Discrete Inversion Formulas for the Design of Lead and Lag Discrete Compensators

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***Abstract*— In the paper, new discrete inversion formulas suitable for the design of lead and lag discrete compensators in the frequency domain are presented. These formulas can be very useful for teaching in Automatic Control courses. The links of these discrete formulas with the continuous inversion formulas are deeply investigated. A simple graphical procedure for the design of discrete compensators on the Nyquist plane is also presented. Finally, some numerical examples illustrate the presented results.**

I. INTRODUCTION

The design of lead and lag compensators for linear systems can be done in a lot of different ways. Many different methods can be found in the text books of Automatic Control: design using Bode, Nyquist or Nichols diagrams, root locus, analytic design, use of Diophantine equations, state space approach, etc. In this paper a new method for the design of lead and lag discrete compensators based on the use of simple discrete inversion formulas is presented. The presented design method is particularly interesting for teaching purposes because it has a simple graphical interpretation on the Nyquist plane.

The inversion formulas presented in this paper are similar to other formulas that can be found in literature (see for example [1], [2], [3] and [4]), but in this case the formulas are simpler, the graphical interpretation on the Nyquist plane is more direct and the use of these formulas is suitable also for the discrete-time case.

The paper is organized as follows. In Section II the inversion formulas for the continuous time case and the related main properties are briefly summarized and discussed. In Section III the "discrete" inversion formulas are presented, the related mathematical result is proved, the shape of the admissible domains is investigated and many remarks stressing the links between the discrete and the continuous inversion formulas are given. Some numerical examples and a brief conclusion end the paper.

II. THE CONTINUOUS TIME CASE

Let us consider the continuous time system shown in Fig. 1 where $G(s)$ is the controlled system and $C(s)$ is the compensator to be designed:

$$C(s) = \frac{1+\tau_1 s}{1+\tau_2 s} \tag{1}$$

Let γ_0 and γ denote, respectively, the steady state gain and

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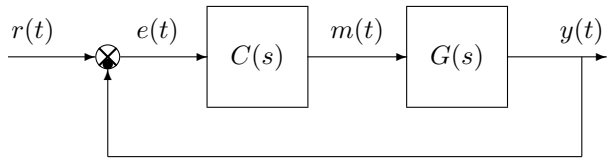

Fig. 1. The considered block scheme for the continuous time case.

the high frequency gain of function $C(s)$:

$$\gamma_0 = \lim_{s\to 0} C(s) = 1, \qquad \gamma = \lim_{s\to\infty} C(s) = \frac{\tau_1}{\tau_2}$$

Function $C(s)$ represents a lead compensator if $\gamma > 1$, a lag compensator if $\gamma < 1$. The zero s_0 and the pole s_p of the compensator $C(s)$ are

$$s_0 = -\frac{1}{\tau_1}, \qquad s_p = -\frac{1}{\tau_2} \tag{2}$$

The maximum (or minimum) phase φ_m of compensator $C(s)$

$$\varphi_m = \arcsin\frac{\gamma-1}{\gamma+1} = \arcsin\frac{\tau_1-\tau_2}{\tau_1+\tau_2}$$

is reached for $\omega = \omega_n$ where

$$\omega_n = \frac{1}{\sqrt{\tau_1\tau_2}} \tag{3}$$

The Bode magnitude and phase plots of compensator $C(s)$ when (using Matlab notation) $\gamma = [0.2 : 0.2 : 1,\ 1./[0.2 : 0.2 : 1]]$ and $\omega_n = 1$ are shown in Fig. 2.

For the continuous time case the design problem can often be formulated as follows.

Design Problem (continuous time): *find the parameters τ_1 and τ_2 of compensator (1) such that*

$$C(j\omega) = \frac{1+j\,\omega\tau_1}{1+j\,\omega\tau_2} = M\ e^{j\varphi} \tag{4}$$

where M and φ are the magnitude and the phase desired at frequency ω.

Inversion formulas (continuous time): the continuous time design problem is solved by using the following *inversion formulas*:

$$\boxed{\tau_1 = \frac{M-\cos\varphi}{\omega\sin\varphi}, \qquad \tau_2 = \frac{\cos\varphi - \frac{1}{M}}{\omega\sin\varphi}} \tag{5}$$

This solution follows directly from the previous works [5], [6] and [7].

From (5) one can easily verify that the parameters τ_1 and τ_2

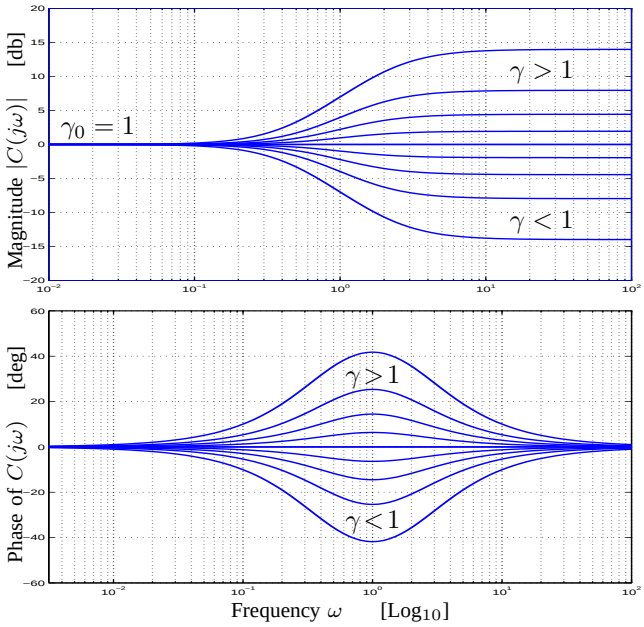

Fig. 2. Bode magnitude and phase plots of compensator $C(s)$ when $\gamma = [0.2 : 0.2 : 1,\ 1./[0.2 : 0.2 : 1]]$ and $\omega_n = 1$.

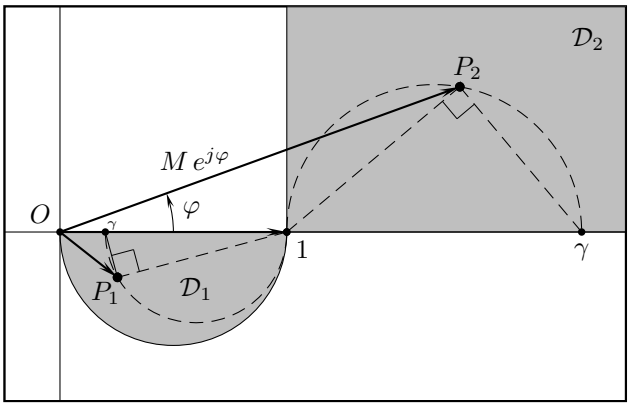

Fig. 3. Admissible domains $\mathcal{D}_1$ and $\mathcal{D}_2$ when $\tau_1 > 0$, $\tau_2 > 0$.

are both positive only when $(M,\ \varphi) \in \mathcal{D}_1 \cup \mathcal{D}_2$, where the domains $\mathcal{D}_1$ and $\mathcal{D}_2$, shown in Fig. 3, are defined as follows:

$$\mathcal{D}_1 = \left\{0 \le \varphi < \frac{\pi}{2},\ M\cos\varphi \ge 1\right\} \tag{6}$$

$$\mathcal{D}_2 = \left\{-\frac{\pi}{2} < \varphi \le 0,\ 0 < M \le \cos\varphi\right\} \tag{7}$$

For all the points $P_2 = M\,e^{j\varphi} \in \mathcal{D}_2$ the inversion formulas (5) provide $\tau_1 > \tau_2 > 0$, $\gamma > 1$ and the obtained function $C(s)$ is a *lead compensator*. For all the points $P_1 = M\,e^{j\varphi} \in \mathcal{D}_1$ the inversion formulas provide $\tau_2 > \tau_1 > 0$, $\gamma < 1$ and the obtained $C(s)$ is a *lag compensator*.

Remark: the parameter γ corresponding to points $P_1 \in \mathcal{D}_1$ and $P_2 \in \mathcal{D}_2$ can also be obtained graphically by using the geometric construction shown in Fig. 3.

Remark: the two inversion formulas (5) are *reciprocal*, in the sense that one formula can be obtained from the other

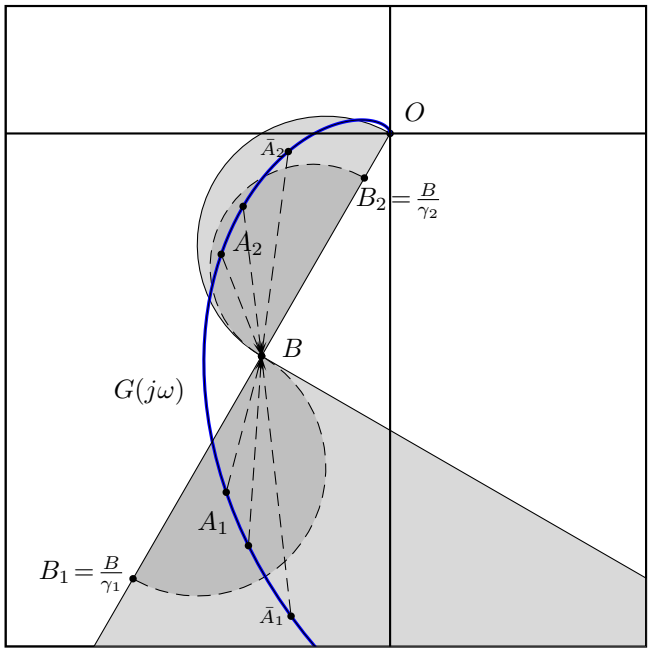

Fig. 4. Nyquist plane: admissible domains for the design of a compensator $c(s)$ which moves the points A_1 and A_2 into point B.

just substituting M with $1/M$ and φ with $-\varphi$:

$$\underbrace{\frac{M - \cos\varphi}{\omega\,\sin\varphi}}_{\tau_1}\Bigg|_{\substack{M\to\frac{1}{M}\\ \varphi\to-\varphi}} = \frac{\cos\varphi - \frac{1}{M}}{\omega\,\sin\varphi} = \tau_2$$

In the same way τ_1 can be obtained from τ_2.

Remark: the two domains $\mathcal{D}_1$ and $\mathcal{D}_2$ are *reciprocal*, that is the reciprocal of each point P_1 belonging to $\mathcal{D}_1$ belongs to $\mathcal{D}_2$, and vice-versa:

$$\mathcal{D}_1 = (\mathcal{D}_2)^{-1} \leftrightarrow (\forall P_1 \in \mathcal{D}_1 \to P_1^{-1} \in \mathcal{D}_2)$$

$$\mathcal{D}_2 = (\mathcal{D}_1)^{-1} \leftrightarrow (\forall P_2 \in \mathcal{D}_2 \to P_2^{-1} \in \mathcal{D}_1)$$

Remark: due to the reciprocity property, the domain $\mathcal{D}_1$ is the set of all the points which can be transformed into point $1 + j\,0$ by using a *lag* compensator, and the domain $\mathcal{D}_2$ is the set of all the points which can be transformed into point $1 + j\,0$ by using a *lead* compensator.

Graphical design procedure. The use of the inversion formulas (5) is particularly useful if the design procedure is graphically performed on the Nyquist plane. Let us refer, for example, to the Nyquist diagram $G(j\omega)$ of system $G(s)$ shown in Fig. 4. Chosen $\gamma_1 < 1$ and $\gamma_2 > 1$, let us design the compensator $C(s)$ such that:

$$\gamma_1 < \gamma < \gamma_2 \tag{8}$$

in order to properly bound the high frequency gain γ of compensator $C(s)$. A point B of the complex plane where to move function $G(j\omega)$ can be easily determined on the basis of the given phase margin or gain margin specifications. From point B one can easily determine points $B_1 = \frac{B}{\gamma_1}$ and

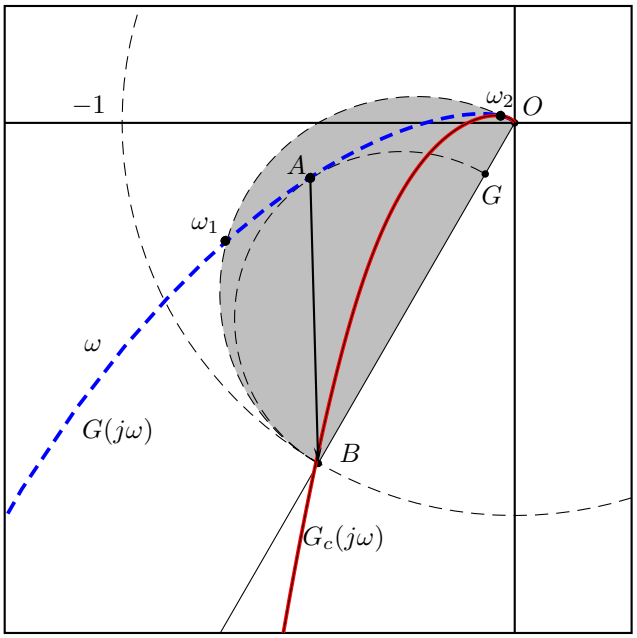

Fig. 5. Design of a lead compensator on the Nyquist plane.

$B_2 = \frac{B}{\gamma_2}$, see Fig. 4. The grey half circle with the diameter coincident with segment $\overline{B\,B_2}$ is the region of all the points A_2 that can be moved in B by using a *lead compensator*, while the grey half circle with the diameter coincident with segment $\overline{B\,B_1}$ is the region of all the points A_1 that can be moved in B by using a *lag compensator*. Points $\bar{A}_1$ and $\bar{A}_2$ can be mapped in B but without satisfying the constraint (8). Chosen a generic point $A = G(j\omega_A)$ belonging to one of the admissible domains, the parameters τ_1 and τ_2 of compensator $C(s)$ that moves point $A = M_A\, e^{j\varphi_A}$ in $B = M_B\, e^{j\varphi_B}$ can be obtained from (5) by using the following parameters:

$$M = \frac{M_B}{M_A}, \qquad \varphi = \varphi_B - \varphi_A, \qquad \omega = \omega_A \tag{9}$$

Numerical example. Given the system:

$$G(s) = \frac{25}{s(s+1)(s+10)},$$

let us design a lead compensator $C_1(s)$ which imposes a phase margin $M_\varphi = 60^o$ and with the gain γ as small as possible. The design specification $M_\varphi = 60^o$ completely defines the position of point $B = M_B\, e^{j\varphi_B}$:

$$M_B = 1, \qquad \varphi_B = \pi + M_\varphi = 240^o,$$

which must be crossed by the frequency response $G_c(j\omega)$ of the compensated system $G_c(s)$. The points $A = G(j\omega)$ that can be moved in B by using a lead compensator belong to the grey admissible region shown in Fig. 5: $\omega_1 < \omega < \omega_2$. The point $A = G(j\omega_A) = M_A e^{j\varphi_A}$ which minimizes the parameter γ can be determined with the graphical construction shown in Fig. 5 (note that $\gamma = 1/|G|$):

$$M_A = 0.538, \qquad \varphi_A = 194.9^o, \qquad \omega_A = 2.02.$$

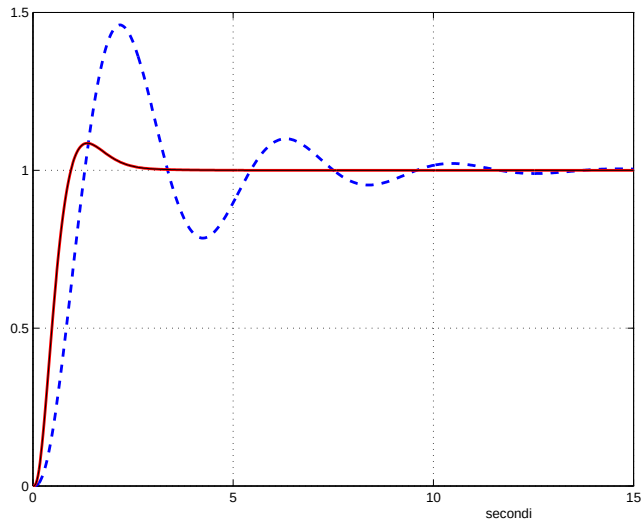

Fig. 6. Step responses of the two systems $G(s)$ and $G_c(s) = C_1(s)G(s)$ controlled in closed loop.

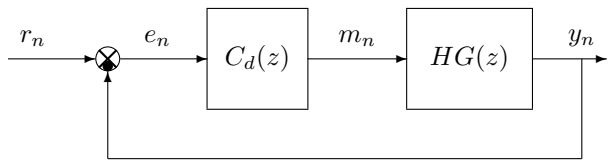

Fig. 7. The considered block scheme for the discrete time case.

The parameters M and φ to be used in (9) are the following:

$$M = \frac{M_B}{M_A} = 1.859, \qquad \varphi = \varphi_B - \varphi_A = 45.1^o.$$

Substituting M, φ and $\omega = \omega_A$ in (5) one obtains $\tau_1 = 0.806$ and $\tau_2 = 0.117$, that is:

$$C_1(s) = \frac{(1 + 0.806\, s)}{(1 + 0.117\, s)}.$$

The step responses of the two systems $G(s)$ and $G_c(s) = C_1(s)G(s)$ controlled in closed loop are shown in Fig. 6.

III. THE DISCRETE TIME CASE

For the discrete time case, let us refer to the block scheme of Fig. 7 where $HG(z)$ is the discrete system to be controlled, $H_0(s)$ is the zero-order hold:

$$HG(z) = \mathcal{Z}[H_0(s)\, G(s)], \qquad H_0(s) = \frac{1 - e^{-T\,s}}{s}$$

and $C_d(z)$ is the compensator to be designed:

$$C_d(z) = \frac{1 + \alpha(z-1)}{1 + \beta(z-1)} \tag{10}$$

This particular structure has been chosen to have the steady-state gain $\gamma_0 = \lim_{z\to 1} C_d(z) = 1$ for compensator $C_d(z)$. The zero z_0 and the pole z_p of function $C_d(z)$ are:

$$z_0 = 1 - \frac{1}{\alpha}, \qquad z_p = 1 - \frac{1}{\beta} \tag{11}$$

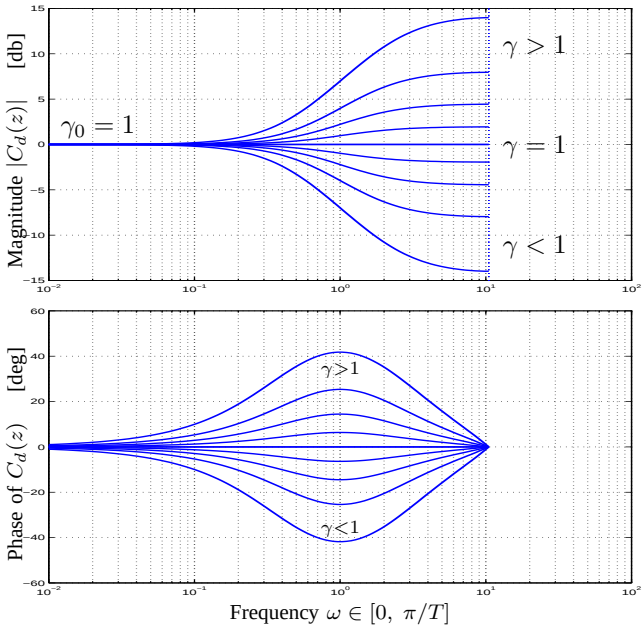

Fig. 8. Bode magnitude and phase plots of $C_d(z)$ when $\gamma_d = [0.2 : 0.2 : 1,\ 1/(0.2 : 0.2 : 1)]$, $\omega_{nd} = 1$ and $T = 0.3$.

The compensator $C_d(z)$ is a *minimum-phase system* only when $|z_0| < 1$ and $|z_p| < 1$, that is when:

$$\alpha > 0.5, \qquad \beta > 0.5. \tag{12}$$

The frequency response of function $C_d(z)$ is obtained from (10) when $z = e^{j\omega T}$ and $\omega \in [0,\ \frac{\pi}{T}]$ where T is the discrete sampling period. The high frequency gain γ_d of compensator $C_d(z)$ when $\omega = \frac{\pi}{T}$ is the following:

$$\gamma_d = C_d(e^{j\omega T})_{\omega=\frac{\pi}{T}} = C_d(-1) = \frac{2\alpha - 1}{2\beta - 1}. \tag{13}$$

For $\omega \in [0,\ \frac{\pi}{T}]$, the maximum (or minimum) phase angle φ_{md} of function $C_d(e^{j\omega T})$ is:

$$\varphi_{md} = \arcsin\left[\frac{\gamma_d - 1}{\gamma_d + 1}\right] = \arcsin\left[\frac{\alpha - \beta}{\alpha + \beta - 1}\right]. \tag{14}$$

This value φ_{md} is reached for $\omega = \omega_{nd}$:

$$\begin{aligned} \omega_{nd} &= \frac{1}{T}\arccos\left[\frac{2\alpha\beta - \alpha - \beta}{2\alpha\beta - \alpha - \beta + 1}\right] \\ &= \frac{1}{T}\arccos\left[\frac{z_0 + z_p}{1 + z_0 z_p}\right]. \end{aligned} \tag{15}$$

The magnitude and phase plots of the compensator $C_d(z)$ when $\gamma_d = [0.2 : 0.2 : 1,\ 1/(0.2 : 0.2 : 1)]$, $\omega_{nd} = 1$ and $T = 0.3$ are shown in Fig. 8. For the discrete time case the compensator design problem can be formulated as follows.

Design Problem (discrete time): *find the parameters α and β of compensator (10) such that:*

$$C_d(e^{j\omega T}) = \frac{1 + \alpha(e^{j\omega T} - 1)}{1 + \beta(e^{j\omega T} - 1)} = M e^{j\varphi}, \tag{16}$$

where M and φ are the magnitude and the phase desired at frequency ω.

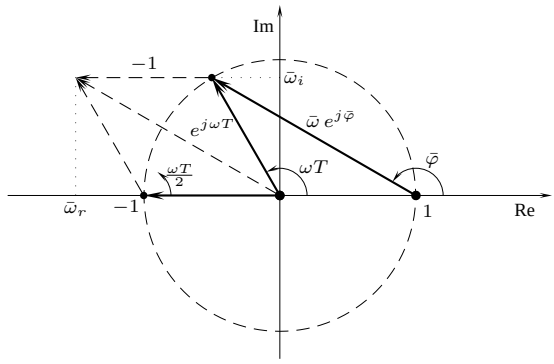

Fig. 9. Graphical representation of vectors $e^{j\omega T}$ and $\bar\omega\, e^{j\bar\varphi} = e^{j\omega T} - 1$.

Inversion formulas (discrete time): the discrete time design problem is solved by the following *inversion formulas*:

$$\boxed{\begin{aligned} \alpha &= \frac{1}{2} + \frac{M - \cos\varphi}{2\sin\varphi\tan\frac{\omega T}{2}} \\ \beta &= \frac{1}{2} + \frac{\cos\varphi - \frac{1}{M}}{2\sin\varphi\tan\frac{\omega T}{2}} \end{aligned}} \tag{17}$$

Proof. Let $\bar\omega_r$ and $\bar\omega_i$ denote the real and imaginary parts of vector $e^{j\omega T} - 1$:

$$e^{j\omega T} - 1 = \bar\omega_r + j\,\bar\omega_i = \bar\omega\, e^{j\bar\varphi}, \tag{18}$$

where

$$\begin{aligned} \bar\omega_r &= \bar\omega\cos\bar\varphi = \cos\omega T - 1 \\ \bar\omega_i &= \bar\omega\sin\bar\varphi = \sin\omega T \end{aligned} \tag{19}$$

One can easily verify, see Fig. 9, that the amplitude $\bar\omega$ and the phase $\bar\varphi$ of vector $\bar\omega\, e^{j\bar\varphi} = e^{j\omega T} - 1$ can also be expressed as follows:

$$\bar\omega = 2\sin\frac{\omega T}{2}, \qquad \bar\varphi = \frac{\pi}{2} + \frac{\omega T}{2}. \tag{20}$$

Substituting (18) in (16) one obtains the equation:

$$\frac{1 + \alpha(\bar\omega_r + j\bar\omega_i)}{1 + \beta(\bar\omega_r + j\bar\omega_i)} = M(\cos\varphi + j\sin\varphi).$$

Separating the real and the imaginary parts, one obtains the system:

$$\begin{aligned} 1 + \alpha\,\bar\omega_r &= M\big[(1 + \beta\,\bar\omega_r)\cos\varphi - \beta\,\bar\omega_i\sin\varphi\big] \\ \alpha\,\bar\omega_i &= M\big[(1 + \beta\,\bar\omega_r)\sin\varphi + \beta\,\bar\omega_i\cos\varphi\big] \end{aligned}$$

which can be rewritten in a matrix form as a linear system to be solved with respect to the parameters α and β:

$$\begin{bmatrix} M(\bar\omega_r\cos\varphi - \bar\omega_i\sin\varphi) & -\bar\omega_r \\ M(\bar\omega_r\sin\varphi + \bar\omega_i\cos\varphi) & -\bar\omega_i \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 - M\cos\varphi \\ -M\sin\varphi \end{bmatrix}$$

The solutions of this system are the followings:

$$\alpha = \frac{\begin{vmatrix} M(\bar{\omega}_r \cos\varphi - \bar{\omega}_i \sin\varphi) & 1 - M\cos\varphi \\ M(\bar{\omega}_r \sin\varphi + \bar{\omega}_i \cos\varphi) & -M\sin\varphi \end{vmatrix}}{M\,(\bar{\omega}_i^2 + \bar{\omega}_r^2)\sin\varphi}$$

$$\beta = \frac{\begin{vmatrix} 1 - M\cos\varphi & -\bar{\omega}_r \\ -M\sin\varphi & -\bar{\omega}_i \end{vmatrix}}{M(\bar{\omega}_i^2 + \bar{\omega}_r^2)\sin\varphi}$$

After some mathematical manipulations, the solutions α and β can be simplified as follows:

$$\alpha = -\frac{\bar{\omega}_r \sin\varphi + \bar{\omega}_i \cos\varphi - M\bar{\omega}_i}{\bar{\omega}^2 \sin\varphi}$$

$$\beta = \frac{-M\bar{\omega}_r \sin\varphi + M\bar{\omega}_i \cos\varphi - \bar{\omega}_i}{M\,\bar{\omega}^2 \sin\varphi}$$

from which, using relations (19) and simplifying, it follows:

$$\alpha = \frac{M\sin\bar{\varphi} - \sin(\bar{\varphi} + \varphi)}{\bar{\omega}\sin\varphi}$$

$$\beta = \frac{M\,\sin(\bar{\varphi} - \varphi) - \sin\bar{\varphi}}{M\,\bar{\omega}\sin\varphi}$$

Using relations (20) one obtains:

$$\alpha = \frac{M\sin(\frac{\pi}{2} + \frac{\omega T}{2}) - \sin(\frac{\pi}{2} + \frac{\omega T}{2} + \varphi)}{2\,\sin\frac{\omega T}{2}\sin\varphi}$$

$$\beta = \frac{\sin(\frac{\pi}{2} + \frac{\omega T}{2} - \varphi) - \frac{1}{M}\sin(\frac{\pi}{2} + \frac{\omega T}{2})}{2\,\sin\frac{\omega T}{2}\sin\varphi}$$

which can be rewritten as follows:

$$\alpha = \frac{M\cos\frac{\omega T}{2} - \cos\frac{\omega T}{2}\,\cos\varphi + \sin\frac{\omega T}{2}\,\sin\varphi}{2\,\sin\frac{\omega T}{2}\,\sin\varphi}$$

$$\beta = \frac{\cos\frac{\omega T}{2}\,\cos\varphi - \sin\frac{\omega T}{2}\,\sin\varphi - \frac{1}{M}\cos(\frac{\omega T}{2})}{2\,\sin\frac{\omega T}{2}\,\sin\varphi}$$

Simplifying one obtains the discrete inversion formulas (17):

$$\alpha = \frac{1}{2} + \frac{M - \cos\varphi}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}$$

$$\beta = \frac{1}{2} + \frac{\cos\varphi - \frac{1}{M}}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}$$

Remark: these formulas can also be rewritten in the following form:

$$\boxed{\begin{aligned} \alpha &= \frac{1}{2} + \frac{\omega\,\tau_1}{2\,\tan\frac{\omega T}{2}} \\ \beta &= \frac{1}{2} + \frac{\omega\,\tau_2}{2\,\tan\frac{\omega T}{2}} \end{aligned}} \qquad (21)$$

where τ_1 and τ_2 are the parameters obtained from the continuous time inversion formulas (5) when the same design parameters M, φ and ω are used.

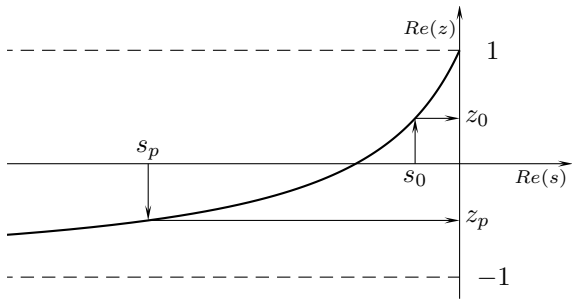

Fig. 10. Graphical correspondence between s-plane and z-plane due to the bilinear transformation.

Remark: substituting (21) in (13) one obtains:

$$\gamma_d = \frac{2\alpha - 1}{2\beta - 1} = \frac{\tau_1}{\tau_2} = \gamma \qquad (22)$$

that is, the high frequency gain γ_d of the discrete compensator $C_d(z)$ obtained using inversion formulas (17) is equal to the high frequency gain γ of the continuous compensator $C(s)$ obtained with formulas (5). Moreover, from (22) and (14) it follows that the maximum phase φ_{md} of the discrete compensator $C_d(z)$ obtained by using the inversion formulas (17) is equal to maximum phase φ_m of the correspondent continuous compensator $C(s)$: $\varphi_{md} = \varphi_m$.

Remark: from (11), (12) and (21) it follows that, for $0 < \omega < \frac{\pi}{T}$, the two parameters α and β are greater that 0.5 *if and only if* the two parameters τ_1 and τ_2 are positive:

$$(\alpha > 0.5,\ \beta > 0.5) \quad \Leftrightarrow \quad (\tau_1 > 0,\ \tau_2 > 0)$$

This result is important because it proofs that in the complex plane the set of all the points $P = M\,e^{j\varphi}$ for which $\alpha > 0.5$ and $\beta > 0.5$ coincides with the set of all the points for which $\tau_1 > 0$ and $\tau_2 > 0$, that is, *the admissible domains for the discrete inversion formulas (17) are equal to the admissible domains $\mathcal{D}_1$ and $\mathcal{D}_2$ shown in Fig. 3 for the continuous time inversion formulas (5)*. Moreover, from (22) it follows that $\mathcal{D}_1$ is the domain for which $C_d(z)$ is a lag compensator, while $\mathcal{D}_2$ is the domain for which $C_d(z)$ is a lead compensator.

Remark: the formulas (21) create a link between parameters τ_1, τ_2 and parameters α, β. Nevertheless, τ_1 and τ_2 determine the position of pole s_p and zero s_0 of function $C(s)$ on the s-plane, while α and β determine the position of pole z_p and zero z_0 of function $C_d(z)$ on the z-plane. Taking into account relations (2), (11) and (21), one obtains the following biunivocal correspondence between s-plane and z-plane:

$$s = \frac{\omega}{\tan\frac{\omega T}{2}}\left[\frac{z-1}{z+1}\right] \quad \leftrightarrow \quad z = \frac{1 + \frac{s}{\omega}\tan\frac{\omega T}{2}}{1 - \frac{s}{\omega}\tan\frac{\omega T}{2}}$$

This correspondence is the well known *bilinear transformation with prewarping*, see Fig. 10.

Numerical example. Let us refer to the same system $G(s) = 25/(s(s+1)(s+10))$ considered in the previous numerical example and let us try to design a discrete lead compensator

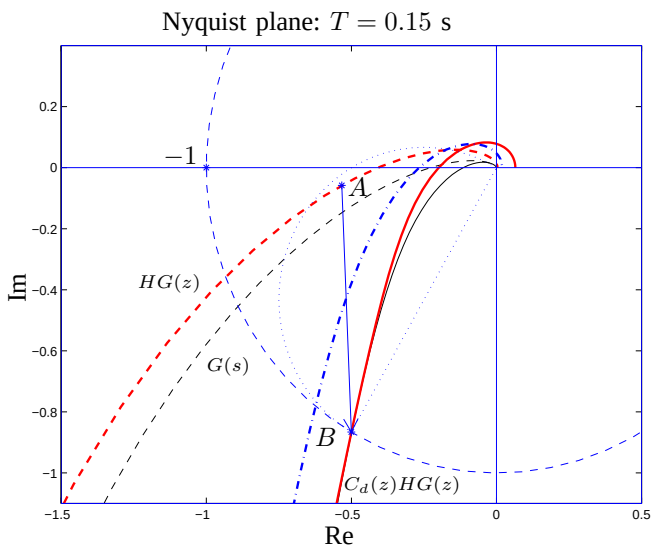

Fig. 11. Design of a lead discrete compensator $C_d(z)$ on the Nyquist plane when $T = 0.15$ s.

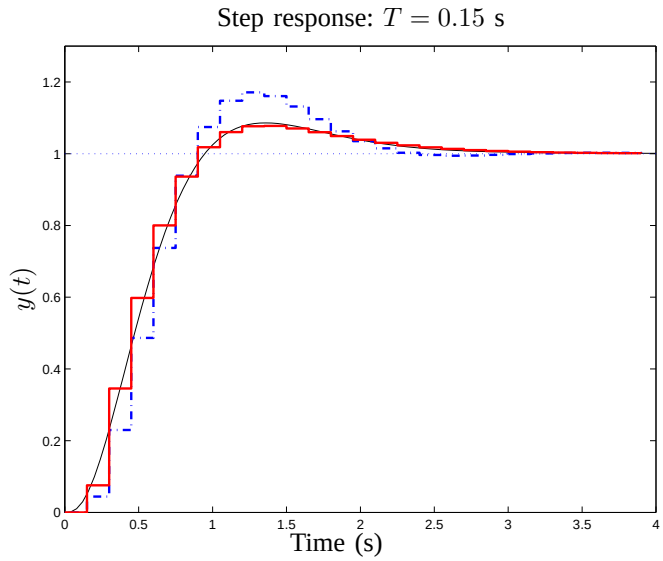

Fig. 12. Step responses of the considered systems controlled in closed loop. Sampling period: $T = 0.15$ s.

$C_d(z)$ which imposes a phase margin $M_\varphi = 60^o$ when $T = 0.15$ s. The discrete system $HG(z)$ to be controlled is the following:

$$HG(z) = \frac{(9.657z^2 + 26.66z + 4.259)10^{-3}}{z^3 - 2.084z^2 + 1.276z - 0.192}.$$

The design specification $M_\varphi = 60^o$ is unchanged, so point $B = M_B\, e^{j\varphi_B}$ is the same used in the previous case: $M_B = 1$, $\varphi_B = \pi + M_\varphi = 240^o$. Point $A = HG(e^{j\omega_A T}) = M_A e^{j\varphi_A}$ is now chosen on the Nyquist diagram of function $HG(z)$ when $\omega = \omega_A$:

$$M_A = 0.5361, \qquad \varphi_A = 186.2^o, \qquad \omega_A = 2.02.$$

From Fig. 11 one can easily verify that point A belongs to the admissible domain for a discrete lead compensator (the dotted half circle shown in Fig. 11). Parameters M and φ are now the following:

$$M = \frac{M_B}{M_A} = 1.865, \qquad \varphi = \varphi_B - \varphi_A = 53.76^o.$$

Substituting M, φ, ω and T in the discrete inversion formulas (17) one obtains $\alpha = 5.673$ and $\beta = 0.723$. The discrete compensator is now the following:

$$C_d(z) = \frac{1 + 5.673(z-1)}{1 + 0.723(z-1)}.$$

The red lines shown in Fig. 11 are the Nyquist diagrams of the discrete functions $HG(z)$ (dashed line) and $C_d(z)HG(z)$ (solid line). For comparison, in Fig. 11 are also reported the Nyquist diagrams of the continuous functions $G(s)$ (black dashed thin line) and $C_l(s)G(s)$ (black solid thin line).

In Fig. 12 are shown the step responses of the two systems $C_l(s)G(s)$ (black solid thin line) and $C_d(z)HG(z)$ (red solid thick line) controlled in closed loop. For comparison, in the same figure it is also reported (blue dash-dotted thick line) the step response of the system $C_b(z)HG(z)$ controlled in closed loop, when the discrete compensator $C_b(z)$ is obtained from $C_l(s)$ by using the bilinear correspondence between the s and z complex variables:

$$C_b(z) = C_l(s)|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{1.762\,z - 1.462}{0.384\,z - 0.084}.$$

From Fig. 12 it is evident that with the discrete inversion formulas (17) one obtains a discrete time behaviour very similar to the one obtained in the continuous time case.

IV. CONCLUSIONS

The discrete inversion formulas presented in this paper are particularly suitable for the design of discrete compensators in the frequency domain. Due to the simplicity of the formulas and the clarity of the related graphical design procedure, we think that these formulas can be very useful also for teaching.

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