{\omega_A} + \frac{K_I}{\omega_A^2}, \tag{8}$$

for all $K_I > 0$, where coefficients $X(A,B)$ and $Y(A,B)$ are obtained using (6).

Prop. 2 can be used to exactly satisfy design specifications on the phase margin or on the gain margin at a given crossover frequency ω_A if point B is chosen on the unitary circle or on the negative real axis. The point $A = G(j\omega_A) = M_A e^{j\varphi_A}$ on the frequency response of the plant at the desired crossover frequency ω_A can be moved by the controller $C(s)$ to the point B only if A belongs to admissible domain $\mathscr{D}^-_B$. Note that in (7) and (8) the position of points A and B affects only X and Y. In this way the parametrization introduced by the Inversion Formulae (6) can be easily applied to all the regulators whose frequency response can be expressed in the form $C(j\omega) = X(\omega) + jY(\omega)$.

From Prop. 1 the following graphical procedure to determine the parameter K_P can be easily proved.

Property 3: *The parameter K_P can be determined on the Nyquist plane as shown in Fig. 3: 1) draw the unique circle ${}^A\mathscr{C}^-_B$ passing through points A and O having its diameter on the straight line r which passes through points 0 and B; 2) the circle ${}^A\mathscr{C}^-_B$ intersects the straight line r in points O and $E = B/K_P$; 3) the parameter K_P is equal to the modulus of point B over the modulus of point E: $K_P = |B|/|E|$.*

A. Design Problems

Let us consider the case of given steady-state specifications that impose the value of the integrative term $K_I > 0$. This case occurs, for example, for type-0 systems and a design specification on velocity error, or for type-1 systems and a design specification on acceleration error. The fact that the integrative term K_I has been fixed does not reduce the amplitude of the admissible domain $\mathscr{D}^-_B$: this domain is still equal to the gray region shown in Fig. 3. The other parameters K_P and K_D of the regulator can be used to impose the phase margin ϕ_m and the gain crossover frequency ω_p.

Design Problem A: (K_I, ϕ_m, ω_p). Given the control scheme of Fig. 1, the transfer function $G(s)$, the steady-state specifications that impose the value of the integrative term $K_I > 0$ and design specifications on the phase margin ϕ_m and gain crossover frequency ω_p, design a PID compensator $C(s)$ such that the loop gain transfer function $C(j\omega)G(j\omega)$ passes through point $B_p = e^{j(\pi+\phi_m)}$ for $\omega = \omega_p$.

Solution A: *If point $A = G(j\omega_A)$ belongs to the admissible domain $\mathscr{D}^-_B$ shown in Fig. 3, the solution follows directly from (8) with K_I imposed by the steady-state specifications.*

Let us now consider the case of steady-state specifications that do not constrain the value of K_I. This degree of freedom in the Prop. 2 can be used to satisfy another specification. Let us now consider for example the following design problem.

Design Problem B: $(\phi_m\ ,\ G_m\ ,\ \omega_p)$. Given the control scheme of Fig. 1, the transfer function $G(s)$ and design specifications on the phase margin ϕ_m, gain margin G_m and gain crossover frequency ω_p, design a PID compensator $C_p(s)$ such that the loop gain transfer function $C_p(j\omega)G(j\omega)$ passes through point $B_p = e^{j(\pi+\phi_m)}$ for $\omega = \omega_p$ and passes through point $B_g = -1/G_m$.

Solution B: *Step 1. Draw the admissible domain $\mathscr{D}^-_{B_p}$ of point $B_p = e^{j(\pi+\phi_m)}$ delimited by the straight line q passing through point O and perpendicular to segment $\overline{B_pO}$, see Fig. 4*
Step 2. Check whether the point $A_p = G(j\omega_p)$ belongs to $\mathscr{D}^-_{B_p}$. If not the problem has not acceptable solutions.
Step 3. Determine the parameters $X_p = X(A_p, B_p)$, $Y_p = Y(A_p, B_p)$ using the inversion formulas (6).
Step 4. Draw the circle $\mathscr{C}^-_{B_gK_P}(j\omega)$ having its diameter on the segment defined by points O and $\frac{B_g}{K_P}$, where it is $B_g = -1/G_m$ and $K_P = X_p$. If there are not intersections points of $\mathscr{C}^-_{B_gK_P}(j\omega)$ with $G(j\omega)$ the problem has not acceptable solutions. Otherwise let A_{gi} denote the intersections points of circle $\mathscr{C}^-_{B_gK_P}(j\omega)$ with $G(j\omega)$ at frequencies ω_{gi}. These points can be equivalently obtained solving the equation

$$K_P = X_g(\omega_g) = \frac{M_{B_g}}{M_{A_g(\omega_g)}}\cos(\varphi_{B_g} - \varphi_{A_g}(\omega_g)). \tag{9}$$

Step 5. For $\omega_g = \omega_{gi}$, calculate K_D and K_I using the following *equations*

$$K_D = \frac{Y_p\omega_p - Y_g\omega_g}{\omega_p^2 - \omega_g^2}, \quad (10)$$

$$K_I = \frac{Y_p\omega_p\omega_g^2 - Y_g\omega_g\omega_p^2}{\omega_p^2 - \omega_g^2}, \quad (11)$$

where $X_g = X(A_g,B_g)$ and $Y_g = Y(A_g,B_g)$ are obtained using (6). The solutions are acceptable only if K_D and K_I are real and positive.

Proof The design specifications define the position of points $B_p = e^{j(\pi+\phi_m)}$, $A_p = G(j\omega_p)$ and $B_g = -1/G_m$. According to Prop. 2, the compensators $C_p(s,K_D)$ which move point $A_p \in \mathscr{D}^-_{B_p}$ to point B_p are obtained using the parameters K_P and K_I in (7). The free parameter K_D can now be used to force the loop gain frequency response $C_p(j\omega,K_D)G(j\omega)$ to pass through point B_g. This condition can be satisfied only if a frequency ω_g exists such that the compensator $C_p(s,K_D)$ moves point $A_g = G(j\omega_g) \in \mathscr{D}^-_{B_g}$ to point B_g:

$$G(j\omega_g)C_p(j\omega_g,K_D) = B_g, \quad (12)$$

where $C_p(j\omega_g,K_D)$ is the PID compensator (1) with the value of parameter $K_P = X_p$, that is only if (9) holds. Relation (12) can also be rewritten as follows

$$G(j\omega) = \frac{B_g}{C_p(j\omega,K_D)} = \mathscr{C}^-_{B_gK_P}(j\omega), \quad (13)$$

with $\omega = \omega_g$, and therefore it can be solved graphically on the Nyquist plane by finding the intersections ω_g of $G(j\omega)$ with $\mathscr{C}^-_{B_gK_P}(j\omega)$. The frequencies ω_g satisfying (9) are acceptable only if the compensator $C_g(s,K_D)$ which moves point A_g to point B_g, obtained using Prop. 2, is equal to the compensator $C_p(s,K_D)$. This condition is satisfied only if the two compensators share the same K_I and K_D, that is only if

$$K_I = \omega_p^2 K_D - Y_p\omega_p = \omega_g^2 K_D - Y_g\omega_g. \quad (14)$$

Solving (14) with respect to K_D leads to the expression of K_D given in (10), which can be substituted in (14) obtaining (11). The solutions are acceptable only if $K_P,K_D,K_I > 0$.

Remark 1: The proposed graphical solution can be easily modified to meet design specifications on phase margin ϕ_m, gain margin G_m and phase crossover frequency ω_g.

III. The discrete-time case

Referring to the block scheme of Fig. 5, $HG(z)$ is the discrete system to be controlled,

$$HG(z) = \mathscr{Z}[H_0(s)\,G(s)],$$

where $H_0(s) = \frac{1-e^{-Ts}}{s}$ is the zero-order hold, while $C_d(z)$ is the discrete-time PID compensator having the following structure:

$$C_d(z) = \overline{K}_P + \overline{K}_D\frac{z-1}{z+1} + \overline{K}_I\frac{z+1}{z-1}. \quad (15)$$

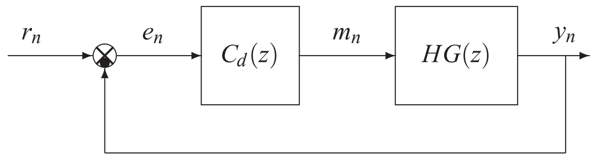

Fig. 5. The considered block scheme for the discrete-time case.

Let us now design the regulator $C_d(z)$ in order to meet the same design specifications considered in Sec. II for the continuous-time case.

The frequency response of (15) for $\omega \in [0, \frac{\pi}{T}]$ and sampling period T is

$$C_d(e^{j\omega T}) = X + jY(\omega,T), \quad (16)$$

where

$$X = \overline{K}_P, \qquad Y = \overline{K}_D\Omega(\omega) - \frac{\overline{K}_I}{\Omega(\omega)}, \qquad \Omega(\omega) = \tan\frac{\omega T}{2}.$$

Let us define $\mathscr{C}_d(\overline{K}_P)$ as the set of all the PID compensators $C_d(z)$ having the same parameter $\overline{K}_P$

$$\mathscr{C}_d(\overline{K}_P) = \left\{ C_d(z) \text{ as in (15)} \;\middle|\; \overline{K}_I > 0, \overline{K}_D > 0 \right\}, \quad (17)$$

and let $\mathscr{C}^-_d(\overline{K}_P)$ denote the set of all the inverse functions $C_d(z)^{-1}$ having the same parameter $\overline{K}_P$:

$$\mathscr{C}^-_d(\overline{K}_P) = \left\{ \frac{1}{C_d(z)} \;\middle|\; C_d(z) \in \mathscr{C}_d(\overline{K}_P) \right\}. \quad (18)$$

As for the continuous-time case, the graphical representation of each element of $\mathscr{C}_d(\overline{K}_P)$ on the Nyquist plane is a vertical straight line which passes through point $(\overline{K}_P, 0)$. Moreover the Nyquist plot of each element of $\mathscr{C}^-_d(\overline{K}_P)$ is a circle with center $C_0 = \frac{1}{2K_P}$ and radius $R_0 = \frac{1}{2K_P}$.

It can be easily shown that the graphical representation of the admissible domain $\mathscr{D}^-_B$ for reaching point B on the Nyquist plane is equal to the domain considered for continuous-time case, see Fig. 3.

Property 4: ***(From** A **to** B**)*** *Given a point $B \in \mathbb{C}$ and chosen a point $A = HG(\omega_A,T) \in \mathscr{D}^-_B$ on the frequency response of the plant at frequency $\omega_A \in [0, \frac{\pi}{T}]$, the sets $C_d(z,\overline{K}_D)$ and $C_d(z,\overline{K}_I)$ of all the PID compensators $C_d(z)$ that move point A to point B are obtained from (15) using, respectively, the parameters*

$$\overline{K}_P = X(A,B), \qquad \overline{K}_I = \overline{K}_D\Omega_A^2 - Y(A,B)\Omega_A, \quad (19)$$

for all $\overline{K}_D > \frac{Y}{\Omega_A}$, or the parameters

$$\overline{K}_P = X(A,B), \qquad \overline{K}_D = \frac{Y(A,B)}{\Omega_A} + \frac{\overline{K}_I}{\Omega_A^2}, \quad (20)$$

for all $\overline{K}_I > 0$, where $X(A,B)$ and $Y(A,B)$ are obtained using the same inversion formulae defined for the continuous-time case (6) and $\Omega_A = \tan\frac{\omega_A T}{2}$.

Property 4 can be used to solve the following Design Problems C and D.

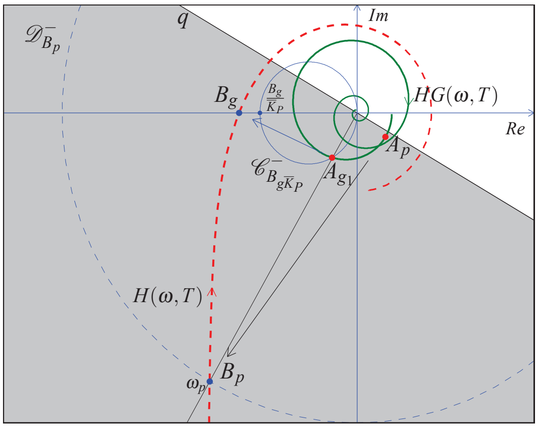

Fig. 6. Graphical solution of Design Problem D on the Nyquist plane.

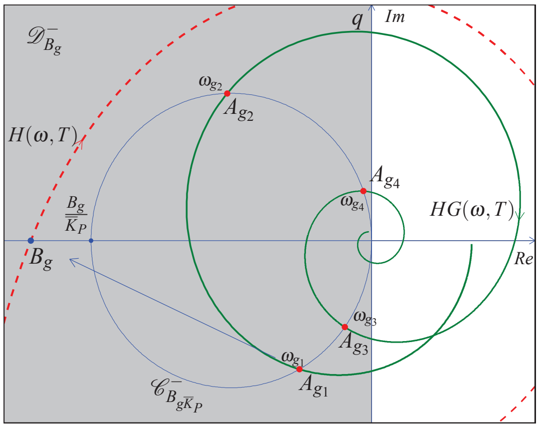

Fig. 7. Zoom of the graphical solution of Design Problem D: set of points $A_{gi} = (A_{g1}, A_{g2}, A_{g3}, A_{g4} \dots)$ that can be moved to point B_g by the controller.

Design Problem C: ($\overline{K}_I$, ϕ_m , ω_p). Given the control scheme of Fig. 5, the transfer function $HG(z)$, the steady-state specifications that impose the value of $\overline{K}_I$ and the design specifications on the phase margin ϕ_m and gain crossover frequency $\omega_p \in [0, \frac{\pi}{T}]$, design a discrete-time compensator $C_d(z)$ such that the loop gain transfer function $C_d(\omega,T)HG(\omega,T)$ passes through point $B = e^{j(\pi+\phi_m)}$ for $\omega = \omega_p$.

Solution C: *As for the continuous-time case, if the point $A_p = HG(\omega_p,T)$ belongs to the admissible domain $\mathscr{D}^-_B$ shown in Fig 6, the values of parameters $\overline{K}_P$ and $\overline{K}_D$ are the same which solve the Design Problem B and given by (20), with $\overline{K}_I > 0$ determined by the steady-state specifications.*

Design Problem D: (ϕ_m , G_m , ω_p). Given the control scheme of Fig. 5, the transfer function $HG(z)$ and the design specifications on the phase margin ϕ_m, gain margin G_m and gain crossover frequency $\omega_p \in [0, \frac{\pi}{T}]$, design a discrete compensator $C_d(z)$ such that the loop gain transfer function $C_d(\omega,T)HG(\omega,T)$ passes through point $B_p = e^{j(\pi+\phi_m)}$ for $\omega = \omega_p$ and passes through point $B_g = -1/G_m$.

Solution D: *Step 1. Draw the admissible domain $\mathscr{D}^-_{B_p}$ of point $B_p = e^{j(\pi+\phi_m)}$ defined by the straight line q passing through point O and perpendicular to segment $\overline{B_pO}$, see Fig. 6.*
Step 2. Check whether the point $A_p = HG(\omega_p,T)$ belongs to $\mathscr{D}^-_{B_p}$. If not the problem has not acceptable solutions.
Step 3. Determine the parameters $X_p = X(A_p,B_p)$, $Y_p = Y(A_p,B_p)$ using the inversion formulas (6).
Step 4. Draw the circle $\mathscr{C}^-_{B_g\overline{K}_P}$ having its diameter on the segment defined by points O and $\frac{B_g}{\overline{K}_P}$, where it is $B_g = -1/G_m$ and $\overline{K}_P = X_p$. If there are not intersections points of $\mathscr{C}^-_{B_g\overline{K}_P}$ with $HG(\omega,T)$, the problem has no acceptable solutions. Otherwise let A_{gi} denote the intersections points of circle $\mathscr{C}^-_{B_g\overline{K}_P}$ with $HG(\omega,T)$ at frequencies ω_{gi}. These points can also be obtained solving the following equation

$$\overline{K}_P = X_g(\omega_g) = \frac{M_{B_g}}{M_{A_g(\omega_g)}} \cos(\varphi_{B_g} - \varphi_{A_g}(\omega_g)), \qquad (21)$$

Step 5. For $\omega_g = \omega_{gi} \in [0, \frac{\pi}{T}]$, the set of all the compensators $C_d(z)$ which solve Design Problem D is obtained from (15) using the parameters

$$\overline{K}_P = X_p > 0, \qquad \overline{K}_D = \frac{Y_p\Omega_p - Y_g\Omega_g}{\Omega_p^2 - \Omega_g^2} > 0, \qquad (22)$$

$$\overline{K}_I = \frac{Y_p\Omega_p\Omega_g^2 - Y_g\Omega_g\Omega_p^2}{\Omega_p^2 - \Omega_g^2} > 0, \qquad (23)$$

where $\Omega_p = \tan\frac{\omega_p T}{2}$, $\Omega_g = \tan\frac{\omega_g T}{2}$ and parameters $X_p = X(A_p,B_p)$, $Y_p = Y(A_p,B_p)$, $X_g = X(A_g,B_g)$ and $Y_g = Y(A_g,B_g)$ are obtained using the inversion formulas (6), with $A_g = HG(\omega_g,T) = M_{A_g}(\omega_g)\, e^{j\varphi_{A_g}(\omega_g)}$. The solution is admissible only if the parameters $\overline{K}_I$ and $\overline{K}_D$ in (22) and (23) are real and positive.

The proof is quite similar to the one given in Solution B.

IV. Numerical Examples

Example 1. Let us consider the following plant

$$G(s) = \frac{0.7}{(s+0.3)(s^2+0.6s+1)}. \qquad (24)$$

a) Design the continuous-time PID compensator (1) satisfying the following design specifications: phase margin $\phi_m = 60^o$, gain margin $G_m = 4.5$ and gain crossover frequency $\omega_p = 0.91$.
b) Design the discrete-time PID compensator $C_d(z)$ for the discrete system $HG(z)$ with $T = 0.1$s to meet the following design specifications: velocity constant $K_v = 3$, phase margin $\phi_m = 60^o$ and gain crossover frequency $\omega_p = 0.91$.

Solution: a) Referring to Fig. 4, the point $A_p = G(j\omega_p) = 1.28e^{-j144^o}$ belongs to the admissible domain $\mathscr{D}^-_{B_p}$ for reaching point $B_p = e^{j240^o}$. From (6) it is $X_p = 0.714$ and $Y_p = 0.322$. The circle $\mathscr{C}^-_{B_gK_P}(j\omega)$ with the diameter $\left(O, \frac{B_g}{K_P}\right)$, $B_g = -1/G_m$ and $K_P = X_p$ intersects $G(j\omega)$ in point $A_{g1} = 0.196e^{j129^o}$ at frequency $\omega_{g1} = 1.68$. From (10) and (11) it is $K_D = 0.5953$ and $K_I = 0.1998$. The loop gain frequency response $H_1(j\omega)$

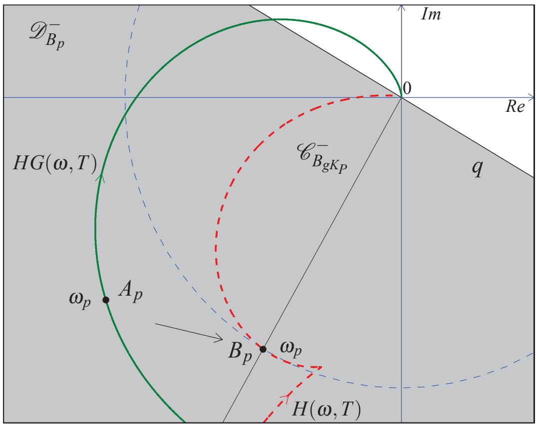

Fig. 8. Graphical solution of Design Problem C.

is the dashed red line shown in Fig. 4.
b) The discrete-time system to be controlled is

$$HG(z) = 10^{-4}\frac{1.14z^2+4.457z+1.09}{z^3-2.903z^2+2.817z-0.9139}. \quad (25)$$

The integral constant K_I is determined by the velocity constant requirement

$$K_v = \lim_{z\to 1}\frac{1-z^{-1}}{T}C_d(z)HG(z) = \frac{2\cdot 6.687\cdot \overline{K}_I}{0.1} = 3,$$

which leads to $\overline{K}_I = 0.0224$. The point $A_p = HG(e^{j\omega_p T}) = 1.28e^{-j147^o}$ belongs to the domain $\mathscr{D}^-_{B_p}$, see Fig. 8. From (6) it follows that $X(A,B) = 0.699$ and $Y(A,B) = 0.354$. Finally from (20) it is $\overline{K}_P = 0.699$ and $\overline{K}_D = 18.6$. The designed compensator is

$$C_d(z) = 0.699 + 18.6\frac{z-1}{z+1} + 0.0224\frac{z+1}{z-1}.$$

The correspondent loop gain frequency response $H(\omega,T) = HG(\omega,T)C_d(\omega,T)$ is the dashed red line in shown in Fig. 8.

Example 2. Given the following nonminimum phase plant proposed in [11]

$$G(s) = \frac{s^3-4s^2+s+2}{s^5+8s^4+32s^3+46s^2+46s+17}e^{-s}, \quad (26)$$

synthesize a discrete-time PID compensator $C_d(z)$ for system $HG(z)$ with $T = 0.3$s to meet the following design specifications: phase margin $\phi_m = 60^o$, gain margin $G_m = 2.5$ and gain crossover frequency $\omega_p = 0.2$.

Solution The point $A_p = HG(e^{j\omega_p T}) = 0.123e^{-j38.6^o}$ belongs to the domain $\mathscr{D}^-_{B_p}$, see Fig. 6. From (6) it follows that $X(A,B) = 1.21$ and $Y(A,B) = -8.03$. The intersections points of $\mathscr{C}^-_{B_g\overline{K}_P}$ with $HG(\omega,T)$ occur at frequencies $\omega_{gi} = (0.6\,,\,1.06\,,\,2.49\,,\,3.88,\ldots)$. One possible solution is obtained moving the point A_{g_1} at frequency (ω_{g1}) to point B_p using relations (22) and (23). The obtained designed parameters are $\overline{K}_P = 1.21$, $\overline{K}_D = 7.10$ and $\overline{K}_I = 0.247$. The corresponding loop gain frequency response $H(\omega,T)$ is plotted in dashed red line in Fig. 6 and 7.

V. COMPARISON WITH OTHER METHODS

One of the main advantages of the graphical solution presented in this paper over other graphical approaches, such as the one in [12], is that it can be easily determined in the complex plane by finding the intersections of the frequency response of the plant with particular design circles. This graphical solution can be easily obtained by students in valuation test without the use of computer. Moreover, the proposed method provides *all* the solutions of the control problem and not only a subset of all the solutions, as it happens in [12]. The proposed method has the advantage to avoid the double transformation as required by the classical indirect method and it is not based on trial-and-error procedure, such as the classical methods based on the use of Bode plot, see [5].

VI. CONCLUSIONS

In this paper, a new graphical procedure for the design of continuous-time PID regulators and new direct numerical and graphical methods for the design of discrete-time PID regulators for a robust control are presented. The solutions are based on the use of Inversion Formulae. The presented simulation results confirm the effectiveness of the proposed methods. The proposed procedures are very useful both on industrial and educational environments for the simplicity of the relations and the graphical solutions.

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