The POG Technique for Modeling Multi-phase Asynchronous Motors

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Abstract—In this paper the dynamic model of a multi-phase asynchronous motor has been obtained using the Power-Oriented Graphs (POG) technique and applying the Lagrangian approach. The “power” internal structure of the motor has been clearly identified. The steady-state behaviour of the motor has been analyzed putting in evidence the mathematical relations between the current and voltage vectors in the transformed rotating frame. In particular an interesting expression for the generated motor torque has been obtained. Some simulation results are finally reported.

I. INTRODUCTION

The dynamic model of a three-phase asynchronous motor is well known in literature, see for example [1], and it is usually described referring only to the steady-state behavior of the motor. In this paper the full dynamic equations of a multi-phase asynchronous motor are obtained using the Lagrangian approach and applying the POG modeling technique. The obtained dynamic model shows a very compact graphical representation that clearly puts in evidence the power internal structure of the motor. The dynamic equations of the motor in a transformed rotating frame are then obtained using a proper time-varying state space transformation. Interesting relations between the voltage and current vectors are finally obtained from the analysis of the steady-state equations of the motor. The paper is organized as follows: Sec. II describes the basic properties of the POG modeling technique. Sec. III shows the details of POG modeling of a multi-phase asynchronous motor and the steady-state analysis of the system. Finally, in Sec. IV some simulation results are reported.

II. POWER-ORIENTED GRAPHS BASIC PRINCIPLES

The Power-Oriented Graphs technique, see [2] and [3], is suitable for modeling physical systems. The POG are normal block diagrams combined with a particular modular structure essentially based on the use of the two blocks shown in Fig. 1.a and Fig. 1.b: the elaboration block (e.b.) stores and/or dissipates energy (i.e. springs, masses, dampers, capacities, inductances, resistances, etc.); the connection block (c.b.) redistributes the power within the system without storing nor dissipating energy (i.e. any type of gear reduction, transformers, etc.). The e.b. transforms the power variables with the constraint \( x_1^T y_1 = x_2^T y_2 \). The e.b. and the c.b. are suitable for representing both scalar and vectorial systems. In the vectorial case, \( G(s) \) and \( K \) are matrices: \( G(s) \) is always square, \( K \) can also be rectangular. The circle present in the e.b. is a summation element and the black spot represents a minus sign that multiplies the entering variable. The main feature of the Power-Oriented Graphs is to keep a direct correspondence between the dashed sections of the graphs and real power sections of the modeled systems: the scalar product \( x^T y \) of the two power vectors \( x \) and \( y \) involved in each dashed line of a power-oriented graph, see Fig. 1, has the physical meaning of the power flowing through that particular section. The Bond Graphs technique, see [4], is based on the same idea, but it uses a different and specific graphical representation.

The main energetic domains encountered in modeling physical systems are the electrical, the mechanical (translational and rotational) and the hydraulic, see Fig. 1.c. Each energetic domain is characterized by two power variables: an across-variable \( v_e \) defined between two points (i.e. the voltage \( V \)),
and a through-variable \( v_f \) defined in each point of the space (i.e. the current \( I \)). Each Physical Element (PE) interacts with the external world through the power sections associated to its terminals. A Physical Element is connected in series when its terminals share the same through-variable \( v_f \), see Fig. 2.a.

This physical element can be modeled by using the POG block shown in Fig. 2.b: note that the summation element \( \Sigma \) can also be modeled by using the POG block shown in Fig. 2.a.

The physical element PE in Fig. 2.a can also be modeled by using the POG block shown in Fig. 2.c. This is an “equivalent” elaboration block (with a different graphical shape) obtained from the POG block of Fig. 2.b inverting the path that goes from \( v_{c1} \) to \( v_f \). The Physical Elements connected in parallel share the same across-variable \( v_e \). It can be easily shown that the POG modeling of the physical elements in parallel can be done in a way quite similar to the way used above for the elements in series.

Another important aspect of the POG technique is the direct correspondence between the POG representations and the corresponding state space descriptions. For example, the POG scheme shown in Fig. 3 can be represented by the state space equations given in (1) where the energy matrix \( L \) is symmetric and positive definite: \( L = L^T > 0 \). For such a system, the stored energy \( E_s \) and the dissipating power \( P_d \) can be expressed as follows: \( E_s = \frac{1}{2} x^T L x \), \( P_d = x^T A x \). When an eigenvalue of matrix \( L \) tends to zero (or to infinity), system (1) degenerates towards a lower dimension dynamic system. In this case, the dynamic model (2) of the “reduced” system can be directly obtained from (1) by using a simple “congruent” transformation \( x = T z \) (\( T \) is constant) where \( L = T^T L T \), \( A = T^T A T \) and \( B = T^T B \). If matrix \( T \) is time-varying, an additional term \( T^T L T z \) appears in the transformed system. When matrix \( T \) is rectangular, the system is transformed and reduced at the same time.

**A. Notations**

In this paper the following notations will be used:

- Full matrices:

\[
[R_{i,j} = \begin{bmatrix}
R_{11} & R_{12} & \cdots & R_{1m} \\
R_{21} & R_{22} & \cdots & R_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
R_{n1} & R_{n2} & \cdots & R_{nm}
\end{bmatrix}_{1:n \times 1:m}]
\]

- The symbols \( j \) and \( e^{j \theta} \) denote the following matrices:

\[
[j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e^{j \theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}]
\]

- The symbol \( I_m \) denotes an identity matrix of order \( m \).

**III. ELECTRICAL MOTORS MODELING**

The structure and the basic elements of a multi-phase asynchronous motor are shown in Fig. 4. The stator and rotor phases are supposed to be star connected. The system is characterized by the following parameters:

- \( m_s \) : number of stator phases;
- \( m_r \) : number of rotor phases;
- \( p \) : number of rotor and stator polar expansions;
- \( \gamma_s \) : stator angular phase displacement (\( \gamma_s = \frac{2 \pi}{m_s} \));
- \( \gamma_r \) : rotor angular phase displacement (\( \gamma_r = \frac{2 \pi}{m_r} \));
- \( \theta_m \) : rotor angular position;
- \( \omega_m \) : rotor angular velocity;
- \( \theta_c \) : stator rotating field angular position;
- \( \omega_c \) : stator rotating field angular velocity;
- \( \theta \) : electric angle (\( \theta = p \theta_m \));
- \( R_s \) : stator phases resistance (\( p = 1 \));
- \( L_s \) : stator phases self inductance (\( p = 1 \));
- \( M_{s0} \) : maximum value of the stator phases mutual inductance (\( p = 1 \));
- \( R_r \) : rotor phases resistance (\( p = 1 \));
- \( L_r \) : rotor phases self inductance (\( p = 1 \));
- \( M_{r0} \) : maximum value of the rotor phases mutual inductance (\( p = 1 \));
- \( M_{sr0} \) : maximum value of the mutual inductance between stator and rotor phases (\( p = 1 \));
- \( J_m \) : rotor moment of inertia;
- \( b_m \) : rotor linear friction coefficient;
- \( \tau_m \) : electromotive torque acting on the rotor;
- \( \tau_e \) : external load torque acting on the rotor.

To obtain the dynamic equations of the multi-phase asynchronous motor, a Lagrangian approach will be used, see [5].
Let us consider \( ^tV_s, \) \( ^tI_s, \) \( ^tV_r \) and \( ^tI_r \) as stator and rotor voltage and current vectors in the original reference frame \( \Sigma_t \): 

\[
^tV_s = \begin{bmatrix} V_{s1} \\ V_{s2} \\ \vdots \\ V_{sm_s} \end{bmatrix}, \quad ^tI_s = \begin{bmatrix} I_{s1} \\ I_{s2} \\ \vdots \\ I_{sm_s} \end{bmatrix}, \quad ^tV_r = \begin{bmatrix} V_{r1} \\ V_{r2} \\ \vdots \\ V_{rm_r} \end{bmatrix}, \quad ^tI_r = \begin{bmatrix} I_{r1} \\ I_{r2} \\ \vdots \\ I_{rm_r} \end{bmatrix},
\]

where \( V_{si} = V_i - V_{s0}, \) \( V_{ri} = V_{r0} - V_r. \) One can now define the generalized state vectors \( ^tq, \) \( ^t\dot{q} \) and the extended input vector \( ^tV: \)

\[
^tq = \begin{bmatrix} ^tq_s \\ ^tq_r \end{bmatrix}, \quad ^t\dot{q} = \begin{bmatrix} ^t\dot{q}_s \\ ^t\dot{q}_r \end{bmatrix}, \quad ^tV = \begin{bmatrix} ^tV_s \\ ^tV_r \end{bmatrix}.
\]

Using the “Lagrangian” approach, the dynamic equations of the multi-phase asynchronous motor can be expressed as:

\[
\frac{d}{dt} \begin{bmatrix} ^tL \ ^tq \ \ ^t\dot{q} \end{bmatrix} = ^tV - ^tR \ ^t\dot{q},
\]

where the “Lagrangian” function \( K(\ ^tq, \ ^t\dot{q}) \) is the following:

\[
K(\ ^tq, \ ^t\dot{q}) = \frac{1}{2} \ ^t\dot{q} \ ^tL (\ ^tq) \ ^t\dot{q}.
\]

From (3) and (4) one obtains the dynamic equations:

\[
\frac{d}{dt} \begin{bmatrix} ^tL \ ^tq \ \ ^t\dot{q} \end{bmatrix} = -\begin{bmatrix} ^tR + ^tW \ ^t\dot{q} \end{bmatrix} + \begin{bmatrix} ^tV \end{bmatrix} \quad \begin{bmatrix} ^tL \ ^tq \ \ ^t\dot{q} \end{bmatrix} = \begin{bmatrix} ^tL \ ^tq \ \ ^t\dot{q} \end{bmatrix} = \begin{bmatrix} ^tL \ ^tq \ \ ^t\dot{q} \end{bmatrix}.
\]

that in compact form have the following representation:

\[
\frac{d}{dt} \begin{bmatrix} ^tL \ ^tq \ \ ^t\dot{q} \end{bmatrix} = -\begin{bmatrix} ^tR + ^tW \ ^t\dot{q} \end{bmatrix} + \begin{bmatrix} ^tV \end{bmatrix}.
\]

The energy matrix \( ^tL(\ ^tq) \), the dissipating matrix \( ^tR \) and the energy redistribution matrix \( ^tW \) have the following structure:

\[
^tL(\ ^tq) = \begin{bmatrix} ^tL_s \\ ^tM_{sr} (\ ^t\theta_m) \\ 0 \\ 0 \end{bmatrix}, \quad ^tR = \begin{bmatrix} ^tR_s \\ ^tR_r \\ 0 \\ 0 \end{bmatrix}, \quad ^tW = \begin{bmatrix} 0 \\ 0 \\ ^tM_{sr} (\ ^t\theta_m) \\ 0 \end{bmatrix}.
\]
Vectors $\dot{\omega}V = t^T\dot{\omega}V$ and $\dot{\omega}q = t^T\dot{q}$ are defined as:

$$\dot{\omega}V = \begin{bmatrix} \dot{\omega}V_s \\ \dot{\omega}V_r \end{bmatrix}, \quad \dot{\omega}q = \begin{bmatrix} \dot{\omega}I_s \\ \dot{\omega}I_r \\ \dot{\omega}I_m \end{bmatrix}$$

where

$$\dot{\omega}V_s = \begin{bmatrix} \dot{\omega}V_{s1} \\ \dot{\omega}V_{s2} \\ \dot{\omega}V_{sx} \end{bmatrix}, \quad \dot{\omega}V_r = \begin{bmatrix} \dot{\omega}V_{r1} \\ \dot{\omega}V_{r2} \\ \dot{\omega}V_{rx} \end{bmatrix}, \quad \dot{\omega}I_s = \begin{bmatrix} \dot{\omega}I_{s1} \\ \dot{\omega}I_{s2} \\ \dot{\omega}I_{sx} \end{bmatrix}, \quad \dot{\omega}I_r = \begin{bmatrix} \dot{\omega}I_{r1} \\ \dot{\omega}I_{r2} \\ \dot{\omega}I_{rx} \end{bmatrix},$$

Since the rotor phases are short-circuited, it can be easily shown that $\dot{\omega}V_r = 0$. Matrices $\dot{\omega}L = t^T\dot{\omega}L$ and $\dot{\omega}W = t^T\dot{\omega}W^Tt\dot{\omega}$ now have the following structure:

$$\dot{\omega}L = \begin{bmatrix} pL_e \mathbf{I}_2 & 0 & 0 & pM_{src} \mathbf{I}_2 & 0 & 0 \\ pM_{src} \mathbf{I}_2 & 0 & 0 & 0 & 0 & 0 \\ pL_{te} \mathbf{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\dot{\omega}W = \begin{bmatrix} \dot{\omega}F_e + \dot{\omega}\Omega_e & \dot{\omega}K_e \\ -\dot{\omega}K_e & 0 \end{bmatrix}$$

Figure 7. POG graphical representation of the asynchronous motor in the transformed rotating frame $\Sigma_\omega$. 

Transformed reference frame are:

$$\begin{bmatrix} \dot{\omega}L_g \\ \dot{\omega}q \\ \dot{\omega}V \end{bmatrix} = -\begin{bmatrix} R_r + \dot{\omega}F_e + \dot{\omega}\Omega_e & \dot{\omega}K_e & \dot{\omega}V \\ \dot{\omega}L_m \end{bmatrix} + \begin{bmatrix} \dot{\omega}V \\ \dot{\omega}q \end{bmatrix}$$

that in compact form can be expressed as:

$$\dot{\omega}L \dot{\omega}q = -(\dot{\omega}R + \dot{\omega}W) \dot{\omega}\dot{q} + \dot{\omega}V. \tag{7}$$

Note that $\dot{\omega}R = t^T\dot{\omega}R^Tt\dot{\omega}$ because $t\dot{\omega}R$ is a diagonal matrix. The energy matrix $\dot{\omega}L$ is now constant and has only $m_s + m_r + 3$ not null elements. It can be easily verified that matrix $\dot{\omega}W$ is skew-symmetric, i.e. $\dot{\omega}W^T = -\dot{\omega}W$, and therefore it represents the “energy redistributions” within the system. A POG graphical representation of system (7) is shown in Fig. 7: the connection block between power sections (1) and (2) represents the frame transformation $\Sigma_i \rightarrow \Sigma_\omega$. The elaboration blocks between power sections (3) and (4) represents the Electrical part of the system, while those between sections (5) and (3) represent the Mechanical part. The connection block between sections (3) and (4) represents the conversion of energy and power (without accumulation nor dissipation) between the electrical and mechanical domains. The expression of the mechanical torque $\tau_m$ in the new transformed space is:

$$\tau_m = \dot{\omega}K_e \dot{\omega}I_e = \left[ \dot{\omega}K_e \dot{\omega}K_e \right] \dot{\omega}I_e = p^2 M_{src} \dot{\omega}T_{sx} \dot{\omega}T_{sx}.$$

A detailed analysis of the dynamic behavior of the system is given in [6]. Let us now consider the following steady-state equations of system (7):

$$\begin{cases}
\dot{\omega}V_{sx} = (pR_s \mathbf{I}_2 + p^2 \omega_c L_{sx} j) \dot{\omega}I_{sx} + p^2 \dot{\omega}_c M_{src} j \dot{\omega}I_{r2} \\
\dot{\omega}V_{sx} = pR_s \dot{\omega}I_{sx} \\
\dot{\omega}V_{sx} = 0 \\
\tau_r = p^2 M_{src} \dot{\omega}T_{sx} \dot{\omega}T_{sx} - \tau_m \tag{8}
\end{cases}$$

where $\tau_r = \tau_c + b_m \omega_m$. Note that only the two current vectors $\dot{\omega}I_{sx}$ and $\dot{\omega}I_{r2}$ generate the mechanical torque $\tau_m$. From the second and the fourth equation of system (8) one obtains:

$$\dot{\omega}I_{sx} = \frac{1}{pR_s} \dot{\omega}V_{sx}, \quad \dot{\omega}I_{r2} = 0.$$
From the third equation of system (8) one obtains the following relation between current vectors $sI_2$ and $rI_2$:

$$sI_2 = -\frac{L_{re}}{M_{sre} \cos \beta_s} e^{-j \beta_s} rI_2$$

(9)

where $\beta_s$ is defined in (11). Substituting (9) in the first equation of system (8) one obtains the following phase displacement vector $v\phi_r$:

$$v\phi_r = \frac{p \omega_s M_{sre} (a - \cos^2 \beta_s)}{\cos \beta_r \cos (\delta_s + \delta_r)} e^{-j (\delta_s + \delta_r)} e^{-j (\beta_s + \gamma)}$$

(10)

which links the stator voltage vector $sV_2$ to the stator current vector $sI_2$: $sV_2 = v\phi_r sI_2$, assuming that:

$$\delta_s + \delta_r = \arctan \left( \frac{a \tan \beta_s}{\cos \beta_s} - 1 \right) \cos \beta_s$$

$$\beta_s = \arctan \frac{R_s}{p \omega_c M_{sre}}$$

$$\beta_r = \arctan \frac{R_r}{p \omega_d L_{re}}.$$  

(11)

A graphical representation of equation (10) is shown in Fig. 8. Inverting equation (10), one obtains the following displacement vector $r\phi_v$:

$$r\phi_v = \frac{\cos \beta_s \cos (\delta_s + \delta_r)}{p \omega_s M_{sre} (a - \cos^2 \beta_s)} e^{j (\delta_s + \delta_r)} e^{j (\beta_s + \gamma)}$$

(12)

which links the stator voltage vector $sV_2$ to the rotor current vector $rI_2$: $rI_2 = r\phi_v rV_2$. A graphical representation of equation (12) is shown in Fig. 9. Using equation (12) one finally obtains the expression of the mechanical torque $\tau_m$ as a function of stator voltage vector $sV_2$:

$$\tau_m = \frac{p R_r}{\omega_d} |sI_2|^2$$

$$= \frac{p R_r}{\omega_d} \left( \frac{\cos \beta_s \cos (\delta_s + \delta_r)}{p \omega_s M_{sre} (a - \cos^2 \beta_s)} \right)^2 |sV_2|^2.$$  

(13)

Deriving equation (13) with respect to $\omega_d$ and putting it equal to zero, one obtains that at frequency $\omega_d$:

$$\omega_d = \frac{R_r}{p L_{re} \sqrt{1 + \frac{(1-2a)}{a^2} \cos^2 \beta_s}}.$$

The maximum values $\tau_{max}$ of the torque $\tau_m$ is:

$$\tau_{max} = \frac{L_{re} \cos^2 \beta_s}{\omega_s^2 M_{sre}^2 2 a [\sqrt{(B^2 + C^2) + B}] |sV_2|^2}$$

where $B = \sin \beta_s \cos \beta_s$ and $C = (a - \cos^2 \beta_s)$.

IV. SIMULATION RESULTS

The described model of the multi-phase asynchronous motor has been implemented in Matlab/Simulink. The simulation results presented in this Section have been obtained using the following electrical and mechanical parameters: $m_s = 5$, $m_r = 5$, $p = 1$, $L_s = 0.1$ H, $M_{s0} = 0.08$ H, $R_s = 2 \Omega$, $L_r = 0.1$ H, $M_{r0} = 0.08$ H, $R_r = 3 \Omega$, $M_{sre} = 0.07$ H, $J_m = 1.6$ kg m$^2$, $b_m = 1.8$ Nm s/rad, $\tau_c = 0$ Nm. The motor is supplied by balanced star-connected voltages with a maximum amplitude of $V_{max} = 100$ V. In Fig. 10 one observes the evolution of stator and rotor currents in the original reference frame $\Sigma_l$, whereas Fig. 11 shows the same currents in the transformed rotating frame $\Sigma_s$. Note that in the new coordinate space only two components of stator and rotor current vectors are not zero ($sI_2$, $rI_2$), and effectively generate the mechanical torque, the others ($sI_3$, $rI_3$) are zero. In Fig. 12 angular velocity $\omega_m$ and mechanical torque $\tau_m$ are shown: after a short transient they settle on steady-state values. In Fig. 13 mechanical torque $\tau_m$ as a function of $\omega_m$ is shown. Note that mechanical torque $\tau_m$ (blue continuous line) settles on the steady-state value of $\tau_m$ (black dashed line), calculated using equation (13). Fig. 14 and Fig. 15 respectively confirm the geometrical interpretations of phase displacement vectors shown in Fig. 8 and Fig. 9.

V. CONCLUSIONS

In this paper a multi-phase asynchronous motor has been modeled using the Power-Oriented Graphs (POG) technique and the “Lagrangian” approach, obtaining a very simple and compact model that can be easily translated into Simulink.
schemes. A coordinate transformation has been used to simplify the dynamic equations and to obtain a simple expression of the mechanical torque. Simulation results have shown the effectiveness of the realized model, proving also the geometric interpretations of the phase displacement vector that links the mechanical torque $\tau_m$ to the stator voltage vector $\mathbf{V}_s$.

REFERENCES