# New Formulae and Graphics for Compensator Design 

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#### Abstract

In this paper, two simple "inversion formulae" for analytic design of lead and lag compensators are proposed, and a graphical interpretation for them is given. Their use in connection with both Bode and Nyquist diagrams is pointed out with some numerical examples. Keywords. Bode Compensator Design, Lead Compensator, Lag Compensator, Nichols diagrams, Nyquist diagrams.


## 1 Introduction

The design of lead and lag compensators for feedback control systems is usually performed in the frequency domain with trial and error procedures on Bode or Nichols diagrams.

In this paper, two "inversion formulae", that express in strict trigonometric terms the $\alpha$ and $\tau$ parameters of a general compensator, are derived with a frequency domain approach. These formulae and their proofs are simpler than the ones available in the literature, that, to the authors knowledge, are due to Wakeland [1], [2], [3] and Phillips and Harbor [4]. Moreover, the formulae here presented have a simple graphical interpretation on the Nyquist plane. This feature can also be used for a complete graphical design of the compensator parameters directly on the Nyquist diagram by only using rule and square.

The paper is organised as follows. In Section 2 the design problem is briefly recalled. In Section 3 the inversion formulae are derived, and the corresponding domains of applicability in the Nyquist and Nichols planes are defined. In Section 4 a graphic interpretation of the inversion formulae is presented, and in Section 5 a straightforward method for graphical design of the lead and lag compensators in the Nyquist plane is described. Numerical examples and conclusions end the paper.

## 2 The standard design

Let us refer to the block-diagram shown in Fig. 1, where $G(s)$ denotes the controlled system and $C(s)$
is the compensator to be designed. Let us suppose


Figure 1: The control problem considered.
that the compensator $C(s)$ is a lead or a lag network:

$$
\begin{equation*}
C_{d}(s)=\frac{1+\tau s}{1+\alpha \tau s}, \quad C_{g}(s)=\frac{1+\alpha \tau s}{1+\tau s} . \tag{1}
\end{equation*}
$$

Such a compensator is usually designed with graphic procedures using Bode or Nichols diagrams. A typical example of lead compensator design based on Nichols diagram is shown in Fig. 2. Imposing a phase mar$\operatorname{gin} M_{\varphi}$ requires that the the frequency response of the compensated system $C(s) G(s)$ passes through the point $B$. Then, a point $A$ at frequency $\omega_{A}$ is chosen on the frequency response plot of $G(j \omega)$ and a suitable compensator is designed to move $A$ to $B$, i.e. such that the new Nichols diagram passes through $B$ at frequency $\omega_{A}$.
It is well known, see, for instance, [1], that this problem can be graphically solved by using the Nichols chart of lead compensators shown in Fig. 3. When the coordinates $M$ and $\varphi$ of vector $\overline{B A}$ are transferred on the Nichols chart of Fig. 3, a point on the chart is determined. The two curves which pass through this point completely define the parameters $\alpha$ and $r$ of the lead compensator.
This design problem is analitically solved by using the inversion formulae presented in next section.

## 3 The inversion formulae

The lead and lag compensators (1) can be described in a unified way as follows:

$$
\begin{equation*}
C(s)=\frac{1+\tau^{\prime} s}{1+\tau s}=\frac{1+\gamma \tau s}{1+\tau s} \tag{2}
\end{equation*}
$$



Figure 2: Design of a lead compensator on the Nichols plane.


Figure 3: Nichols chart of a lead compensator family.
where

$$
\begin{equation*}
\tau^{\prime}=\gamma \tau \quad \rightarrow \quad \gamma=\frac{\tau^{\prime}}{\tau} \tag{3}
\end{equation*}
$$

When $\tau^{\prime}>\tau(\gamma>1), C(s)$ is a lead compensator, when $\tau^{\prime}<\tau(\gamma<1)$, it is a lag compensator.

Result 1 Consider the function $(\gamma, \omega \tau) \rightarrow(M, \varphi)$

$$
\begin{equation*}
M(\gamma, \omega \tau) e^{j \varphi(\gamma, \omega \tau)}=\frac{1+j \gamma(\omega \tau)}{1+j(\omega \tau)} \tag{4}
\end{equation*}
$$

defined in the domain $(\gamma, \omega \tau) \in R^{+} \times R^{+}$. The inverse function $(M, \varphi) \rightarrow(\gamma, \omega \tau)$ is given by:

$$
\begin{gather*}
\gamma=\frac{M(\cos \varphi-M)}{1-M \cos \varphi}  \tag{5}\\
\omega \tau=\frac{M \cos \varphi-1}{M \sin \varphi} \tag{6}
\end{gather*}
$$

and is defined in the domain $D_{1} \cup D_{2}$ with

$$
\begin{align*}
& D_{1}=\left\{(M, \varphi):-\frac{\pi}{2}<\varphi \leq 0,0<M \leq \cos \varphi\right\}(7)  \tag{7}\\
& D_{2}=\left\{(M, \varphi): 0 \leq \varphi<\frac{\pi}{2}, M \geq \frac{1}{\cos \varphi}\right\} \tag{8}
\end{align*}
$$

The domains $D_{1}$ and $D_{2}$ correspond, respectively, to the lead and lag compensators. Their layouts in the Nyquist and Nichols planes are shown in Fig. 4. $D_{1}$ is the reciprocal of domain $D_{2}$ and vice versa: $D_{2}=$ $\left(D_{1}\right)^{-1}$.


Figure 4: Admissible domains $D_{1}$ and $D_{2}$ on Nyquist and Nichols planes.

Proof. In the following, we will refer to the case of a lag compensator. The Nyquist diagram of function $C(j \omega)$ when $\omega \in R^{+}$and $\gamma<1$ is shown in Fig. 5. It


Figure 5: Nyquist diagrams of a lag compensator.
is a semi-circumference with centre $C$ and radius $\rho$. The coordinates $\left(x_{c}, y_{c}\right)$ of $C$ and the value of $\rho$ are

$$
x_{c}=\frac{1+\gamma}{2}, \quad y_{c}=0, \quad \rho=\frac{|1-\gamma|}{2}
$$

Let $A=(x, y)$ denote the generic point of function $C(j \omega)$ on the Nyquist diagram:

$$
A=(x, y)=(M \cos \varphi, M \sin \varphi)
$$

The Pitagora theorem applied to triangle rectangle $\widehat{A B C}$ yields $\left(x-x_{c}\right)^{2}+y^{2}=\rho^{2}$, hence

$$
\left(M \cos \varphi-\frac{1+\gamma}{2}\right)^{2}+M^{2} \sin ^{2} \varphi=\frac{(1-\gamma)^{2}}{4}
$$

Then

$$
M^{2}-M(1+\gamma) \cos \varphi=\frac{(1-\gamma)^{2}}{4}-\frac{(1+\gamma)^{2}}{4}
$$

and

$$
M^{2}-M(1+\gamma) \cos \varphi=-\gamma
$$

By solving for $\gamma$, we directly obtain the first inversion formula (5), while taking the absolute value of relation (4) yields

$$
M^{2}\left(1+\omega^{2} \tau^{2}\right)=1+\gamma^{2} \omega^{2} \tau^{2}
$$

that provides $\omega \tau$ as

$$
\begin{equation*}
\omega \tau=\sqrt{\frac{1-M^{2}}{M^{2}-\gamma^{2}}} \tag{9}
\end{equation*}
$$

In (9), the term $M^{2}-\gamma^{2}$ can be expressed as:

$$
\begin{align*}
M^{2}-\gamma^{2} & =M^{2} \frac{(1-M \cos \varphi)^{2}-(\cos \varphi-M)^{2}}{(1-M \cos \varphi)^{2}} \\
& =M^{2} \frac{\left(1-M^{2}\right) \sin ^{2} \varphi}{(1-M \cos \varphi)^{2}} \tag{10}
\end{align*}
$$

By substituting (10) in (9) the second inversion formula (6) is directly obtained. Moreover, since $\tau^{\prime}=$ $\gamma \tau$, from relations (5) and (6) it follows that:

$$
\begin{equation*}
\omega \tau^{\prime}=\frac{M-\cos \varphi}{\sin \varphi} \tag{11}
\end{equation*}
$$

Formulae (5) and (6) are suitable for the design of both the lead and lag compensators, but they can only be used when parameters $\gamma$ and $\tau$ are positive since only in this case $C(s)$ is stable. Imposing $\gamma>0$ and $\tau>0$ in (5) and (6), directly implies that pair ( $M, \varphi$ ) belongs to the admissible domains $D_{1}$ and $D_{2}$ given in (7) and (8).

## 4 Graphical interpretation

Result 2 Referring to the graphical constructions shown in Fig. 6 (lag compensator) and Fig. 7 (lead compensator), let $S_{1}, S_{2}$ and $S_{3}$ denote the lengths of segments $\overline{B 1}, \overline{B A}$ and $\overline{B \gamma}$, and $\theta$ the angle formed by segments $\overline{C 1}$ and $\overline{C A}$.
Formulae (6) and (11) have the following graphical interpretation

$$
\begin{align*}
& \omega \tau=\frac{S_{1}}{S_{2}}=\frac{S_{2}}{S_{3}}=\tan \frac{\theta}{2}  \tag{12}\\
& \omega \tau^{\prime}=\frac{\left|\overline{A B^{\prime}}\right|}{\left|\overline{B^{\prime} 1}\right|}=\gamma \tan \frac{\theta}{2} \tag{13}
\end{align*}
$$

Moreover, the parameters $\gamma, \omega \tau^{\prime}$ and $1 /(\omega \tau)$ are determined on the Nyquist plane by means of simple graphical constructions.


Figure 6: Lag compensator: graphic interpretation.

Proof. From Fig. 6 and 7, it follows that

$$
\begin{equation*}
S_{1}=|M \cos \varphi-1|, \quad S_{2}=|M \sin \varphi| \tag{14}
\end{equation*}
$$

Hence, from (6),

$$
\begin{equation*}
\omega \tau=\frac{S_{1}}{S_{2}} \tag{15}
\end{equation*}
$$

Triangle $\widehat{\gamma A 1}$ is right-angled (inscribed into a circumference), hence

$$
\begin{equation*}
\frac{S_{1}}{S_{2}}=\frac{S_{2}}{S_{3}}=\tan \frac{\theta}{2} \tag{16}
\end{equation*}
$$

Relation (12) is simply obtained from (15) and (16). On the other hand, the numerator and the denominator of the right of (11) have the following geometrical interpretation (see Fig. 6 or Fig. 7):

$$
\begin{equation*}
|M-\cos \varphi|=\left|\overline{A B^{\prime}}\right|, \quad|\sin \varphi|=\left|\overline{B^{\prime} 1}\right| \tag{17}
\end{equation*}
$$

Relation (13) directly follows from (2), (11) and (17).
The graphical construction for the determination of parameters $\gamma, \omega \tau^{\prime}$ and $1 /(\omega \tau)$ on the Nyquist diagram is shown in Fig. 6 and Fig. 7. Vector $M e^{j \varphi}$ determines the point $A$ on the plane. Let us denote by $r_{2}$ the line passing through points 1 and $A$ and by $r_{1}$ the line orthogonal to $r_{2}$ in $A$. The term $1 /(\omega \tau)$ is the absolute value of the $y$-coordinate of the point where $r_{2}$ intersects the imaginary axis; the parameters $\gamma$ and $\omega \tau^{\prime}$ are the interections of $r_{1}$ with the real and imaginary axes.


Figure 7: Lead compensator: graphic interpretation.

## 5 Design on the Nyquist plane

The design of lead and lag compensators can also be performed graphically on the Nyquist plane as follows.

Result 3 Let $M_{\varphi}$ be the desired phase margin. The parameters $\alpha$ and $\tau$ of the lead and lag compensators

$$
C_{d}(s)=\frac{1+\tau s}{1+\alpha \tau s} \quad C_{g}(s)=\frac{1+\alpha \tau s}{1+\tau s}
$$

can be determined on the Nyquist plane with the graphical constructions shown in Fig. 8 for the lead compensator $C_{d}(s)$ and in Fig. 9 for the lag compensator $C_{g}(s)$, respectively.

Proof. For the lead compensator let us refer to Fig. 8. Phase margin $M_{\varphi}$ determines point $B$. The semi-circumference shown in grey is the region $D_{1}$ of all the points which can be moved to $B$ by using a lead compensator $C_{d}(s)$. This is true because the admissible domains $D_{1}$ and $D_{2}$ are reciprocal of each other. In fact, let $A$ be a point in the domain $D_{1}$; a lag compensator $C_{g}(s)$ which transforms vector $\overline{B 0}$ into $\bar{A} \overline{0}$ always exists; clearly, its reciprocal $C_{d}(s)=\left(C_{g}(s)\right)^{-1}$ is the lead compensator which transforms vector $\overline{A 0}$ into vector $\overline{B 0}$.

The design of a lead compensator proceeds as follows. Let $r_{q}$ denote the line passing through $A$ and $O$, and let $\boldsymbol{r}_{p}$ be the line passing through the origin $O$ and orthogonal to $r_{q}$. Choose a point $A$ on the frequency diagram $G(j \omega)$ belonging to the admissible domain $D_{1}$. Then, draw a circumference with the centre on segment $\overline{B 0}$ and passing through $A$ and $B$.


Figure 8: Graphical costruction on the Nyquist plane for the design of a lead compensator $C_{d}(s)$.

The parameter $\alpha$ is derived as the distance from the origin of the point where this circumference intersects the segment $\overline{B 0}$.

The line which passes through points $A$ and $\alpha$ intersects line $r_{p}$ at $\omega \alpha \tau$, and the line which passes through $B$ and $A$ intersects $r_{p}$ at point $1 /(\omega \tau)$. This holds since it is the graphical construction is the same shown in Fig. 6 for the lag compensator when $\gamma=\alpha$.


Figure 9: Graphical construction on the Nyquist plane for the design of a lag compensator $C_{g}(s)$.

The graphical design for a lag compensator is shown in Fig. 9. Point $B$ is the same. The admissible domain ( $D_{2}$ in this case) is the quarter of plane
shown in grey. In fact, for each point $A \in D_{2}$, a lag compensator $C_{g}(s)$ which transforms $A$ into $B$ exists. The graphical construction shown in Fig. 9 is the same described above. In this case the points determined are: $1 /(\alpha \omega \tau)$ and $\omega \tau$ on line $r_{p}$, and $1 / \alpha$ on line $r_{q}$.
Remark. Point $B$ is not constrained to belong the unit circle: it can be any point of the complex plane. In this case, the graphical construction is the same described above with the only caution that the points determined on lines $r_{p}$ and $r_{q}$ must be divided by the length of segment $\overline{B 0}$.

## 6 Numerical examples

Example 1. Given the system $G(s)=\frac{25}{s(s+1)(s+10)}$ design a lead compensator $C_{d}(s)$ providing a phase $\operatorname{margin} M_{\varphi}=60^{\circ}$. Let us refer to Fig. 10. Point $B$ is


Figure 10: Example of lead compensator design.
determined by the desired phase margin. By using a rule and a square, one can easily draw line $r_{q}$ passing through points 0 and $B$, and line $r_{p}$ orthogonal in 0 to line $r_{q}$. Chosen a point $A$ on curve $G(j \omega)$, it is straightforward to determine the segment $\overline{A P}$ which is orthogonal to segment $\overline{A B}$. Points $Q$ and $P$ are the intersections of segment $\overline{A P}$ with lines $r_{q}$ and $r_{p}$. The lenghts of segments $\overline{Q 0}$ and $\overline{P 0}$ are equal to $\alpha$ and $\omega \tau$ respectively. The obtained lead compensator is $C_{d}(s)=\frac{(1+0.9827 s)}{(1+0.1303 s)}$.
Example 2. For system $G(s)=\frac{60000}{(s+1)(s+2)(s+10)(s+30)}$ design a lag compensator $C_{g}(s)$ providing a phase $\operatorname{margin} M_{\varphi}=60^{\circ}$. Let us refer to Fig. 11. The specified phase margin directly determines line $r_{q}$. Line $r_{p}$ passes through the origin and is orthogonal to line $r_{q}$. Chosen a point $A$ on curve $G(j \omega)$, the segment $\overline{Q P}$ can be easily determined as the segment pass-


Figure 11: Example of lag compensator design.
ing through point $A$ and orthogonal to segment $\overline{A B}$ where $B$ is the intersection of line $r_{q}$ with the unit circle. In this case, the lenghts of segments $\overline{Q 0}$ and $\overline{P 0}$ are equal, respectively, to $1 / \alpha$ and $\omega \tau$. The obtained lag compensator is $C_{g}(s)=\frac{(1+1.9683 s)}{(1+986.3 s)}$. In Fig. 11 it is also reported a zoom of the region in the vicinity of the origin which clearly shows that for the compensated system $G_{c}(s)=G(s) C_{g}(s)$ the desired phase $\operatorname{margin} M_{\varphi}=60^{\circ}$ has been obtained.

## 7 Conclusions

In this paper, two simple inversion formulae for the analitic design of lead and lag compensators are presented. A geometric interpretations and a graphic procedure for the design of both the lead and lag compensators on the Nyquist plane are also given. Because of the simplicity of the formulae and the clarity of the related graphical constructions we think that the results presented can be very useful in teaching.

## References

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