Direct method for digital lead-lag design: analytical and graphical solutions

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***Abstract*— The paper presents a direct digital method to the design of a discrete lead-lag compensator for a robust control. The design specifications on phase margin, gain margin and crossover frequency of the close-loop system can be obtained by using numerical and graphical solutions. The discrete design method proposed in the paper is also compared with the continuous-time case. Some numerical examples are presented. The proposed method can be useful both on educational and industrial environments.**

I. INTRODUCTION

Nowadays digital control of automatic systems is widely used for its benefits over analog control, including reduced parts count and greater flexibility. The discrete-time control design methods can be classified as indirect and direct. The first are widely used because require only limited knowledge of the discrete-time control theory and are based on the vast background of the continuous-time control. Conversely, direct design of classical discrete regulators receives far less attention than indirect design in control textbooks [1]-[3]. However, the discretization of a continuous-time control system creates new phenomena not present in the original continuous-time control system, such as considerable inaccuracies in the the locations of poles and zeros [4]. This paper presents a method for the direct design of lead-lag regulators for the robust control. The recent literature shows a renewed interest in the design of this type of regulators [5]. As known, PID controllers are the most widely used controllers in industry because of their simplicity. However in some cases lead-lag controllers, compared with PID regulators, lead to a better tradeoff between the static accuracy, system stability and insensibility to disturbance in frequency domain [6]. The three parameters of the presented second order lead-lag regulator can be synthesized in order to meet design specifications on the phase, the gain margins and the gain crossover frequency. In classical control design, the gain and the phase margins are important frequency-domain measures used to assess robustness and performance, while the gain crossover frequency affects the rise time and the bandwidth of the close-loop system [7]. A survey of different methods for the classical control design based on these frequency domain specifications can be found in [8]. The solution of this robust control design for lead-lag regulators leads to a set of nonlinear and coupled equations difficult to solve. In the literature has been recently proposed a new control technique

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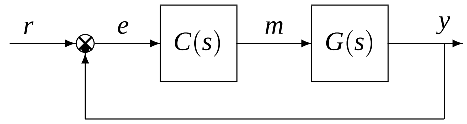

Fig. 1. Unity feedback control structure.

for the design of these classical compensators. It is based on the so called inversion formulae that express the frequency response of the compensator in polar form. This technique has been applied so far to the continuous and discrete lead and lag regulators [9] and to the continuous lead-lag compensators [10]. This paper presents the extension of this method to the design of discrete lead-lag compensators. An effective graphical method on the Nyquist plane is also presented. The similarity of the continuous and discrete design methods allows on easy use of the discrete direct procedure.

The paper is organized as follows. In Section II, the fundamental characteristics of the continuous-time lead-lag compensator and the design method to meet the given frequency domain specifications are recalled. In Section III the general structure and the properties of the discrete-time lead-lag compensator are described. In Section IV, the direct synthesis of the lead-lag controller in discrete-time domain is presented. Numerical examples and the comparison with other methods end the paper.

II. THE CONTINUOUS-TIME CASE

Consider the block diagram of the continuous-time system shown in Fig. 1 where $G(s)$ denotes the transfer function of the LTI plant to be controlled, which may include the gain and the integration terms required to meet the steady-state accuracy specifications and $C(s)$ denotes the lead-lag compensator to be design. The form of the considered regulator includes real or complex zeros and poles:

$$C(s) = \frac{s^2 + 2\gamma\delta\,\omega_n s + \omega_n^2}{s^2 + 2\delta\,\omega_n s + \omega_n^2}, \tag{1}$$

where the parameters γ, δ and ω_n are real and positive. The synthesis of these parameters does not change the static behavior of the controlled system, since the static gain of $C(s)$ is unity. The frequency response $C(j\omega)$ can be written as

$$C(j\omega) = \frac{1 + jX(\omega)}{1 + jY(\omega)}, \tag{2}$$

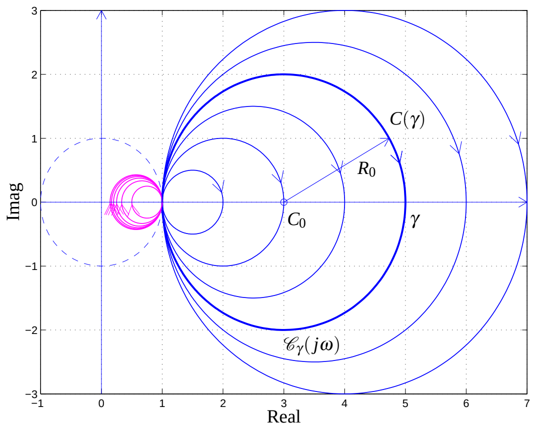

Fig. 2. Nyquist diagrams of $C(j\omega)$ when $\omega_n = 1$, ($\delta = 1.5$, $\gamma = [2:1:7]$, blue lines) and ($\delta = 1.5$, $\gamma = 1./[2:1:7]$, magenta lines). The thick blue line corresponds to $\delta = 1.5$ and $\gamma = 5$.

where

$$X(\omega) = \frac{2\gamma\delta\omega\,\omega_n}{\omega_n^2-\omega^2}, \qquad Y(\omega) = \frac{2\delta\omega\,\omega_n}{\omega_n^2-\omega^2}. \tag{3}$$

Since γ, δ and ω_n are assumed to be real and positive, $X(\omega)$ and $Y(\omega)$ are positive when $\omega < \omega_n$ and negative when $\omega > \omega_n$. Since the distance of the generic point $C(j\omega)$ from the point $C_0 = \frac{\gamma+1}{2}$ is constant and equal to $R_0 = \frac{|\gamma-1|}{2}$, the Nyquist diagram of $C(j\omega)$ is a circle $C(\gamma) = C_0 + R_0\, e^{j\theta}$ with center C_0, radius R_0 and $\theta \in [0, 2\pi]$, see Fig 2. The shape of $C(\gamma)$ does not depend on $\delta > 0$ and $\omega_n > 0$. The intersections of $C(j\omega)$ with the real axis occur at points 1 and γ. Notice that the value of γ denotes the minimum (or maximum) amplitude of $C(j\omega)$ when $\gamma < 1$ (or $\gamma > 1$) and is the gain of the compensator $C(s)$ for $\omega = \omega_n$:

$$\gamma = C(j\omega_n) = \lim_{\omega\to\omega_n} \frac{X(\omega)}{Y(\omega)}. \tag{4}$$

In Fig. 2, the circle $\mathscr{C}_\gamma(j\omega)$ denotes the set of all the frequency responses $C(j\omega)$ having the same parameter γ.

A. Synthesis of continuous-time lead-lag compensator

Let us consider the following Design Problem: *find the parameters γ, δ and ω_n of $C(s)$ in order to exactly satisfy the phase margin ϕ_m or the gain margin G_m specifications at a given crossover frequency ω_0.*

The solution can be obtained using the modified Zigler-Nichols method, see [11] pp. $140-142$. This method tries to find the controller that moves the point $A = M_A\, e^{j\varphi_A}$ of plant $G(s)$ at the frequency ω_0 to a suitable point $B = M_B\, e^{j\varphi_B}$ of the the loop gain transfer function in order to satisfy the given margin specification. Let $C(j\omega_0) = M_0\, e^{j\varphi_0}$ denote the value of the frequency response $C(j\omega) = M(\omega)e^{j\varphi(\omega)}$ at frequency ω_0, where $M_0 = M(\omega_0)$ and $\varphi_0 = \varphi(\omega_0)$. Referring to Fig. 3, we say that *point A is controllable to point B* (or equivalently that *point A can be moved to point B*) if a value $C(j\omega_0)$ exists

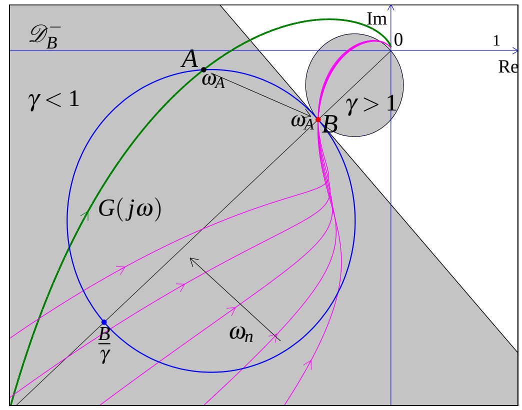

Fig. 3. Controllable domain $\mathscr{D}_B^-$ of lead-lag compensator $C(s)$.

such that $B = C(j\omega_0)\cdot A$, that is if and only if the following conditions hold:

$$M_B = M_A M_0, \qquad \varphi_B = \varphi_A + \varphi_0. \tag{5}$$

Definition 1: **($\mathscr{D}_B^-$)** given a point $B \in \mathbb{C}$, the "*controllable domain $\mathscr{D}_B^-$ of lead-lag compensator $C(s)$ to point B*" is defined as follows

$$\mathscr{D}_B^- = \Big\{ A \in \mathbb{C} \,\Big|\, \exists\, \gamma, \delta, \omega_n > 0, \exists\, \omega \ge 0 : C(j\omega)\cdot A = B \Big\}.$$

The shape of region $\mathscr{D}_B^-$ is shown in gray in Fig. 3. △

Definition 2: **(*Inversion Formulae*)** given two points $A = M_A\, e^{j\varphi_A}$ and $B = M_B\, e^{j\varphi_B}$ of the complex plane $\mathbb{C}$, the inversion formulae $X(A,B)$ and $Y(A,B)$ are defined as

$$\begin{aligned} X(A,B) &= \frac{M-\cos\varphi}{\sin\varphi}, \\ Y(A,B) &= \frac{\cos\varphi - \frac{1}{M}}{\sin\varphi}, \end{aligned} \tag{6}$$

where $M = \frac{M_B}{M_A}$ and $\varphi = \varphi_B - \varphi_A$. △

These equations are the same inversion formulae introduced in [12] and used in [13] and [9].

Property 1: ***(From** A **to** B**)*** *given a point $B \in \mathbb{C}$ and chosen a point A of the frequency response $G(j\omega)$ at frequency ω_A belonging to the controllable domain $\mathscr{D}_B^-$, i.e., $A = G(j\omega_A) \in \mathscr{D}_B^-$, the set $C(s,\omega_n)$ of all the lead-lag compensators $C(s)$ that move point A to point B is obtained from (1) using the parameters*

$$\gamma = \frac{X(A,B)}{Y(A,B)} > 0, \qquad \delta = Y(A,B)\,\frac{\omega_n^2-\omega_A^2}{2\omega_n\omega_A} > 0 \tag{7}$$

for all $\omega_n > 0$ such that $\delta > 0$ and with parameters $X(A,B)$ and $Y(A,B)$ obtained using the inversion formulae (6).

Proof: for $\omega = \omega_A$, relations (3) can be rewritten as

$$\gamma = \frac{X(\omega_A)}{Y(\omega_A)}, \qquad \delta = Y(\omega_A)\,\frac{\omega_n^2-\omega_A^2}{2\omega_n\omega_A}.$$

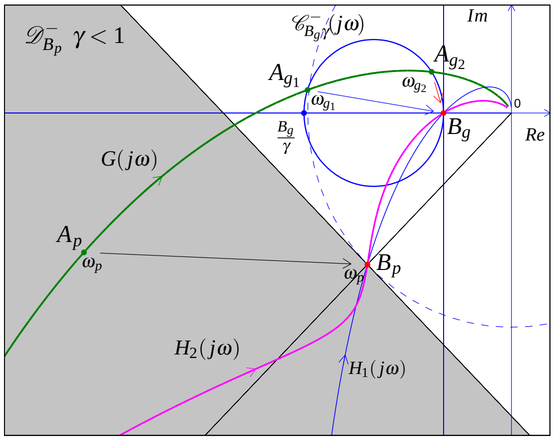

Fig. 4. Graphical solution of Design Problem A on the Nyquist plane.

Substituting in (1) yields

$$C(s,\omega_n)=\frac{s^2+X(\omega_A)\,\frac{\omega_n^2-\omega_A^2}{\omega_A}s+\omega_n^2}{s^2+Y(\omega_A)\,\frac{\omega_n^2-\omega_A^2}{\omega_A}s+\omega_n^2}.$$

One can easily verify that the frequency response of $C(s,\omega_n)$ at frequency ω_A is

$$C(j\omega_A,\omega_n)=C(j\omega_A)=\frac{1+jX(\omega_A)}{1+jY(\omega_A)}. \tag{8}$$

From equations $B=C(j\omega_0)\cdot A$ and (5) it is evident that point $A=G(j\omega_A)=M_A\,e^{j\varphi_A}$ can be moved to point $B=M_B\,e^{j\varphi_B}$ if and only if

$$C(j\omega_A)=M\,e^{j\varphi}=\frac{M_B}{M_A}e^{j(\varphi_B-\varphi_A)}. \tag{9}$$

Solving equations (8) and (9) with respect to $X(\omega_A)$ and $Y(\omega_A)$, one obtains the *Inversion Formulae* $X(\omega_A)=X(A,B)$ and $Y(\omega_A)=Y(A,B)$ introduced in Definition 2. The hypothesis that point A belongs to the controllable domain $\mathscr{D}^-_B$ ensures, see Definition 1, that there exist admissible lead-lag controllers $C(s,\omega_n)$ moving point A to point B which are characterized by positive parameters γ, δ and ω_n. All the admissible values of ω_n are those satisfying $\delta>0$ in (7). □

Prop. 1 can be used to solve the following Design Problem.

Design Problem A: $(\phi_m\,,\,G_m\,,\,\omega_p)$. Given the control scheme of Fig. 1, the transfer function $G(s)$ and design specifications on the phase margin ϕ_m, gain margin G_m and gain crossover frequency ω_p, design a lead-lag compensator $C(s)$ such that the loop gain transfer function $C(j\omega)G(j\omega)$ passes through point $B_p=e^{j(\pi+\phi_m)}$ for $\omega=\omega_p$ and passes through point $B_g=-1/G_m$.

Solution A: *The solution can be obtained graphically and numerically as follows:*
Step 1. Draw the controllable domain $\mathscr{D}^-_{B_p}$ of point $B_p=e^{j(\pi+\phi_m)}$ as shown in Fig. 4 and check whether the point $A_p=G(j\omega_p)$ belongs to $\mathscr{D}^-_{B_p}$. If not the Design Problem has no acceptable solutions.
Step 2. Determine the parameters $X_p=X(A_p,B_p)$, $Y_p=Y(A_p,B_p)$ using the inversion formulas (6).
Step 3. Draw the circle $\mathscr{C}^-_{B_g\gamma}(j\omega)$ having its diameter on the segment defined by points B_g and $\frac{B_g}{\gamma}$, where $B_g=-1/G_m$ and $\gamma=\frac{X_p}{Y_p}$, see Fig. 4. If there are no intersections between $\mathscr{C}^-_{B_g\gamma}(j\omega)$ and $G(j\omega)$ the Design Problem has no solutions. Otherwise let $A_{g_i}=\{A_{g_1},A_{g_2},\ldots\}$ denotes the set of intersections points of circle $\mathscr{C}^-_{B_g\gamma}(j\omega)$ with $G(j\omega)$ at frequencies $\omega_{g_i}=\{\omega_{g_1},\omega_{g_2},\ldots\}$. These points can also be obtained numerically solving the following relation

$$\gamma=\gamma_g(\omega_g)=\frac{\frac{M_{B_g}}{M_{A_g}(\omega_g)}-\cos(\varphi_{B_g}-\varphi_{A_g}(\omega_g))}{\cos(\varphi_{B_g}-\varphi_{A_g}(\omega_g))-\frac{M_{A_g}(\omega_g)}{M_{B_g}}}, \tag{10}$$

or graphically as shown in Fig. 5.
Step 4. For each $\omega_g=\omega_{g_i}$, calculate δ and ω_n as follows

$$\delta=Y_p\frac{\omega_n^2-\omega_p^2}{2\omega_n\omega_p}, \tag{11}$$

$$\omega_n=\sqrt{\frac{X_g\,\omega_g-X_p\,\omega_p}{\frac{X_g}{\omega_g}-\frac{X_p}{\omega_p}}}=\sqrt{\frac{Y_g\,\omega_g-Y_p\,\omega_p}{\frac{Y_g}{\omega_g}-\frac{Y_p}{\omega_p}}}, \tag{12}$$

where $X_g=X(A_g,B_g)$ and $Y_g=Y(A_g,B_g)$ are obtained using (6) and $A_g=G(j\omega_g)$. The solutions are acceptable only if δ and ω_n are real and positive.

Proof: The design specifications define the position of points $B_p=e^{j(\pi+\phi_m)}$, $A_p=G(j\omega_p)$ and $B_g=-1/G_m$. According to Property 1, the compensators $C_p(s,\omega_n)$ which move point $A_p\in\mathscr{D}^-_{B_p}$ to point B_p are obtained using the parameters γ and δ given in (7). The free parameter ω_n can now be used to force the loop gain frequency response $C_p(j\omega,\omega_n)G(j\omega)$ to pass through point B_g. This condition can be satisfied only if a frequency ω_g exists such that compensator $C_p(s,\omega_n)$ moves point $A_g=G(j\omega_g)\in\mathscr{D}^-_{B_g}$ to point B_g:

$$G(j\omega_g)C_\gamma(j\omega_g)=B_g, \tag{13}$$

where $C_\gamma(s)$ is the compensator with $\gamma=\frac{X_p}{Y_p}$, that is only if (10) holds. Relation (13) can also be rewritten as follows

$$G(j\omega)=\frac{B_g}{C_\gamma(j\omega)}=\mathscr{C}^-_{B_g\gamma}(j\omega), \tag{14}$$

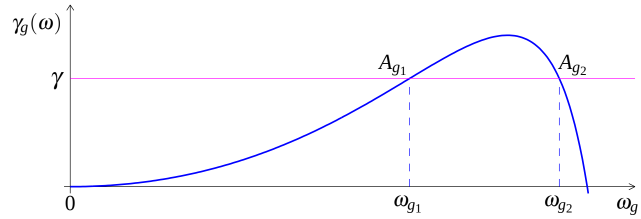

Fig. 5. Plot of function $\gamma_g(\omega)$ versus ω_g.

with $\omega=\omega_g$, and therefore it can be solved graphically on the Nyquist plane by finding the intersections ω_g of $G(j\omega)$ with $\mathscr{C}^-_{B_g\gamma}(j\omega)$. The compensator $C_g(s,\omega_n)$ moving point A_g to point B_g, which can be obtained using Property 1, has to be equal to the compensator $C_p(s,\omega_n)$ which moves point A_p to point B_p. This condition is satisfied only if the two compensators share the same γ, δ and ω_n, that is only if (11) and (12) hold. The solutions are acceptable only if $\gamma,\delta,\omega_n>0$. □

Remark 1: In a similar way the proposed graphical solution can be easily rewritten to meet the design specifications on the phase margin ϕ_m, gain margin G_m and phase crossover frequency ω_g, see [10].

B. Numerical example

Design Problem: given the plant

$$G(s)=\frac{36(s+1.1)}{s(s+1.5)^2(s+3)}, \tag{15}$$

design the compensator (1) in order to meet a phase margin $M_\varphi=45^o$, a gain margin $G_m=3$ and a gain crossover frequency $\omega_p=1.8$. *Solution:*
Step 1. The point $B_p=e^{j225^o}$ defines the shape of the controllable domain $\mathscr{D}^-_{B_p}$ shown in Fig. 4. The point $A_p=G(j\omega_p)=2.2e^{-j162^o}$ belongs to $\mathscr{D}^-_{B_p}$.
Step 2. Using (6) it is: $X_p=-0.921$ and $Y_p=-0.921$.
Step 3. The parameter $\gamma=0.327$ and point $B_g=0.333e^{j180^o}$ define the blue circle $\mathscr{C}^-_{B_g\gamma}(j\omega)$ in Fig. 4. The intersections of $\mathscr{C}^-_{B_g\gamma}(j\omega)$ with $G(j\omega)$ are $A_{g_1}=1.007e^{j174^o}$ and $A_{g_2}=0.436e^{j154^o}$ at frequencies $\omega_{g_1}=2.704$ and $\omega_{g_2}=3.90$. These points can also be obtained solving relation (10), see Fig. 5.
Step 4. The solution for $\omega_g=\omega_{g_1}$ is not acceptable because $\delta<0$. The corresponding loop transfer function $H_1(j\omega)$ is plotted in blue in Fig. 4. The second solution is obtained for $\omega_g=\omega_{g_2}$ and leads to $\delta=1.63>0$ and $\omega_n=1.04>0$. The obtained regulator is

$$C(s)=\frac{s^2+1.11\,s+1.07}{s^2+3.39\,s+1.07}. \tag{16}$$

The corresponding loop transfer function $H_2(j\omega)=G(j\omega)C(j\omega)$ is plotted in red in Fig. 4.

III. GENERAL STRUCTURES OF DISCRETE-TIME LEAD-LAG COMPENSATORS

Let us now consider the following structure of a discrete-time lead-lag compensator

$$C(z)=\frac{(z-1)^2+2\gamma_d\delta_d\Omega_n(z^2-1)+\Omega_n^2(z+1)^2}{(z-1)^2+2\delta_d\Omega_n(z^2-1)+\Omega_n^2(z+1)^2}, \tag{17}$$

where γ_d, Ω_n and δ_d are real and positive. Since

$$\frac{e^{j\omega T}-1}{e^{j\omega T}+1}=j\Omega(\omega), \quad \text{where} \quad \Omega(\omega)=\tan\frac{\omega T}{2},$$

the frequency response of (17) for $\omega\in[0,\frac{\pi}{T}]$ can be written as

$$C(\omega,T)=C(e^{j\omega T})=\frac{1+jX(\omega,T)}{1+jY(\omega,T)}, \tag{18}$$

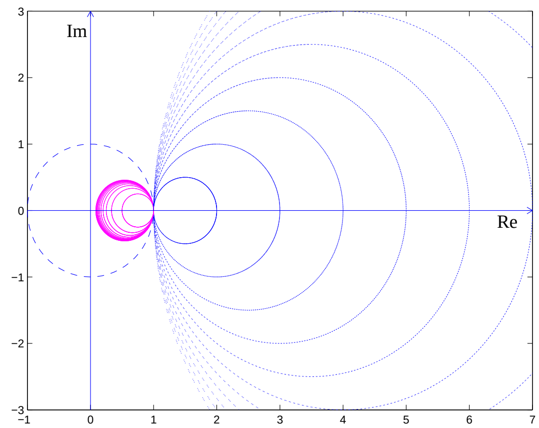

Fig. 6. Nyquist diagrams of function $C(\omega,T)$ when $\Omega_n=1$, $\delta_d=1.5$, $\gamma_d=[2:1:12]$ (blue lines) and $\delta_d=1.5$, $\gamma_d=1./[2:1:12]$ (magenta lines).

where T is the sampling period and

$$X(\omega,T)=\frac{2\gamma_d\delta_d\Omega_n\Omega(\omega)}{\Omega_n^2-\Omega(\omega)^2}, \qquad Y(\omega,T)=\frac{2\delta_d\Omega_n\Omega(\omega)}{\Omega_n^2-\Omega(\omega)^2}. \tag{19}$$

The steady-state gain of the discrete compensator (17) is $\gamma_{d0}=\lim_{z\to1}C(z)=1$. From (18) and (19) it follows that

$$\gamma_d=C(e^{j\omega T})\Big|_{\omega=\frac{2}{T}\arctan\Omega_n}=\frac{X(\omega,T)}{Y(\omega,T)}. \tag{20}$$

In analogy with the continuous time case, the following property holds.

Property 2: *the shape of the frequency response (18) on the Nyquist plane is a circle $C(\gamma_d)$ with center C_0 and radius R_0:*

$$C(\gamma_d)=C_0+R_0\,e^{j\theta}, \quad C_0=\frac{\gamma_d+1}{2}, \quad R_0=\frac{|\gamma_d-1|}{2}, \tag{21}$$

where $\theta\in[0,2\pi]$. The intersections of $C(\gamma_d)$ with the real axis occur at points 1 and γ_d. The shape does not depend on parameters $\delta_d>0$ and $\Omega_n>0$.

From Property 2 it follows that γ_d is the minimum (or maximum) amplitude of $C(\omega,T)$ when $\gamma_d>1$ (or $\gamma_d<1$). When $\Omega_n=1$, the frequency at the point $(\gamma_d,0)$ is $\omega=\frac{\pi}{2T}$.
Let $\mathscr{C}(\gamma_d)$ denote the set of all the lead-lag compensators $C(z)$ having the same parameter γ_d and the same shape on the Nyquist plane, that is

$$\mathscr{C}(\gamma_d)=\Big\{C(z) \text{ as in (17)} \;\Big|\; \delta_d>0,\Omega_n>0\Big\}. \tag{22}$$

Moreover, let $\mathscr{C}_{\gamma_d}(z)\in\mathscr{C}(\gamma_d)$ denote one element of set $\mathscr{C}(\gamma_d)$ chosen arbitrarily.

IV. SYNTHESIS OF DISCRETE-TIME LEAD-LAG

Let us refer to the block scheme of Fig. 7 where $HG(z)$ is the discrete system to be controlled, $H_0(s)$ is the zero-order hold

$$HG(z)=\mathscr{Z}[H_0(s)\,G(s)], \qquad H_0(s)=\frac{1-e^{-Ts}}{s},$$

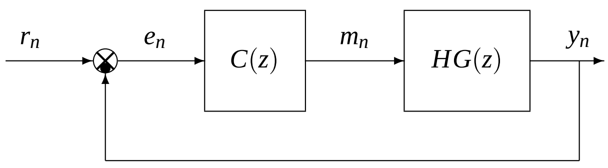

Fig. 7. The considered block scheme for the discrete time case.

T is the sampling period and $C(z)$ is the compensator (17) to be designed.

Definition 3: Given a point $B \in \mathbb{C}$, let us define "*controllable domain of discrete lead-lag compensator $C(z)$ to point B*" the set $\mathscr{D}^-_{dB}$ defined as follows

$$\mathscr{D}^-_{dB} = \left\{ A \in \mathbb{C} \,\middle|\, \exists\, \gamma_d, \delta_d, \Omega_n > 0, \exists\, \omega \geq 0 : C(\omega, T) \cdot A = B \right\}.$$

$\triangle$

It can be easily shown that the domain $\mathscr{D}^-_{dB}$ on the Nyquist plane is equal to the lead-lag controllable domain $\mathscr{D}^-_B$ in the continuous time case, see the gray area shown in Fig. 3.

Property 3: ***(From** A **to** B**)*** *given a point $B \in \mathbb{C}$ and chosen a point A of the frequency response $HG(\omega, T)$ at frequency $\omega_A \in [0, \frac{\pi}{T}]$ belonging to the controllable domain $\mathscr{D}^-_{dB}$, the set $C(z, \Omega_n)$ of all the discrete lead-lag compensators $C(z)$ that move point A to point B is obtained from (17) using the parameters*

$$\gamma_d = \frac{X(A,B)}{Y(A,B)} > 0, \qquad \delta_d = Y(A,B)\frac{\Omega_n^2 - \Omega_A^2}{2\Omega_n \Omega_A} > 0 \qquad (23)$$

for all $\Omega_n > 0$ such that $\delta_d > 0$, with $X(A,B)$ and $Y(A,B)$ obtained using (6) and $\Omega_A = \tan \frac{\omega_A T}{2}$.

Proof: The frequency response (18) at frequency ω_A can be written as follows:

$$C(\omega_A, T) = \frac{1 + jX(\omega_A, T)}{1 + jY(\omega_A, T)} = M e^{j\varphi}, \qquad (24)$$

where M and φ can be expressed as in (6). Due to the similar structure of (2) and (18) the proof is similar to the one given for the continuous-time case.

$\square$

Let us now design the regulator $C(z)$ in order to meet specifications for a robust control. In the continuous and discrete time domains the gain and the phase margins provide a two-points measure of how close the Nyquist plot is to the point -1 [7].

Design Problem B: (ϕ_m , G_m , ω_p). Given the control scheme of Fig. 7, the transfer function $HG(z)$, the sampling period T and design specifications on the phase margin ϕ_m, gain margin G_m and gain crossover frequency ω_p, design a lead-lag compensator $C(z)$ such that the loop gain transfer function $C(\omega, T)HG(\omega, T)$ passes through point $B_p = e^{j(\pi+\phi_m)}$ for $\omega = \omega_p \in [0, \frac{\pi}{T}]$ and passes through point $B_g = -1/G_m$.

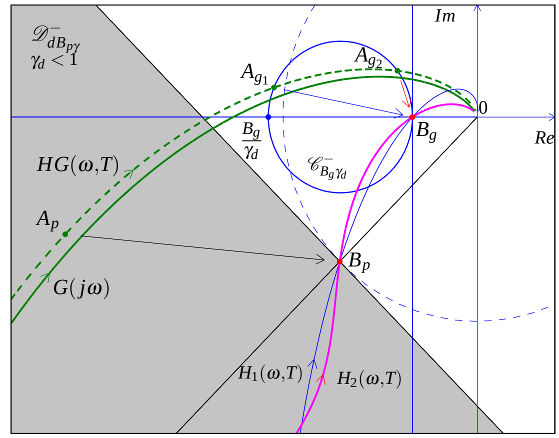

Fig. 8. Graphical solution of Design Problem B on the Nyquist plane.

Solution B:
Step 1. Draw the controllable domain $\mathscr{D}^-_{dB_p}$ of point $B_p = e^{j(\pi+\phi_m)}$ as shown in Fig. 8 and check whether point $A_p = HG(\omega_p, T)$ belongs to $\mathscr{D}^-_{dB_p}$.
Step 2. Determine the parameters $X_p = X(A_p, B_p)$ and $Y_p = Y(A_p, B_p)$ using the inversion formulas (6).
Step 3. Draw the circle $\mathscr{C}^-_{B_g\gamma_d}(j\omega)$ having its diameter on the segment defined by points B_g and $\frac{B_g}{\gamma_d}$, where $B_g = -1/G_m$ and $\gamma_d = \frac{X_p}{Y_p}$. If there are no intersections points of $\mathscr{C}^-_{B_g\gamma_d}(j\omega)$ with $HG(\omega, T)$, the Design Problem has no solutions. Otherwise let $A_{g_i} = \{A_{g_1}, A_{g_2}, \ldots\}$ denotes the set of the intersections points of circle $\mathscr{C}^-_{B_g\gamma_d}(j\omega)$ with $HG(\omega, T)$ at frequencies $\omega_{g_i} = \{\omega_{g_1}, \omega_{g_2}, \ldots\}$. These points can also be obtained solving the following relation (see Fig. 9)

$$\gamma_d = \gamma_{dg}(\omega_g) = \frac{\frac{M_{B_g}}{M_{A_g}(\omega_g)} - \cos(\varphi_{B_g} - \varphi_{A_g}(\omega_g))}{\cos(\varphi_{B_g} - \varphi_{A_g}(\omega_g)) - \frac{M_{A_g}(\omega_g)}{M_{B_g}}}. \qquad (25)$$

Step 4. For each $\omega_g = \omega_{g_i}$ belonging to $[0, \frac{\pi}{T}]$ calculate δ_d and Ω_n as follows

$$\delta_d = Y_p \frac{\Omega_n^2 - \Omega_p^2}{2\Omega_n \Omega_p} > 0, \qquad (26)$$

$$\Omega_n = \sqrt{\frac{Y_g \Omega_g - Y_p \Omega_p}{\frac{Y_g}{\Omega_g} - \frac{Y_p}{\Omega_p}}} > 0 \qquad (27)$$

where $X_g = X(A_g, B_g)$ and $Y_g = Y(A_g, B_g)$ are obtained using (6) and $A_g = HG(\omega_g, T)$. The solutions are acceptable only if δ_d and Ω_n are real and positive.

The proof can be obtained in a similar way as in the continuous time case. Moreover, Solution B can be easily modified in order to meet the design specifications on the phase margin ϕ_m, gain margin G_m and phase crossover frequency ω_g.

V. NUMERICAL EXAMPLE

Design Problem: referring to the continuous system (15), design the discrete lead-lag compensator (17) to meet

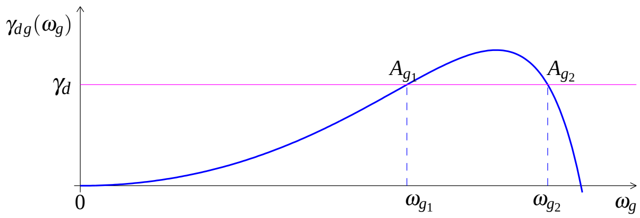

Fig. 9. Plot of function $\gamma_{dg}(\omega_g)$ versus ω_g.

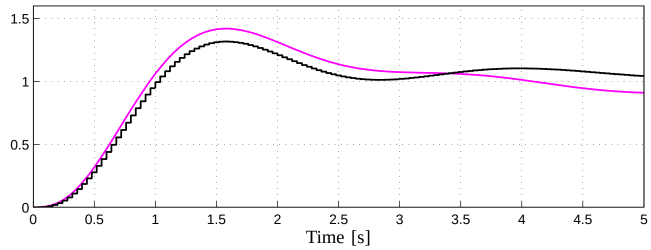

Fig. 10. Step responses of the closed loop system in continuous time (magenta line) and discrete time (black line) cases.

the same design specifications required in the previous numerical example, assuming a sampling period $T = 0.04$ s.

<u>*Solution*</u>: The discrete system $HG(z)$ to be controlled is the following

$$HG(z) = \frac{3.66\,10^{-4}z^3 + 1.04\,10^{-3}z^2 - 10^{-3}z - 3.17 10^{-4}}{z^4 - 3.77z^3 + 5.33z^2 - 3.34z + 0.787}.$$

The points B_p and B_g are completely defined by the design specifications and are the same of the previous example. The point $A_p \in HG(e^{j\omega T})$ at frequency ω_p is $A_p = 2.2e^{-j165^o}$ and it belongs to the controllable domain $\mathscr{D}^-_{dB_p}$. From (6) it is: $X_p = -0.828$ and $Y_p = -2.67$.

The parameter $\gamma_d = 0.310$ and point B_g define the circle $\mathscr{C}^-_{B_g\gamma_d}(j\omega)$ (see blue circle in Fig. 8). The intersections points with $HG(\omega, T)$ are $A_{g_1} = 1.06e^{j172^o}$ and $A_{g_2} = 0.469e^{j151^o}$ at frequencies $\omega_{g_1} = 2.64$ and $\omega_{g_2} = 3.78$. These points can be equivalently obtained solving relation (25), see Fig. 9. The solution for $\omega_g = \omega_{g_1}$ is not acceptable because $\delta_d < 0$. The second solution is obtained for $\omega_g = \omega_{g_2}$, $\delta_d = 2.7163 > 0$ and $\gamma_d = 0.0147 > 0$. The following regulator is obtained

$$C_\gamma(z, \omega_{g_2}) = \frac{(z-1)^2 + 2.48\,10^{-2}(z^2-1) + 2.17\,10^{-4}(z+1)^2}{(z-1)^2 + 8.01\,10^{-2}(z^2-1) + 2.17\,10^{-4}(z+1)^2}.$$

The corresponding loop transfer function $H_2(\omega, T)$ is plotted in magenta in Fig. 8. The step responses of the closed loop system in continuous and discrete time cases are shown in Fig. 10.

VI. COMPARISON WITH OTHER METHODS

A graphical solution to exactly meet specifications on phase margin, gain margin and crossover frequency for the discrete and continuous design of all the compensators whose frequency response can be expressed as $C\frac{1+jB}{1+jA}$ is presented in [14]. From (18) it follows that this method can also be applied to the discrete lead-lag regulator (17), however it is no directly described in [14]. One of the main advantages of the graphical solution presented in this paper is that the point A_g is directly determined in the complex plane by finding the intersections of the frequency response of the plant and particular design circles. The presented graphical solution can be easily done on the Nyquist plane by ruler and compass, while the graphic construction given in [14] is based on the use of a special design chart. Moreover the proposed method provides *all* the solutions of the control problem, whereas other graphical approaches, such as the one in [14], can only provide a subset of all the solutions. As regard the numerical solution of the Design Problem B, on the best of the authors' knowledge, an exact and direct discrete method has never been proposed in literature.

VII. CONCLUSIONS

A numerical and graphical direct method for the design of discrete lead-lag regulators to obtain a robust control of the system is presented. This method has the advantage to avoid the double transformation in continuous and in discrete time domains as required by the classical indirect method. The presented technique is based on the use of inversion formulae similar to the ones used for the lead-lag regulators design in the continuous-time case. The simplicity of the presented design relations and the graphical solution on the Nyquist plane makes the method very useful both on industrial and educational environments.

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